ABSTRACT. Syllogisms with or without negative terms are studied by using Gergonne's ideas. Soundness, completeness, and decidability results are given.

1. BACKGROUND AND MOTIVATION

Gergonne [2] relates the familiar A, E, I, and O sentences without negative terms to five basic sentences that express the "Gergonne relations." These relations are: exclusion, identity, overlap, proper containment, and proper inclusion. What makes these relations especially interesting is that for any pair of non-empty class terms exactly one of them holds.

Faris [1] develops a formal system that takes the Gergonne relations as basic. His system takes advantage of Łukasiewicz's [4], which attempts to formalize the Aristotelian syllogistic. The following paper results from two ideas: 1) If Gergonne had been interested in studying A, E, I, and O sentences with negative terms, the count of Gergonne relations would be seven rather than five; and 2) The most Aristotelian way to develop a syllogistic system based on the these seven relations is by following Smiley's [5] rather than Łukasiewicz's [4].

After developing the Aristotelian "full syllogistic" based on seven relations, we will discuss a subsystem that is adequate for representing AEIO-syllogisms with or without negative terms.

2. THE SYSTEM

Sentences are defined by referring to:

- terms: A, B, C, ...
- simple quantifiers: =, =+, C++, C+-, C-, C--, Z
- comma: ,

Q₁, ..., Qₙ is a quantifier provided i) each Qᵢ (1 ≤ i ≤ n) is a simple quantifier, ii) Qᵢ precedes Qⱼ if i < j, where precedence among simple

quantifiers is indicated by the above ordering of simple quantifiers, and
iii) at least one quantifier is not a $Q_i$. No expressions are quantifiers other
than those generated by the above three conditions. So, for example,
$=, c^{++}, =, c^{+}, =, c^{--}, =, c^{+}, c^{--}, ZAB$ are sentences, but $=, c^{++}, AA$ is not. $Qab$ is a simple
sentence iff $Qab$ is a sentence and $Q$ is a simple quantifier. Read simple
sentences as follows: $=ab$ as “The $a$ are the $b$,” $=ab$ as “The $a$ are
the non-$b$,” $c^{++}ab$ as “The $a$ are properly included in the $b$,” $c^{+-}ab$ as
“The a are properly included in the non-$b$,” $c^{--}ab$ as “The non-$a$ are
properly included in the $b$,” $c^{+-}ab$ as “The non-$a$ are properly included
in the non-$b$,” and $Zab$ as “Some $a$ are $b$, some $a$ are non-$b$, some non-$a$
are $b$, and some non-$a$ are non-$b$.” Read $Q_1, \ldots, Q_nab$ by putting “or”
between sentences that correspond to $Q_iab$. So, read $=, c^{++}ab$ as “The
$a$ are the $b$,” or the $a$ are properly included in the $b$” (or “All $a$ are $b$.”)
$=, c^{--}, c^{+}, c^{--}, Zab$ may be read as “Some $a$ are not $b$.”

The deducibility relation ($\vdash$), relating sets of sentences to sentences,
is defined recursively. Read “$X \vdash y$” as “$y$ is deducible from $X$.” Set
brackets are omitted in the statement of the following definition. “$X, y$”
is short for “$X \cup \{y\}$” and “$x, y$” is short for “$\{x\} \cup \{y\}$.” “a”, “b”, . . .
range over terms; and “$p$”, “q”, . . . range over “+”, and “-”. $p^*$ is “+”
iff $p$ is “-”. $cd(Pab) = Qab$ iff every quantifier that does not occur in $P$
occurrs in $Q$. Read “$cd$” as “the contradictory of.”

\[(B1) \quad =ab \vdash =ba\]
\[(B2) \quad =ab \vdash =ba\]
\[(B3) \quad c^{pq}ab \vdash c^{q^*p^*}ba\]
\[(B4) \quad Zab \vdash Zba\]
\[(B5) \quad =ab, Qbc \vdash Qac, \quad \text{where } Q \text{ is } =, =-, \text{ or } c^{pq}\]
\[(B6) \quad =ab, =bc \vdash =ac\]
\[(B7) \quad =ab, c^{pq}bc \vdash c^{p^*q}ac\]
\[(B8) \quad c^{pq}ab, c^{qr}bc \vdash c^{pr}ac\]
\[(R1) \quad \text{If } X \vdash y \text{ and } y, z \vdash w \text{ then } X, z \vdash w\]
\[(R2) \quad \text{If } X, y \vdash Pab \text{ then } X, Qab \vdash cd(y) \text{ if no quantifier in } P \text{ is a}
\text{quantifier in } Q\]
(R3) If X, Pab ⊨ y and X, Qab ⊨ y then X, Rab ⊨ y if each quantifier in R is in P or Q.

(L1) X ⊨ y iff X ⊨ y in virtue of B1–R3.

So, for example, =¬AB, c++BC ⊨ c--AC (by B7) and c--AC, c++CD ⊨ c--AD (by B8). So =¬AB, c++BC, c++CD ⊨ =, =--C, c++, c--C, ZCD (by R2).

THEOREM 1. (D1) If X, y ⊨ Pab then X, y ⊨ cd(Qab) if no simple quantifier occurs in both P and Q. (D2) If X, y ⊨ cd(Pab) and X, y ⊨ cd(Qab) then X, y ⊨ cd(Rab) if each quantifier in R is in P or Q. (D3) If X, y ⊨ z and v, w ⊨ y then X, v, w ⊨ z.

Proof. Begin each proof by assuming the antecedent. (D1) Then X, Qab ⊨ cd(y) (by R2). Then X, y ⊨ cd(Qab) (by R2). (D2) Then X, Pab ⊨ cd(y) and X, Qab ⊨ cd(y) (by R2). Then X, Rab ⊨ cd(y) (by R3). Then X, y ⊨ cd(Rab) (by R2). (D3) Then X, cd(z) ⊨ cd(y) and v, cd(y) ⊨ cd(w) (by R2). Then X, v, cd(z) ⊨ cd(w) (by R1). Then X, v, w ⊨ z (by R2).

A model is a quadruple (W, v+, v−, v), where i) W is a non-empty set, ii) v+ and v− are functions that assign non-empty subsets of W to terms such that v+(a) ∪ v−(a) = W and v+(a) ∩ v−(a) = ∅, and iii) v is a function that assigns t or f to sentences such that the following conditions are met:

(i) v(=ab) = t iff v+(a) = v+(b)
(ii) v(=¬ab) = t iff v+(a) = v−(b)
(iii) v(≤pqab) = t iff v+(a) ⊂ v+(b)
(iv) v(Zab) = t iff v+(a) ∩ v+(b) ≠ ∅ for each p and q
(v) v(Q1, …, Qnab) = t iff for some i (1 ≤ i ≤ n) v(Qiab) = t

y is a semantic consequence of X (X ⊨ y) iff there is no model (W, …, v) such that v assigns t to every member of X and v assigns f to y. X is consistent iff there is a model (W, …, v) such that v assigns t to every member of X. X is inconsistent iff X is not consistent.

THEOREM 2 (Soundness). If X ⊨ y then X ⊨ y.
Proof. Straightforward. (For B1, note that for any model \( \langle W, \ldots, \nu \rangle \), if \( \nu_+(a) = \nu_+(b) \) then \( \nu_+(b) = \nu_+(a) \). For R2, suppose no quantifier in \( P \) is a quantifier in \( Q \), and suppose that \( X, Qab \not\in cd(y) \). Then there is a model \( \langle W, \ldots, \nu \rangle \) in which \( \nu \) assigns \( t \) to every member of \( X \), \( \nu(Qab) = t \), and \( \nu(cd(y)) = f \). Note that \( \nu(cd(y)) = f \) iff \( \nu(y) = t \). And note that since no quantifier in \( P \) is a quantifier in \( Q \), \( \nu(Pab) = f \). So \( X, y \not\equiv Pab \).

A chain is a set of sentences whose members can be arranged as a sequence \( \langle Q_1[a_1a_2], Q_2[a_2a_3], \ldots, Q_n[a_na_1] \rangle \), where \( Q_i[a_ia_j] \) is either \( Q_iaiaj \) or \( Q_iajai \) and where \( a_i \neq a_j \) if \( i \neq j \). So, for example, \( \{=AB, =-, C++CB, ZCA\} \) is a chain. A pair \( \langle X, y \rangle \) is a syllogism iff \( X \cup \{y\} \) is a chain. So \( \{AB, =-, C++CB, \} \) is a syllogism.

A normal chain is a set of sentences whose members can be arranged as a sequence \( \langle Q_1[a_1a_2], Q_2[a_2a_3], \ldots, Q_n[a_na_1] \rangle \), where \( a_i \neq a_j \) if \( i \neq j \). A simple normal chain is a normal chain in which each quantifier is simple. So, for example, \( \{=, =-AB, =BA\} \) is a normal chain. And \( \{=AB, =BA\} \) is a simple normal chain.

By definition, \( e(=ab) \) is \( =ba \), \( e(=ab) \) is \( =ab \), \( e(Cpqab) \) is \( C^r*a*b \), and \( e(Zab) \) is \( Zba \).

\( \{Q_1ab, Q_2bc\} \) a-reduces to \( Q_3ac \) iff the triple \( \langle Q_1ab, Q_2bc, Q_3ac \rangle \) is recorded on the following Table of Reductions:

\[
\begin{array}{ccc}
Q_1ab & = & =- \\
- & = & =- \\
- & = & =- \\
C^{pq} & C^{pq} & C^{pq*} \\
C^{pr} & C^{pq} & C^{pq*} \\
\end{array}
\]

So, for example, \( \{=AB, =BC\} \) a-reduces to \( =AC \), and \( \{C++AB, C+-BC\} \) a-reduces to \( C+-AC \).

If \( X_1 \) is a simple chain then a sequence of chains \( X_1, \ldots, X_m \) \( (=Y_1) \), \( \ldots, Y_n \) is a full reduction of \( X_1 \) to \( Y_n \) iff: i) \( X_m \) is a normal chain and if \( m > 1 \) then, for \( 1 \leq i < m \), if \( X_i \) has form \( \{Qab\} \cup Z \) then \( X_{i+1} \) has form \( \{e(Qab)\} \cup Z \), and ii) there is no pair in \( Y_n \) that a-reduces to a sentence and if \( n > 1 \) then, for \( 1 \leq i < n \), if \( Y_i \) has form \( \{Q_1ab, Q_2bc\} \cup Z \) then \( Y_{i+1} \) has form \( \{Q_3ac\} \cup Z \). \( X \) fully reduces to \( Y \) iff there is a full reduction of \( X \) to \( Y \).

THEOREM 3. Every simple chain fully reduces to a simple normal chain.

Proof. Assume \( X_1 \) is a simple chain. We construct a sequence of chains that is a full reduction of \( X_1 \) to \( Y_n \). Step 1: If \( X_1 \) is a simple
normal chain let \( X_1 = Y_1 \) and go to Step 2. If \( X_1 \) is not a simple normal chain find the alphabetically first pair of sentences in \( X_1 \) of form \((Qab, Qcb)\) and replace \( Qcb \) with \( e(Qcb) \), forming \( X_2 \). Repeat Step 1 (with \( "X_1" \) in place of \( "X_1" \)).

Step 2: If no pair of sentences in \( Y_1 \) a-reduces to a sentence, then \( X_1 \) fully reduces to \( Y_1 \). If a pair of sentences in \( Y_1 \) a-reduces to a sentence \( x \) find the alphabetically first pair that a-reduces to \( x \) and form \( Y_2 \) by replacing this pair with \( x \). Repeat Step 2 (with \( "Y_1" \) in place of \( "Y_1" \)). \( \Box \).

So, for example, given the sequence \( \langle \{=AB\}, \{=AB, =BA\} \rangle \), \( \{=AB\} \) fully reduces to \( \{=AB, =BA\} \). And, given the sequence \( \langle \{C++AB, C--BC, C++CA\}, \{C++AB, C++BC, C++CA\}, \{C++AC, C++CA\} \rangle \), \( \{C++AB, C--BC, C++CA\} \) fully reduces to \( \{C++AC, C++CA\} \). Some chains fully reduce to themselves. \( \{C++AB, C--BC, ZCA\} \) is an example.

\( \{P_1[a_1a_2], \ldots, P_n[a_n a_1]\} \) is a strand of \( \{Q_1[a_1 a_2], \ldots, Q_n[a_n a_1]\} \) iff each \( P_i \) is a simple quantifier in \( Q_i \) and \( a_i \) is the first term in \( P_i[a_{i+1}a_i] \) iff \( a_i \) is the first term in \( Q_i[a_{i+1}a_i] \), where \( P[a] \) is \( Pab \) or \( Pba \). So, for example, \( \{=AB, =- AB\} \) is a strand of \( \{=, C++AB, =-, C++AB\} \).

A simple normal chain is a \( cd\)-pair if it has one of the following forms:

\[
\{=ab, =- ba \text{ (or } Cpqba \text{ or } Zba\}\}, \{= ab, Cpqba \text{ (or } Zba\}\},
\text{or}\ \{Cpqab, Cqrba \text{ (or } Zba\}\}.
\]

THEOREM 4 (Syntactic decision procedure). If \( \langle X, y \rangle \) is a syllogism then \( X \vdash y \) iff every strand of \( X \cup \{cd(y)\} \) fully reduces to a \( cd\)-pair.

Proof. Assume \( \langle X, y \rangle \) is a syllogism. We use Lemmas 1–3, below. (If) Suppose every strand of \( X \), \( cd(y) \) fully reduces to a \( cd\)-pair. Then by Lemmas 1 and 2, \( X, cd(y) \) is inconsistent. Then \( X \vdash y \). (Only if) Suppose some strand of \( X \), \( cd(y) \) does not fully reduce to a \( cd\)-pair. Then, by Theorem 3, some strand of \( X \), \( cd(y) \) fully reduces to a simple normal chain that is not a \( cd\)-pair. Then, by Lemmas 1 and 3, \( X \), \( cd(y) \) is consistent. Then \( X \not\vdash y \). \( \Box \)

LEMMA 1. A chain is inconsistent iff each of its strands is inconsistent.

Proof. Note that a model satisfies \( \{Q_1ab\} \cup X \) and \( \{Q_2ab\} \cup X \) if it satisfies \( \{Q_3ab\} \cup X \), where the quantifiers in \( Q_3 \) are the quantifiers in \( Q_1 \) and \( Q_2 \). \( \Box \)

LEMMA 2. If a simple chain \( X \) fully reduces to a \( cd\)-pair, then \( X \) is inconsistent.
LEMMA 2.1. Each cd-pair is inconsistent.

LEMMA 2.2. If a simple normal chain \( \{Q_3ac\} \cup X \) is inconsistent and \( \{Q_1ab, Q_3bc\} \) a-reduces to \( Q_3ac \), then \( \{Q_1ab, Q_3bc\} \cup X \) is inconsistent.

LEMMA 2.3. If a simple chain \( \{Qab\} \cup X \) is inconsistent, then \( \{e(Qab)\} \cup X \) is inconsistent.

LEMMA 3. If a simple chain \( X \) fully reduces to a simple normal chain that is not a cd-pair, then \( X \) is satisfied in an \( m \)-model, where \( m \leq n + 2 \) and \( n \) is the number of terms in \( X \).

Proof. Use the following three lemmas.

LEMMA 3.1. If a simple chain fully reduces to a simple normal chain \( X \) that is not a cd-pair, then \( X \) is satisfied in an \( m \)-model, where \( m \leq n + 2 \) and \( n \) is the number of terms in \( X \).

Proof. Assume the antecedent. We consider three cases determined by the number of occurrences of "Z" in \( X \).

Case 1: "Z" does not occur in \( X \). If either "=" or "=−" occurs in \( X \) then \( X \) has form \( \{=ab, =ba\} \) or \( \{=−ab, =−ba\} \). Use \( \langle \{1, 2\}, \ldots, \nu \rangle \), where, for each term \( x \), \( \nu_+(x) = \{1\} \). If neither "=" or "=−" occurs in \( X \) then \( X \) has form \( \{\nu(P_1a_1a_2), \ldots, \nu(P_{n-1}a_{i+1}), \ldots, \nu(P_{n-1}a_na_1)\} \), where \( p_{2i} = p_{2i+1}^* \), for \( 1 \leq i < n \), and \( p_{2n} = p_1^* \). We use induction on the number \( n \) of terms in \( X \) to show that \( X \) is satisfied in a 3-model. Basis step: \( n = 2 \). \( X \) has form \( \{\nu(P_1a_2a_2), \nu(P_2a_2a_1)\} \). Use \( \langle \{1, 2, 3\}, \ldots, \nu \rangle \), where \( \nu_{p_1}(a) = \{1\} \), and, for terms \( x \) other than \( a \), \( \nu_+(x) = \{1, 2\} \). Induction step: \( n > 2 \). By the induction hypothesis \( \{\nu(P_{i-1}a_1a_3), \ldots, \nu(P_{i-1}a_{i+1}), \ldots, \nu(P_{n-1}a_na_1)\} \) is satisfied in a 3-model \( \langle W_i, \ldots, \nu \rangle \), where \( p_{2i} = p_{2i+1}^* \), for \( 2 \leq i < n \), and \( p_{2n} = p_1^* \). Construct a model \( \langle W_i, \ldots, \nu' \rangle \), \( \nu'_2(a_2) = \nu_{p_1}(a_1) \cup \nu'_3(a_3) \), and, for other terms \( x \), \( \nu'_+(x) = \nu_+(x) \). Then \( \nu'(\nu(P_1a_2a_2)) = t. \nu'_3(a_2) = \nu_{p_4}(a_3) - \nu_{p_1}(a_1) \) and \( p_2^* = p_3 \). So \( \nu'(\nu(P_3a_2a_3)) = t. \)

Case 2: "Z" occurs exactly once in \( X \). Then \( X \) has at least three members and has form \( \{Zab\} \cup \{\nu(P_1bc, \ldots, \nu(r^4a)\} \). We use induction on the number of terms in \( X \) to show that \( X \) is satisfied in a 4-model. Basis step: \( n = 3 \). \( X \) has form \( \{Zab\} \cup \{\nu(P_1bc, \nu(r^4ca)\} \). Construct a model \( \langle \{1, 2, 3, 4\}, \ldots, \nu \rangle \), where \( \nu_r(a) = \{1, 2\} \), \( \nu_p(b) = \{1, 3\} \), and, for other terms \( x, \nu_q(x) = \{1, 3, 4\} \). Induction step: \( n > 3 \). Follow the model construction in the induction step in Case 1.
LEMMA 3.2. Suppose QI is $v'(e)$, $v_q(e)$, $Q$, and for terms $x$ other than $b$, $\nu_+^t(x) = \nu_+(x)$. Suppose Q is “$=$”. Construct model $\langle W, \ldots, \nu' \rangle$, where $\nu'_+ (b) = \nu_+(c)$ and, for terms $x$ other than $b$, $\nu_+^t(x) = \nu_+(x)$. Suppose Q is “$=$”. Construct model $\langle W, \ldots, \nu' \rangle$, where $\nu'_+ (b) = \nu_+(a) \cap \nu_+(c) \cup (\nu_-(a) \cap \nu_-(c))$, and, for other terms $x$, $\nu_+^t(x) = \nu_+(x)$. Finally, suppose that Q is “$\subseteq_{pq}$”. The strategy is to construct a model $\langle W', \ldots, \nu' \rangle$ such that X is satisfied in it by letting $\nu'_+ (b) = \nu_q(e) \cup \{M\}$, and, for terms $x$ other than $b$, $\nu'_+ (x) = \nu_+(x)$. Then $\nu'(Zab) = t$ and $\nu''(\subseteq_{pq}bc) = t$.

We construct $\langle W', \ldots, \nu' \rangle$. If $a$ and $c$ are the only terms in X, let $\alpha = \nu_+(a) \cap \nu_q(c)$ (and, thus, $a$ has at least one member. If terms $d_1, \ldots, d_n$ occur in X, where these terms are other than “a” or “c”, pick $p_1 - p_n$ such that $\alpha$ has at least one member, where $\alpha = \nu_+(a) \cap \nu_q(c) \cap \nu_p(d_1) \cap \cdots \cap \nu_p(d_n)$. Let $W' = W \cup \{M\}$, where $M \notin W$. Let $\nu'_+ (x) = \nu_+(x) \cup \{M\}$ if $\alpha \subseteq \nu_+(x)$; otherwise, let $\nu'_+ (x) = \nu_+(x)$. Then $\nu'_- (x) = \nu_-(x) \cup \{M\}$ if $\alpha \subseteq \nu_-(x)$; otherwise, $\nu'_- (x) = \nu_-(x)$. We show that X is satisfied in $\langle W', \ldots, \nu' \rangle$. Suppose $\nu(Qde) = t$. Suppose Q is “$=$”. Then $\nu'_+ (d) = \nu_+(d) \cup \{M\}$ and $\nu'_+ (e) = \nu_+(e) \cup \{M\}$ or $\nu'_+ (d) = \nu_+(d)$ and $\nu'_+ (e) = \nu_+(e)$. Then $\nu' (\text{de}) = t$. Suppose Q is “$=$”. Then $\nu'_+ (d) = \nu_+ (d) \cup \{M\}$ and $\nu'_+ (e) = \nu_+(e)$ or $\nu'_+ (d) = \nu_+(d)$ and $\nu'_+ (e) = \nu_-(e) \cup \{M\}$. Then $\nu' (-\text{de}) = t$. Suppose Q is “$\subseteq_{pq}$”. If $\alpha \subseteq \nu_p(d)$ then $\nu'_+ (d) = \nu_p(d) \cup \{M\}$ and $\nu'_+ (e) = \nu_q(e) \cup \{M\}$. If $\alpha \notin \nu_p(d)$ then $\nu'_+ (d) = \nu_p(d)$ and either $\nu'_+ (e) = \nu_q(e)$ or $\nu'_+ (e) = \nu_q(e) \cup \{M\}$. Then $\nu' (\text{pq} \text{de}) = t$. Finally, suppose Q is “$Z$”. Then, for any $p$ and $q$, $\nu_p(d) \cap \nu_q(e) \subseteq \nu'_+ (d) \cap \nu'_+ (e)$. Then $\nu'(Zde) = t$. 

LEMMA 3.2. If a simple chain $\{Q_3ac\} \cup X$ is satisfied in an n-model $\langle W, \ldots, \nu \rangle$, where $n$ is the number of terms in $\{Q_3ac\} \cup X$, if $\{Q_1ab, Q_2bc\}$ a-reduces to $Q_3ac$, then $\{Q_1ab, Q_2bc\} \cup X$ is satisfied in an m-model, where $m \leq n$ and $n$ is the number of terms in $\{Q_1ab, Q_2bc\} \cup X$.

Proof. Assume the antecedent. Suppose Q1 is “$=$”. Construct $\langle W, \ldots, \nu' \rangle$, where $\nu'_+ (b) = \nu_+(a)$, and, for terms $x$ other than $b$, $\nu'_+ (x) = \nu_+(x)$. Suppose Q1 is “$=$”. Construct $\langle W, \ldots, \nu' \rangle$, where $\nu'_+ (b) = \nu_-(a)$, and,
for terms \( x \) other than \( b \), \( \nu'_+(x) = \nu_+(x) \). Use similar constructions if \( Q_2 \) is \("="\) or \("\sim\"\). So, the only \( a \)-reduction left is this: \( \{C^{pq}ab, C^{pq}bc\} \) \( a \)-reduces to \( C^{pq}ac \). Construct a model \( \langle W', \ldots, \nu' \rangle \) such that \( W' = W \cup \{M\}, M \notin W \), and \( \nu'_{p'}(a) \cap \nu'_+(c) \) has at least two members, including \( M \). To do this follow the procedure in Case 3 of Lemma 3.1. Then construct a model \( \langle W', \ldots, \nu'' \rangle \) such that \( \nu''(b) = \nu'_{p'}(a) \cup \{M\} \) and, for other terms \( x \), \( \nu''_{p'}(x) = \nu'_+(x) \).

**LEMMA 3.3.** If a simple chain \( \{Qab\} \cup X \) is satisfied in an \( n \)-model, where \( n \) is the number of terms in \( \{Qab\} \cup X \), then \( \{e(Qab)\} \cup X \) is satisfied in an \( n \)-model, where \( n \) is the number of terms in \( \{e(Qab)\} \cup X \).

**Proof.** Straightforward.

**THEOREM 5** (Semantic decision procedure). If \( \langle X, y \rangle \) is a syllogism then \( X \vdash y \) iff \( X, cd(y) \) is not satisfied in an \( m \)-model, where \( m \leq n + 2 \) and \( n \) is the number of terms in \( X \).

**Proof.** Assume \( \langle X, y \rangle \) is a syllogism. (Only if) Immediate. (If) Assume \( X, cd(y) \) is not satisfied in an \( m \)-model, where \( m \leq n + 2 \) and \( n \) is the number of terms in \( X \). Then every strand of \( X, cd(y) \) is not satisfied in an \( m \)-model where \( m \leq n + 2 \) and \( n \) is the number of terms in \( X, cd(y) \). Then every strand of \( X, cd(y) \) fully reduces to a cd-pair (by Theorem 3 and Lemma 3 of Theorem 4). Then \( X \vdash y \) (by Theorem 4).

Given Theorem 5, it is natural to ask whether, for any \( n \), there is an \( n \)-termed syllogism that requires an \( n + 2 \) model to show that it is invalid. The answer is Yes. If \( n = 2 \), use \( \{(Z_1a_2), cd(Z_2a_1)\} \). If \( n > 2 \), use \( \{(Z_1a_2, C^{++}a_2a_3, \ldots, C^{++}a_{n-1}a_n), cd(Z_{an}a_1)\} \). Consider a model \( \langle W, \ldots, \nu \rangle \) in which \( \{Z_1a_2, C^{++}a_2a_3, \ldots, C^{++}a_{n-1}a_n, Z_{an}a_1\} \) is satisfied. Note that \( \nu_+(a_1) \) has at least two members, since \( \nu(Z_1a_2) = t \). So \( \nu_+(a_n) \) has at least \( n \) members. \( \nu_-(a_n) \) has at least two members since \( \nu(Z_{an}a_1) = t \).

**THEOREM 6** (Completeness). If \( \langle X, y \rangle \) is a syllogism and \( X \vdash y \) then \( X \vdash y \).

**Proof.** Assume the antecedent. Then, by Theorem 4, every strand of \( X, cd(y) \) fully reduces to a cd-pair. So, by Lemmas 1–4, below, \( X \vdash cd(cd(y)) \). That is \( X \vdash y \).

**LEMMA 1.** If \( \{x, y\} \) is a cd-pair, then \( x \vdash cd(y) \).

**Proof.** 1) \( =ab \vdash =ba \) (by B1). So \( =ab \vdash cd(=ba) \) (and \( cd(C^{pq}ba) \) and \( cd(Zba) \)) (by D1). 2) \( =ba \vdash =ab \) (by B2). So \( =ba \vdash cd(=ab) \) (by D1). And \( =ab \vdash =ba \) (by B2). So \( =ab \vdash cd(C^{pq}ba) \) (and
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cd(Zba) (by D1). 3) \( C^{pq}ba \vdash C^{qr}\neg ab \) (by B3). So \( C^{pq}ba \vdash cd(\neg ab) \) (and \( cd(\neg ab) \)) (by D1). \( C^{pq}ba \vdash C^{qr}ba \) (by B3). So \( C^{pq}ba \vdash cd(C^{qr}ba) \) (by D1). \( C^{qr}ba \vdash C^{sr}\neg ab \) (by B3). So \( C^{qr}ba \vdash cd(C^{pq}ba) \) (by D1). 4) \( Zba \vdash Zab \) (by B4). So \( Zba \vdash cd(\neg ab) \) (and \( cd(\neg ab) \)) (by D1).

**LEMMA 2.** If \( X = \{Q_{3ac}\} \cup Z, Y = \{Q_{1ab},Q_{2bc}\} \cup Z, \{Q_{1ab},Q_{2bc}\} \) a-reduces to \( Q_{3ac} \), and \( X - \{x\} \vdash cd(x) \), for every \( x \) such that \( x \in X \), then \( Y - \{y\} \vdash cd(y) \), for every \( y \) such that \( y \in Y \).

**Proof.** Assume the antecedent. Case 1: \( y \in Z \). \( \{Q_{3ac}\} \cup Z - \{y\} \vdash cd(y) \). We use

**LEMMA 2.1.** If \( \{Q_{1ab},Q_{2bc}\} \) a-reduces to \( Q_{3ac} \) then \( Q_{1ab}, Q_{2bc} \vdash Q_{3ac} \).

**Proof.** Given B5–B8, we only need to show that: i) \( \neg ab = bc \vdash \neg ac \); ii) \( C^{pq}ab, =bc \vdash C^{pq}ac \); and iii) \( C^{pq}ab, =bc \vdash C^{pq}ac \). For i), \( =bc \vdash =cb \) (by B1) and \( \neg ab \vdash \neg ba \) (by B2). \( =bc, =\neg ba \vdash \neg ca \) (by B5). So \( \neg ab, =bc \vdash \neg ca \) (by D3). \( =ca \vdash \neg ac \) (by B2). So \( \neg ab, =bc \vdash \neg ac \) (by R1). Use similar reasoning for ii) and iii).

So \( Q_{1ab}, Q_{2bc} \vdash Q_{3ac} \) (by Lemma 2.1). So \( \{Q_{1ab},Q_{2bc}\} \cup Z - \{y\} \vdash cd(y) \) (by D3).

Case 2: \( y = Q_{1ab} \). \( Z \vdash cd(Q_{3ac}) \). \( Q_{2bc}, cd(Q_{3ac}) \vdash cd(Q_{1ab}) \) (by Lemma 2.1 and R2). So \( Z, Q_{2bc} \vdash cd(Q_{1ab}) \) (by R1).

Case 3: \( y = Q_{2bc} \). Use reasoning similar to that for Case 2. 

**LEMMA 3.** If \( X = \{Qab\} \cup Z, Y = \{e(Qab)\} \cup Z, and X - \{x\} \vdash cd(c) \), for every \( x \) such that \( x \in X \), then \( Y - \{y\} \vdash cd(y) \), for every \( y \) such that \( y \in Y \).

**Proof.** Assume the antecedent. Case 1: \( y \in Z \). \( \{Qab\} \cup Z - \{y\} \vdash cd(y) \). \( e(Qab) \vdash Qab \) (by B1–B4). So \( \{e(Qab)\} \cup Z - \{y\} \vdash cd(y) \) (by D3). Case 2: \( y = e(Qab) \). \( Z \vdash cd(Qab) \). \( cd(Qab) \vdash cd(e(Qab)) \) (by B1–B4 and R2). So \( Z \vdash cd(e(Qab)) \) (by R1).

**LEMMA 4.** If each strand \( Y \cup \{z\} \) of \( X \cup \{y\} \) is such that \( Y \vdash cd(z) \), then \( X \vdash cd(y) \).

**Proof.** Use D2 and R3. (The proof is illustrated below.)

The proof of the above theorem provides a mechanical procedure for showing that \( X \vdash y \) given that \( X \vdash y \). We illustrate by showing that \( \neg, C^{++}AB, =BC \vdash cd(\neg, C^{++}AC) \). First, fully reduce the following strands as indicated: i) \( \{\neg AB, =BC, =\neg AC\} \) to \( \{\neg AB, =BC, =\neg CA\} \); ii) \( \{\neg AB, =BC, =AC\} \) to \( \{\neg AB, =BC, =\neg CA\} \).
to \( \{=AC, C^{+}CA\} \); iii) \( \{C^{++}AB, =BC, =AC\} \) to \( \{C^{++}AB, =BC, =CA\} \); and iv) \( \{C^{++}AB, =BC, C^{+}AC\} \) to \( \{C^{++}AC, C^{+}CA\} \). By the proof of Lemma 1: \( =AC \vdash cd(=CA); =AC \vdash cd(C^{+}CA) \); \( C^{++}AC \vdash cd(=CA) \); and \( C^{++}AC \vdash cd(C^{+}CA) \). By the proof of Lemma 2: \( =AB, =AC \vdash cd(=AC); =AB, =AC \vdash cd(C^{+}CA) \); \( C^{++}AB, =BC \vdash cd(=AC) \); and \( C^{++}AB, =BC \vdash cd(C^{+}CA) \). By the proof of Lemma 3: \( =AB, =AC \vdash cd(=AC); =AB, =AC \vdash cd(C^{+}AC) \); \( C^{++}AB, =BC \vdash cd(=AC) \); and \( C^{++}AB, =BC \vdash cd(C^{+}AC) \). By D2, \( =AB, =AC \vdash cd(=, C^{+}AC) \) and \( C^{++}AB, =BC \vdash cd(=, C^{+}AC) \). By R3, \( =, C^{++}AB, =AC \vdash cd(=, C^{+}AC) \).

3. GERGONNE SYLLOGISMS

Faris [1] is motivated by an interest in providing a decision procedure for Gergonne syllogisms. Faris construes syllogisms as sentences, following Łukasiewicz’s [4], rather than as inferences, as in Smiley’s [5]. For us, a Gergonne syllogism is a syllogism consisting of Gergonne sentences, which are defined as follows, using Gergonne’s symbols in [2]. The Gergonne-quantifiers are: \( H =_{df} =, C^{+}C; X =_{df} C^{+}Z; I =_{df} =; C =_{df} C^{++}, \) and \( Z =_{df} C^{+}, C^{+}C, C^{+}Z \). A Gergonne-sentence is any sentence of form \( Ql, \ldots, Qmab \), where \( Qi \) is a Gergonne-quantifier. So Theorem 4 above gives an alternative solution to the problem that motivated Faris’ [1], since every Gergonne syllogism may be expressed in our system. Note, for example, that “\( H, XAB \)” is expressed as “\( =, C^{+}C, C^{+}C, ZAB \)”.

4. SYSTEM B

In this section we develop a subsystem B which expresses no sentences other than those that may be expressed by using sentences of form “All . . . are \(- - \)-”, “No . . . are \(- - \)-”, “Some . . . are \(- - \)-”, or “Some . . . are not \(- - \)-”, where the blanks are filled by expressions of form \( x \) or non-\( x \) (the “A, E, I, and O sentences, respectively, with or without negative terms.”)

The B-quantifiers (“B” for “basic”) are: \( =, C^{+}A(A^{+}) \); \( =, C^{+}C(A^{++}) \); \( =, C^{+}C(A^{+}) \); \( =, C^{-}A(A^{-}) \); \( =, C^{-}C(A^{-}) \); \( =, C^{-}C(A^{+}) \); \( =, C^{+}C(A^{+}) \); \( =, C^{+}C(A^{+}) \); \( =, C^{+}C(A^{+}) \); \( =, C^{+}C(A^{+}) \); \( =, C^{+}C(A^{+}) \); \( =, C^{+}C(A^{+}) \); and \( =, C^{+}C(A^{+}) \). \( Qab \) is a B-sentence iff \( Qab \) is a sentence and \( Q \) is a B-quantifier. So, for example, \( A^{+}A \) is a B-sentence. And a B-syllogism is a syllogism composed of B-sentences.
We define \( y \) is \( B \)-deducible from \( X \) (\( X \vdash_B y \)), where \( X, y \) is a set of \( B \)-sentences, and where \( \text{ct}(A^{pq}ab) = A^{pq*}ba \), \( \text{cd}(A^{pq}ab) = O^{pq}ab \), and \( \text{cd}(O^{pq}ab) = A^{pq}ab \):

\[(B1) \quad A^{pq}ab \vdash_B A^{q*r}ba\]
\[(B2) \quad A^{pq}ab, A^{qr}bc \vdash_B A^{pr}ac\]
\[(R1) \quad \text{If } X \vdash_B y \text{ and } y, z \vdash_B w \text{ then } X, z \vdash_B w\]
\[(R2) \quad \text{If } X, y \vdash_B \text{ct}(z) \text{ or } \text{cd}(z) \text{ then } X, z \vdash_B \text{cd}(y)\]
\[(L1) \quad \text{If } X \vdash y, \text{ then } X \vdash y \text{ in virtue of } B1-R2.\]

**THEOREM 7.** (D1) \text{If } X, y \vdash_B z \text{ and } u, v \vdash_B y \text{ then } X, u, v \vdash_B z.\]

*Proof.* Use the reasoning for the proof of Theorem 1. \(\square\)

**THEOREM 8** (Soundness). \text{If } X \vdash_B y \text{ then } X \models y.

*Proof.* Straightforward. \(\square\)

By definition, \( e(A^{pq}ab) \) is \( A^{q*r}p*ba \) and \( e(O^{pq}ab) \) is \( O^{q*r}p*ba \). And, by definition, a set \( X \) of sentences \( b \)-reduces to a sentence \( y \) iff \( \langle X, y \rangle \) has form \( \{A^{pq}ab, A^{qr}bc\}, A^{pr}ac \).

If \( X \) is a chain of \( B \)-sentences then a sequence of chains \( X_1, \ldots, X_m \) \( (=Y_1), \ldots, Y_n \) is a full \( B \)-reduction of \( X \) to \( Y_n \) iff: i) \( X_m \) is a normal chain and if \( m > 1 \), then, for \( 1 \leq i < m \), if \( X_i \) has form \( \{Qab\} \cup Y \), then \( X_H 1 \) has form \( \{e(Qab)\} \cup Y \); and ii) there is no pair of sentences in \( Y_n \) that \( b \)-reduces to a sentence and if \( n > 1 \) then, for \( 1 \leq i < n \), \( Y_i \) has form \( \{A^{pq}ab, A^{qr}bc\} \cup X \) and \( Y_{i+1} \) has form \( \{A^{pr}ac\} \cup X \). \( X \) fully \( B \)-reduces to \( Y \) iff there is a full \( B \)-reduction of \( X \) to \( Y \).

**THEOREM 9.** Every chain of \( B \)-sentences fully \( B \)-reduces to a normal chain of \( B \)-sentences.

*Proof.* Imitate the proof of Theorem 3. \(\square\)

A normal chain of \( B \)-sentences is a \( cd \)-\( B \)-pair iff it has one of the following forms: \( \{A^{pq}ab, A^{q*r}ba\} \) or \( \{A^{pq}ab, O^{q*r}p*ba\} \).

**THEOREM 10** (Syntactic decision procedure). \( \langle X, y \rangle \) is a B-syllogism then \( X \models y \) iff \( X, \text{cd}(y) \) fully \( B \)-reduces to a \( cd \)-\( B \)-pair.

*Proof.* Assume \( \langle X, y \rangle \) is a B-syllogism. We use Lemmas 1 and 2, below. (If) Suppose \( X, \text{cd}(y) \) fully \( B \)-reduces to a \( cd \)-\( B \)-pair. Then, by Lemma 1, \( X, \text{cd}(y) \) is consistent. Then \( X \models y \). (Only if) Suppose \( X \models y \). Then \( X, \text{cd}(y) \) is inconsistent. Then \( X, \text{cd}(y) \) fully \( B \)-reduces to a \( cd \)-\( B \)-pair (by Lemma 2 and Theorem 9). \(\square\)
LEMMA 1. If a chain $X$ of $B$-sentences fully $B$-reduces to a $cd$-$B$-pair then $X$ is inconsistent.

Proof. Imitate the proof of Lemma 2 of Theorem 4.

LEMMA 2. If a chain $X$ of $B$-sentences fully $B$-reduces to a normal chain of $B$-sentences that is not a $cd$-$B$-pair, then $X$ is satisfied in a $3$-model.

LEMMA 2.1. If a chain of $B$-sentences fully $B$-reduces to a normal chain of $B$-sentences $X$ that is not a $cd$-$B$-pair, then $X$ is satisfied in a $3$-model.

Proof. Assume the antecedent. We consider three cases determined by the number of occurrences of “$O$” in $X$.

Case 1: “$O$” does not occur in $X$. We use induction on the number $n$ of terms in $X$. Basis step: $n = 2$. Then $X$ has form $\{A^pab, A^pba\}$ or form $\{A^pqa, A^qpb\}$. If $p = q$, use $\{(1, 2, 3), \ldots, \nu\}$, where $\nu_+(x) = \{1\}$, and, for terms $x$ other than $a$, $\nu_+(x) = \{2, 3\}$. Induction step: $n > 2$. Then $X$ has form $\{A^p2a_1a_2, \ldots, A^p2i-1a_i+1, \ldots, A^p2n-1a_na_1\}$, where $p_{2i} = p_{2i+1}$. By Case 1 of Lemma 3.1 of Theorem 4, $\{c^p2a_1a_2, \ldots, c^p2i-1a_i+1, \ldots, c^p2n-1a_na_1\}$, where $p_{2i} = p_{2i+1}$, for $1 \leq i < n$, and $p_{2n} = p_1$, is satisfied in a $3$-model. So $X$ is satisfied in a $3$-model.

Case 2: “$O$” occurs exactly once in $X$. Suppose there are exactly two terms in $X$. Then $X$ has form $A^pqa, O^qrb$ (or $O^qa, A^qrb$). $3$-models are easily constructed to show that $X$ is consistent. Suppose there are more than two terms in $X$. We use induction on the number $n$ of terms in $X$ to show that $X$ is satisfied in a $3$-model. Basis step: $n = 3$. Then $X$ has form $\{O^pqa, A^r^sb, A^s^uca\}$. So there is a strand of $X$ with one of the following forms: $\{c^pqa, c^r^sb, c^s^uca\}$, $\{c^pqa, c^r^sb, c^s^uca\}$, and $\{c^pqa, c^r^sb, c^s^uca\}$. So, by Case 1 of Lemma 3.1 of Theorem 4, $X$ is consistent if $p = u$ or $p = r$. Suppose $p \neq u$ and $q \neq r$. Then $X$ has form $\{O^pqa, A^r^sb, A^s^uca\}$. If $p = q$, there is a strand of $X$ with form $\{=ab, =bc, =ca\}$ or form $\{=ab, =bc, =ca\}$. If $p \neq q$, there is a strand of $X$ with form $\{=ab, =bc, =ca\}$ or form $\{=ab, =bc, =ca\}$. Each of these four chains can easily be shown to be satisfied in a $3$-model. Induction step: $n > 3$. $X$ has form $\{O^pqa, A^r^sb, A^s^uca, \ldots\}$. By the induction hypothesis, $O^pqa, A^rs^uca, \ldots$ is satisfied in a $3$-model $(W, \ldots, \nu)$. Construct $\langle W, \ldots, \nu'\rangle$, where $\nu'_+(c) = \nu_+(b)$, and, for terms $x$ other than $c$, $\nu'_+(x) = \nu_+(x)$. Note that $\nu'(A^r^sb) = t$, since $\nu'_+(b) = \nu_+(c)$, and $\nu'(A^s^uca) = t$, since $\nu'_+(c) = \nu_+(b)$.

Case 3: “$O$” occurs at least twice in $X$. We use induction on the number of terms $n$ in $X$. Basis step: $n = 2$. $X$ has form $\{O^pqa, O^r^rb\}$. It is
easy to show that X is satisfied in a 3-model. Induction step: \( n > 2 \). X has form \( \{O^{pq}ab, Q^{rs}bc, \ldots, O^{uv}de, \ldots\} \). Suppose Q is "A" and \( r = s \) or Q is "O" and \( r \neq s \). By the induction hypothesis, \( \{O^{pq}ac, \ldots, O^{uv}de, \ldots\} \) is satisfied in a 3-model \( \langle W, \ldots, \nu \rangle \). Construct 3-model \( \langle W, \ldots, \nu' \rangle \), where \( \nu'_q(b) = \nu_q(c) \), and, for terms \( x \) other than \( c \), \( \nu'_+(x) = \nu_+(x) \). Suppose Q is "A" and \( r \neq s \) or Q is "O" and \( r = s \). By the induction hypothesis, \( \{O^{pq}ac, \ldots, O^{uv}de, \ldots\} \) is satisfied in a 3-model \( \langle W, \ldots, \nu \rangle \). Construct 3-model \( \langle W, \ldots, \nu' \rangle \), where \( \nu'_q(b) = \nu_q(c) \), and, for terms \( x \) other than \( c \), \( \nu'_+(x) = \nu_+(x) \).

**LEMMA 2.2.** If \( \{A^{pq}ac\} \cup Y \) is satisfied in a 3-model and if term \( b \) does not occur in a member of \( Y \), then \( \{A^{pq}ab, A^{qr}bc\} \cup Y \) is satisfied in a 3-model.

**Proof.** Assume that \( \{A^{pq}ac\} \cup Y \) is satisfied in a 3-model \( \langle W, \ldots, \nu \rangle \). Construct \( \langle W, \ldots, \nu' \rangle \), where \( \nu'_p(b) = \nu_+(a) \), and, for terms \( x \) other than \( b \), \( \nu'_+(x) = \nu_+(x) \).

**LEMMA 2.3.** If \( \{Qab\} \cup Y \) is satisfied in a 3-model, then \( \{e(Qab)\} \cup Y \) is satisfied in a 3-model.

**Proof.** Straightforward.

**THEOREM 11** (Semantic decision procedure). If \( \langle X, y \rangle \) is a B-syllogism then \( X 1= Y \) iff \( X, cd(y) \) is not satisfied in a 3-model.

**Proof.** Assume \( \langle X, y \rangle \) is a B-syllogism. (Only if) Immediate. (If) Suppose \( X, cd(y) \) is not satisfied in a 3-model. Then, by Theorem 9 and Lemma 2 of Theorem 10, \( X, cd(y) \) fully B-reduces to a cd-B-pair. So, by Theorem 10, \( X 1= y \).

Theorem 11 extends the result in Johnson's [3]. There it is shown, in effect, that any invalid syllogism constructed by using B-sentences other than those of form \( A \rightarrow ab \) or \( 0 \rightarrow ab \) is satisfied in a 3-model. There are invalid B-syllogisms that require a domain with at least three members to show their invalidity. This is an example: \( \{\{A \rightarrow AB, A \rightarrow BC\}, 0 \rightarrow AC\} \).

**THEOREM 12** (Completeness). If \( \langle X, y \rangle \) is a B-syllogism and \( X 1= y \) then \( X \vdash_B y \).

**Proof.** Assume the antecedent. Then, by Theorem 10, \( X \cup \{cd(y)\} \) fully B-reduces to a cd-B-pair. Use the following three lemmas.

**LEMMA 1.** If \( \{x, y\} \) is a cd-B-pair, then \( x \vdash_B cd(y) \) and \( y \vdash_B cd(x) \).

**Proof.** (1) \( A^{pq}ba \vdash_B A^{pq}ab \), that is, \( ct(A^{pq}ab) \) (by B₁). So \( A^{pq}ab \vdash_B cd(A^{pq}ba) \) (by R₂). So \( A^{qp}ba \vdash_B cd(A^{pq}ab) \) (by R₂). (2) \( A^{pq}ab \vdash_B
LEMMA 2. If $X = \{A^{pr}ac\} \cup Z$, $Y = \{A^{pq}ab, A^{qr}bc\} \cup Z$, and $X - \{x\} \vdash_B cd(x)$, for each sentence $x$ in $X$, then $Y - \{y\} \vdash_B cd(y)$, for each sentence $y$ in $Y$.

Proof. Imitate the proof of Lemma 2 of Theorem 6.

LEMMA 3. If $X = \{Qab\} \cup Z$, $Y = \{e(Qab)\} \cup Z$, and $X - \{x\} \vdash_B cd(x)$, for each sentence $x$ in $X$, then $Y - \{y\} \vdash_B cd(y)$, for each sentence $y$ in $Y$.

Proof. Imitate the proof of Lemma 3 of Theorem 6.

5. CONCLUSION

Our interest has been in extending the Aristotelian syllogistic. But, in conclusion, we mention Smiley's classic result in [5] about the Aristotelian syllogistic, which follows from the results obtained above. First, delete sentences of form $A^{--}ab$ and $O^{--}ab$ from system $B$. Let $Aa - b = \varnothing$ if $a = b$; otherwise, let $Aa - b$ be a set of sentences that can be arranged as follows: $(A^{++}a_1a_2 \text{ or } A^{--}a_2a_1), \ldots, A^{++}a_na_{n+1} \text{ or } A^{--}a_{n+1}a_n$, where $a_1 = a$ and $a_{n+1} = b$. Then, by Theorem 10, a chain of sentences in this subsystem is inconsistent iff it has one of the following forms: i) $Aa - b, O^{++}ab \text{ or } O^{--}ab$; ii) $Aa - b, A^{+-}bc, Ac - a$; or iii) $Aa - b, A^{+-}bc, Ad - c, O^{+-}da \text{ or } O^{+-}ad$. Next, delete sentences of form $A^{--}ab$ and $O^{--}ab$ from this system. The resulting system can express all of the Aristotelian syllogisms. So, as Smiley [5] says, an Aristotelian syllogism $(X, y)$ is valid iff $X, cd(y)$ has one of the following forms: i') $Aa - b, O^{++}ab$, ii), or iii). (Smiley uses $A, E, I, O$ instead of our $A^{++}, A^{+-}, O^{+-}, O^{++}$, respectively.) So, for example, "$A^{++}BC, A^{++}BA$; so $O^{+-}AC$" (Darapti) is valid since "$A^{++}BC, A^{++}BA, A^{+-}AC$" has form ii).

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*Department of Philosophy,*  
*Colorado State University,*  
*Fort Collins, Colorado 80523,*  
*U.S.A.*