Dissertation

Group Action on Neighborhood Complexes of Cayley Graphs

Submitted by

Justin Hughes

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2014

Doctoral Committee:

Advisor: Alexander Hulpke
Co-Advisor: Chris Peterson

Bruce Berger
Renzo Cavalieri
James Wilson

Abstract

Group Action on Neighborhood Complexes of Cayley Graphs

Given G a group generated by $S \doteq \{g_1, .., g_n\}$, one can construct the Cayley Graph $\mathrm{Cayley}(G, S)$. Given a distance set $D \subset \mathbb{Z}_{\geq 0}$ and $\mathrm{Cayley}(G, S)$ one can construct a D-neighborhood complex. This neighborhood complex is a simplicial complex to which we can associate a chain complex. The group G acts on this chain complex and this leads to an action on the homology of the chain complex. These group actions decompose into several representations of G. This thesis uses tools from group theory, representation theory, homological algebra, and topology to further our understanding of the interplay between generated groups (i.e. a group together with a set of generators), corresponding representations on their associated D-neighborhood complexes, and the homology of the D-neighborhood complexes.

Table of Contents

CHAPTER 1

Introduction

The goal of this thesis is to find relationships between a generated group, i.e. a group along with a chosen generating set, and a graph on which the generated group acts transitively on the set of vertices. This thesis will focus on the case where the graph is a Cayley graph. Cayley graphs give us a large collection of graphs which are easy to define and construct. However in future work we are planning to replace the Cayley graph with an arbitrary graph which has a transitive group action defined on it. In searching for these relationships we shall use tools from group theory, representation theory, homological algebra, and topology.

From the Cayley graph we construct a simplicial complex called the neighborhood complex. The neighborhood complex was originally created by Lovász as a tool to study the chromatic number of a graph and to prove Kneser's conjecture [1]. For further applications of simplicial complexes we refer the reader to *Simplicial Complexs of Graphs* by Jakob Jonsson [2] and *Combinatorial Commutative Algebra* by Miller and Sturmfels [3].

While studying the neighborhood complex we make use of a chain complex which provides information about how the neighborhoods interact. The chain complex also provides several linear representations of the generated group which we make use of to understand the homology of the complex.

We make use of the computer algebra system, GAP4, to compute these linear representations. See Appendix B for examples of the output created. Note that we use Python code to format the data into an easily readable pdf file.

A few of the culminating results are given below.

Theorem. *Let* $S = \{g_1, g_2, ..., g_n\}$ *be an irredundant generating set for a group* G. *Define the set* $\mathcal{S} = S \cup \left\{g_i g_j^{-1} | g_i, g_j \in S \text{ and } i < j\right\}$. *The representation given by the group action on the set of edges,* $F_1(\Delta)$, *consists of*

- *one copy of the regular representation for each element in* $\mathcal{S}$ *which is not order two*
- *a representation of degree* $\frac{|G|}{2}$ *that is a direct sum of constituents of the regular representation for each element in* $\mathcal{S}$ *which has order two.*

Moreover for $a \in \mathcal{S}$ *of order two and* χ *an irreducible character of* G, *the number of copies of* χ *which appear in the representation of degree* $\frac{|G|}{2}$ *is given by the formula:*

$$\left(\chi, \rho\uparrow^G\right) = \left(\chi\downarrow_{\langle a\rangle}, \rho\right) = \frac{1}{|\langle a\rangle|}\sum_{x\in\langle a\rangle}\chi(x)\overline{\rho(x)} = \frac{1}{2}\left(\chi(1)\overline{\rho(1)} + \chi(a)\overline{\rho(a)}\right) = \frac{1}{2}\left(\chi(1) - \chi(a)\right)$$

where ρ *is the sign character of* $\langle a\rangle$.

Let $S = \{g_1, g_2, .., g_n\}$ be an irredundant generating set for a group G and $\mathcal{S} = S \cup \left\{g_i g_j^{-1} | g_i, g_j \in S \text{ and } i < j\right\}$. We say a unordered pair of elements of $\mathcal{S}$, $\{a, b\}$, is a *square pair* if $|a| = |b| = |ab| = 2$.

Theorem. *Let* $S = \{g_1, g_2, .., g_n\}$ *be an irredundant generating set for a group* G *and* $\mathcal{S} = S \cup \left\{g_i g_j^{-1} | g_i, g_j \in S \text{ and } i < j\right\}$. *If* $x, y \in \mathcal{S}$ *form a square pair then* $\dim\left(\widetilde{H_2}\right) \geq \frac{|G|}{4}$.

CHAPTER 2

The Objects: Cayley Graphs, Neighborhood Complexes, Chain Complexes, and Homology

Let G be a *generated group* that is a group together with a fixed generating set $\{g_1, ..., g_n\}$. A directed *Cayley graph* for G, denoted Cayley $(G, \{g_1, .., g_n\})$, is a graph whose vertices, denoted $V(\text{Cayley}(G, \{g_1, .., g_n\}))$, are the elements of the group and whose edge set, denoted $E(\text{Cayley}(G, \{g_1, .., g_n\}))$ is defined by the generating set as follows. For any $a, b \in G$ there exist a directed edge from a to b if there exist $g \in \{g_1, ..., g_n\}$ such that $g \cdot a = b$. An *irredundant generating set* is a set of generators for a group with the property that no proper subset of the generators will generate the group. Therefore, our generating sets will not contain inverses if the order of g_i is greater than two for every i. Although any generating set of G will give rise to a Cayley graph, we will focus on irredundant generating sets and finite groups.

We call a set $D \subset \mathbb{Z}_{\geq 0}$ a *distance set.* Let the distance $d(x, x_i)$ be the length of the shortest path from x to x_i. Given such a distance set and $d(x, x_i)$ we define the neighborhood, $N_D(x)$, for the vertex x as follows.

$$N_D(x) = \{x_i | x_i \in V(\text{Cayley}(G, \{g_1, .., g_n\})) \text{ and } d(x, x_i) \in D\}.$$

A *simplicial complex* Δ is a set such that if $\rho \in \Delta$ and $\sigma \subseteq \rho$ then $\sigma \in \Delta$, i.e. closed under subsets [3, p.4]. All of the sets in Δ are called *simplices* or faces. Define the neighborhood complex of a graph to be the simplicial complex consisting of the subsets of the sets $N_D(x)$, $\forall x \in V$. The neighborhoods are viewed as the largest dimensional faces in a simplicial complex. In this paper unless otherwise stated take $D = \{0, 1\}$.

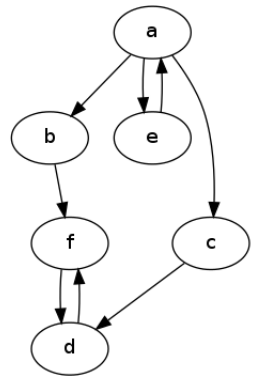


FIGURE 2.1

EXAMPLE 2.1. *Let the distance set be* $D = \{0, 1\}$. *For the graph in Figure 2.1 we obtain the neighborhoods:*

$$N_D(a) = \{a, b, c, e\}, \qquad N_D(b) = \{b, f\},$$

$$N_D(c) = \{c, d\}, \qquad N_D(d) = \{d, f\},$$

$$N_D(e) = \{e, a\}, \qquad N_D(f) = \{f, d\}.$$

So we have the simplicial complex Δ consisting of the all the subsets of the elements in

$$\{\{a, b, c, e\}, \{b, f\}, \{c, d\}, \{d, f\}, \{a, e\}\}.$$

For convenience, we will denote the sets as words in further examples, e.g. $\{a, b, c, e\} = abce$. Notice that the graph in the example is directed and as such the neighborhood $N_D(c)$ contains d, yet the neighborhood $N_D(d)$ does not contain c.

The neighborhood complex leads to a chain complex which is defined has follows. Let Δ be a simplicial complex. Define $F_i(\Delta)$ to be the set of simplices in Δ of dimension i. So $|F_i(\Delta)|$ is the number of simplices of dimension i. We let $|F_{-1}(\Delta)| = |\{\varnothing\}| = 1$.

Let $\mathbb{C}^{|F_i(\Delta)|}$ be a vector space with basis elements indexed by the i-simplices in $F_i(\Delta)$. The map ∂_i from $\mathbb{C}^{|F_i(\Delta)|}$ to $\mathbb{C}^{|F_{i-1}(\Delta)|}$ is defined as follows. Let α be a simplex in $F_i(\Delta)$ and e_α be the corresponding basis vector in $\mathbb{C}^{|F_i(\Delta)|}$. Define $\alpha \smallsetminus j$ to be the simplex α with the j^{th} element removed and $\operatorname{sgn}(j,\alpha) \doteq (-1)^{r-1}$ if j is the r^{th} element of α written in increasing (lexicographical) order. Then the maps in the chain complex are given by

$$\partial_i(e_\alpha) = \sum_{j\in\alpha} \operatorname{sgn}(j,\alpha) e_{\alpha\smallsetminus j}.$$

A (reduced) chain complex of Δ is then as follows:

$$0 \longrightarrow \mathbb{C}^{|F_n(\Delta)|} \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} \mathbb{C}^{|F_1(\Delta)|} \xrightarrow{\partial_1} \mathbb{C}^{|F_0(\Delta)|} \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0.$$

This is a chain complex since the above maps have the property that $\partial_i \circ \partial_{i+1} = 0$ i.e. $\operatorname{im}(\partial_{i+1}) \subseteq \ker(\partial_i)$ [3, p.7]. Recall that a chain complex is said to be exact if all the maps in the sequence have the relation that the image of the i^{th} map is equal to the kernel of the $(i+1)^{th}$ map. So the following sequence,

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is exact if $\operatorname{im}(\varphi) = \ker(\psi)$, φ is injective, and ψ is surjective.

The homology of a chain complex is a measure of how far the sequence is from being exact. The (reduced) i^{th} homology of the chain complex, defined as above from Δ, is the vector space $\ker(\partial_i)/\operatorname{im}(\partial_{i+1})$ and is denoted $\widetilde{H}_i(\Delta)$.

Example 2.2. *Recall Example 2.1. Δ was the simplical complex generated by the maximal faces $\{abce, bf, cd, df, ae\}$. Since the simplical complex is closed under subsets we have:*
$F_3(\Delta) = \{abce\},\ F_2(\Delta) = \{abc, abe, ace, bce\},\ F_1(\Delta) = \{ab, ac, ae, bc, be, bf, cd, ce, df\},$
$F_0(\Delta) = \{a, b, c, d, e, f\},\ F_{-1}(\Delta) = \{\varnothing\}$

$|F_3(\Delta)| = 1,\ |F_2(\Delta)| = 4,\ |F_1(\Delta)| = 9,\ |F_0(\Delta)| = 6,\ |F_{-1}(\Delta)| = 1$

$$0 \longrightarrow \mathbb{C} \xrightarrow{\partial_3} \mathbb{C}^4 \xrightarrow{\partial_2} \mathbb{C}^9 \xrightarrow{\partial_1} \mathbb{C}^6 \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0$$

$$\partial_0 = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \partial_1 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline ab & -1 & 1 & 0 & 0 & 0 & 0 \\ ac & -1 & 0 & 1 & 0 & 0 & 0 \\ ae & -1 & 0 & 0 & 0 & 1 & 0 \\ bc & 0 & -1 & 1 & 0 & 0 & 0 \\ be & 0 & -1 & 0 & 0 & 1 & 0 \\ bf & 0 & -1 & 0 & 0 & 0 & 1 \\ cd & 0 & 0 & -1 & 1 & 0 & 0 \\ ce & 0 & 0 & -1 & 0 & 1 & 0 \\ df & 0 & 0 & 0 & -1 & 0 & 1 \end{array}$$

$$\partial_2 = \begin{array}{c|ccccccccc} & ab & ac & ae & bc & be & bf & cd & ce & df \\ \hline abc & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ abe & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ ace & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ bce & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \end{array}$$

$$\partial_3 = \begin{array}{c|cccc} & abc & abe & ace & bce \\ \hline abce & -1 & 1 & -1 & 1 \end{array}$$

Calculating the rank of the matrices we get that

$\widetilde{H_3}(\Delta) = ker(\partial_3)/im(\partial_4) = 0,\ \widetilde{H_2}(\Delta) = ker(\partial_2)/im(\partial_3) = 0$

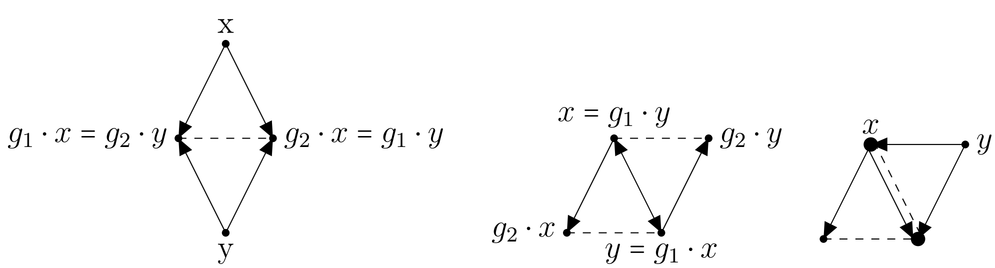
$\widetilde{H_1}(\Delta) = ker(\partial_1)/im(\partial_2) \cong \mathbb{C},\ \widetilde{H_0}(\Delta) = ker(\partial_0)/im(\partial_1) = 0$

CHAPTER 3

The Objects: Results

Since the neighborhood complex comes from a Cayley graph there are structural restrictions on the complex. In this section we will give some of the restrictions on the neighborhood complex and also general results for homology implied by these restrictions.

We need to define some notation for the first theorem. Let $\mathrm{F}_n(x_0, x_1, ..., x_n)$ be an n-dimensional face of the neighborhood complex Δ with vertices $x_0, x_1, ..., x_n$. We will say that an edge, $\mathrm{F}_1(a, b)$, corresponds to an edge in the Cayley graph if there exists an element g in the generating set such that $g \cdot a = b$. More concretely, the neighborhood centered at a for a group generated by $\{g_1, g_2\}$ is $N_{\{0,1\}}(a) = \{a, g_1 \cdot a, g_2 \cdot a\}$ and the edges $\mathrm{F}_1(a, g_1 \cdot a)$ and $\mathrm{F}_1(a, g_2 \cdot a)$ are the edges which correspond to an edge in the Cayley graph while $\mathrm{F}_1(g_1 \cdot a, g_2 \cdot a)$ is the edge which does not correspond to an edge in the Cayley graph.



(1) (2) (3)

(1) and (2) are valid ways for $N_{\{0,1\}}(x)$ and $N_{\{0,1\}}(y)$ to share an edge while (3) is not.

Figures for Theorem 3.1.

THEOREM 3.1. *Let* $S = \{g_1, ..., g_n\}$ *be an irredundant generating set for a group* G. *Suppose* $x \neq y$ *and that there is a 1-simplex* $\varepsilon \in N_{\{0,1\}}(x) \cap N_{\{0,1\}}(y)$. *Then either* $\varepsilon = F_1(x, g_i \cdot x) = F_1(y, g_i \cdot y)$ *and* $|g_i| = 2$ *or* $\varepsilon = F_1(g_i \cdot x, g_j \cdot x) = F_1(g_j \cdot y, g_i \cdot y)$ *and* $|g_i g_j^{-1}| = 2$.

PROOF. Suppose that our two neighborhoods, $N_{\{0,1\}}(x)$ and $N_{\{0,1\}}(y)$, intersect at a 1-simplex which corresponds to an edge in the Cayley graph for $N_{\{0,1\}}(x)$ but not for $N_{\{0,1\}}(y)$. Since the 1-simplex in $N_{\{0,1\}}(x)$ corresponds to an edge in the Cayley graph the simplex has the form $\mathrm{F}_1(x, g_i \cdot x)$ for some g_i in S. The form of the 1-simplex in $N_{\{0,1\}}(y)$ is $\mathrm{F}_1(g_j \cdot y, g_k \cdot y)$ for some g_j and g_k in S since it does not correspond to an edge in the Cayley Graph (see above figure (3)).

The equality $\mathrm{F}_1(x, g_i \cdot x) = \mathrm{F}_1(g_j \cdot y, g_k \cdot y)$ implies either $x = g_j \cdot y$ and $g_i \cdot x = g_k \cdot y$ or $x = g_k \cdot y$ and $g_i \cdot x = g_j \cdot y$. Suppose that $x = g_j \cdot y$ and $g_i \cdot x = g_k \cdot y$. Then $g_i g_j = g_k$. However, $g_i g_j = g_k$ contradicts S being irredundant. (Note: $j \neq k$ but i could equal either j or k. If $i \neq j$ and $i \neq k$ then $g_i g_j = g_k$ implies g_k is redundant. If $i = j$ and $i \neq k$ then $g_i g_j = g_k$ implies $g_i^2 = g_k$ and so again g_k is redundant. If $i \neq j$ and $i = k$ then $g_i g_j = g_k$ implies g_j is the identity which is a redundant generator by definition.) On the other hand, $x = g_k \cdot y$ and $g_i \cdot x = g_j \cdot y$ implies $g_i g_k = g_j$ which again contradicts S being irredundant.

By the above argument, we have shown that either $\varepsilon = \mathrm{F}_1(x, g_i \cdot x) = \mathrm{F}_1(y, g_j \cdot y)$ or $\varepsilon = \mathrm{F}_1(g_i \cdot x, g_j \cdot x) = \mathrm{F}_1(g_k \cdot y, g_l \cdot y)$ i.e. either $\varepsilon \in E(\mathrm{Cayley}(G, S))$ for both neighborhoods or $\varepsilon \notin E(\mathrm{Cayley}(G, S))$ for both neighborhoods (see above figures (1) and (2)). Consider $\varepsilon = \mathrm{F}_1(x, g_i \cdot x) = \mathrm{F}_1(y, g_j \cdot y)$. So $x = g_j \cdot y$ and $y = g_i \cdot x$ since $x \neq y$. Thus $x = (g_i g_j) \cdot x \implies g_i g_j = \mathrm{id}$ but again S is irredundant and so $g_j = g_i$. Therefore $|g_i| = 2$ and $\varepsilon = x(g_i \cdot x) = y(g_i \cdot y)$.

Next consider $\varepsilon = \mathrm{F}_1(g_i \cdot x, g_j \cdot x) = \mathrm{F}_1(g_k \cdot y, g_l \cdot y)$. So up to labeling $g_i \cdot x = g_k \cdot y$ and $g_j \cdot x = g_l \cdot y$. First notice that $i \neq j$ and $k \neq l$ since $\mathrm{F}_1(g_i \cdot x, g_j \cdot x)$ and $\mathrm{F}_1(g_k \cdot y, g_l \cdot y)$ are edges. $g_i \cdot x = g_k \cdot y$ and $g_j \cdot x = g_l \cdot y$ imply that $g_i = g_k g_l^{-1} g_j$. However S is irredundant so $i = k$ or $i = l$.

Suppose $i = l$. Then $g_i = g_k g_l^{-1} g_j \implies g_i = g_k g_i^{-1} g_j \implies g_j^{-1} = g_i^{-1} g_k g_i^{-1}$. Again S is irredundant, so $j = k$. Thus $g_i \cdot x = g_j \cdot y$ and $g_j \cdot x = g_i \cdot y$, which implies $\left|g_i g_j^{-1}\right| = 2$. Suppose instead that $i = k$. Then $g_i \cdot x = g_i \cdot y \implies x = y$, which is a contradiction. Thus we conclude that $\varepsilon = \mathrm{F}_1\left(g_i \cdot x, g_j \cdot x\right) = \mathrm{F}_1\left(g_j \cdot y, g_i \cdot y\right)$ and $\left|g_i g_j^{-1}\right| = 2$. □

Let $S = \{g_1, g_2, .., g_n\}$ be a irredundant generating set for a group then we define the *extended generating set* to be $\mathcal{S} = S \cup \left\{g_i g_j^{-1} | g_i, g_j \in S \text{ and } i < j\right\}$.

Lemma 3.2. *[4] The number of i-dimensional simplices in an n-dimensional simplex is* $\binom{n+1}{i+1}$.

Corollary 3.3. *Let $S = \{g_1, ..., g_n\}$ be an irredundant generating set for a group G and Δ be the neighborhood complex. Define $\alpha = |\{g \in S| \, |g| = 2\}|$ and $\beta = \left|\left\{g_i g_j^{-1} | g_i, g_j \in S, \left|g_i g_j^{-1}\right| = 2, \text{ and } i < j\right\}\right|$. Then*

$$|F_1(\Delta)| = \left(\binom{n+1}{2} - \frac{1}{2}(\alpha + \beta)\right) |G| .$$

Proof. The number of edges in one neighborhood is $\binom{n+1}{2}$. By Theorem 3.1 we know that two distinct neighborhoods share an edge if and only if $|g_i| = 2$ for some i or $\left|g_i g_j^{-1}\right| = 2$ for some $i < j$. Therefore, $\binom{n+1}{2} |G|$ will count twice an edge for every generator of order two and pair of generators such that $\left|g_i g_j^{-1}\right| = 2$.

Note that if $\alpha + \beta > 0$ then $|G|$ is even since there must be an order two element in the extended generating set. □

It should be noted that the above theorem gives the condition for two neighborhoods to share an edge. This is not to say that an edge could not be shared between two simplices contained in one neighborhood. In fact, an edge can be shared between two i-simplices if both i-simplices are highest dimensional simplices, i.e. the neighborhoods, as described by Theorem 3.1 or if both i-simplices are contained in one neighborhood in which case the edge does not need to be given by an order two element.

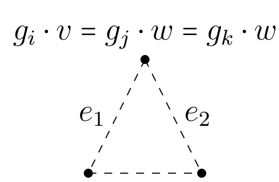
We can also show that two neighborhoods cannot intersect in a triangle.

Theorem 3.4. *Two neighborhoods of a Cayley graph defined by the group G and an irredundant generating set $S = \{g_1, ..., g_n\}$ cannot intersect in a 2-simplex.*

Proof. Let $v, w \in V(\text{Cayley}(G, S)) = G$ and suppose that $N_D(v)$ and $N_D(w)$ intersect in a 2-simplex. There must exist edges $e_1, e_2 \in N_D(v) \cap N_D(w)$ such that e_1 and e_2 intersect in a vertex. By Theorem 3.1, $e_1 = \mathrm{F}_1(g_i \cdot v, g_j \cdot v) = \mathrm{F}_1(g_j \cdot w, g_i \cdot w)$ or $e_1 = \mathrm{F}_1(v, g_i \cdot v) = \mathrm{F}_1(w, g_i \cdot w)$.

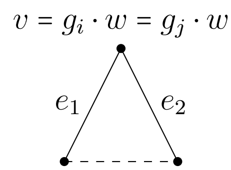
Any 2-simplex in a neighborhood complex consists of exactly one edge of type $\mathrm{F}_1(g_i \cdot v, g_j \cdot v)$ or all three edges are of type $\mathrm{F}_1(g_i \cdot v, g_j \cdot v)$. Therefore we need only consider the following two cases.

Case 1:



Assume $e_1 = \mathrm{F}_1(g_i \cdot v, g_j \cdot v) = \mathrm{F}_1(g_j \cdot w, g_i \cdot w)$, $e_2 = \mathrm{F}_1(g_i \cdot v, g_k \cdot v) = \mathrm{F}_1(g_k \cdot w, g_i \cdot w)$ and $e_1 \neq e_2$. Then $g_i \cdot v = g_j \cdot w$, and $g_i \cdot v = g_k \cdot w$. These two relations imply that $g_j \cdot w = g_k \cdot w$ and so $g_j = g_k$ which contradicts that S is an irredundant generating set.

Case 2:



Assume $e_1 = \mathrm{F}_1(v, g_i \cdot v) = \mathrm{F}_1(w, g_i \cdot w)$, $e_2 = \mathrm{F}_1(v, g_j \cdot v) = \mathrm{F}_1(w, g_j \cdot w)$, and $e_1 \neq e_2$. Then $v = g_i \cdot w$ and $v = g_j \cdot w$. These two relations imply that $g_j \cdot w = g_k \cdot w$ and so $g_j = g_k$ which contradicts that S is an irredundant generating set. $\square$

Notice that if two neighborhoods intersect in a higher dimensional simplex (greater than 2) then the 2-simplices in the intersection must be shared in the two neighborhoods. Thus we immediately have the following corollaries.

Corollary 3.5. *Two neighborhoods of a Cayley graph defined by the group G and an irredundant generating set $S = \{g_1, \ldots, g_n\}$ cannot intersect in an i-simplex for $i \geq 2$.*

Corollary 3.6. *Let $S = \{g_1, \ldots, g_n\}$ be an irredundant generating set for a group G. The number of m-simplices in the corresponding neighborhood complex is $\binom{n+1}{m+1}|G|$ for $n \geq m \geq 2$.*

The impact of how simplices can intersect is quite profound as can be seen in the next theorem.

Theorem 3.7. *Let S be a irredundant generating set of size n for G. Then $\widetilde{H_n}(N_D(Cayley(G, S))) = 0$ except when*

(1) *$n = 1$ in which case G is cyclic and thus $\widetilde{H_1} = \mathbb{C}$, or*

(2) *$n = 2$ and G is the Klein 4-group in which case $\widetilde{H_2} = \mathbb{C}$.*

We shall prove Theorem 3.7 by showing the result for $n = 2$ and $n \geq 3$ as lemmata. Notice that a cyclic group generated by one element will give a cyclic graph. Thus its neighborhood complex is simply a loop and $\widetilde{H_1} = \mathbb{C}$.

Lemma 3.8. *Let $S = \langle g_1, g_2 \rangle$ be an irredundant generating set for a group G. If G is not isomorphic to the Klein 4 group then $\widetilde{H_2} = 0$.*

Proof. (Geometric proof.) This lemma could be considered as a corollary of Theorem 3.1. Let $G = \langle g_1, g_2 \rangle$ and Δ denote the corresponding neighborhood complex. If we are to have nontrivial $\widetilde{H_2}$ we must enclose a volume with triangles. To see why this corresponds to enclosing a volume notice for $x \in \mathbb{C}^{|\mathrm{F}_2(\Delta)|}$ to be in the kernel of ∂_2, every triangle in the linear combination of triangles, x, must share each of its edges with other triangles in such a way that they cancel after mapping by ∂_2.

Let $\mathrm{F}_2\left(v, g_1 \cdot v, g_2 \cdot v\right) \in \mathrm{F}_2(\Delta)$. To enclose a volume every edge of $\mathrm{F}_2\left(v, g_1 \cdot v, g_2 \cdot v\right)$ must be contained in another triangle in $\mathrm{F}_2(\Delta)$. However, by Theorem 3.1, we know that the edges of $\mathrm{F}_2\left(v, g_1 \cdot v, g_2 \cdot v\right)$ which correspond to the edges in $\mathrm{Cayley}\left(G, \{g_1, g_2\}\right)$ are shared if and only if $|g_i| = 2$. Thus if the edge $\mathrm{F}_1\left(v, g_1 \cdot v\right)$ is shared then $|g_1| = 2$. Similarly, $|g_2| = 2$ for $\mathrm{F}_1\left(v, g_2 \cdot v\right)$. Again by Theorem 3.1, $\mathrm{F}_1\left(g_1 \cdot v, g_2 \cdot v\right)$ is shared if and only if $|g_1 g_2^{-1}| = 2$. So we see that the only way to have nontrivial $\widetilde{H_2}$ for a group with generating set of size two is for the group to be the Klein 4 group. □

An alternate proof is:

Proof. (Algebra proof.) Since there is a bijection between our basis for $\mathbb{C}^{|\mathrm{F}_2(\Delta)|}$ and $\mathrm{F}_2(\Delta)$, we can use the set $\mathrm{F}_2(\Delta)$ to show that $\ker\left(\partial_2\right) = 0$. Let x be a linear combination of 2-simplexes in $\mathrm{F}_2(\Delta)$ such that $\partial_2(x) = 0$ and $x \neq 0$. We shall say that a 2-simplex is in x if that 2-simplex has nonzero coefficient in x.

Let $v \in V(\mathrm{Cayley}(G,S))$, then the neighborhood $N_D(v) = \mathrm{F}_2(v, g_1 \cdot v, g_2 \cdot v)$ is mapped by ∂_2 to $\mathrm{F}_1(g_1 \cdot v, g_2 \cdot v) - \mathrm{F}_1(v, g_2 \cdot v) + \mathrm{F}_1(v, g_1 \cdot v)$. If $\mathrm{F}_2(v, g_1 \cdot v, g_2 \cdot v)$ is in x then there must be another neighborhood $N_D(w) \in x$ such that $\mathrm{F}_1(v, g_2 \cdot v)$ is in $\mathrm{F}_1(g_1 \cdot w, g_2 \cdot w) - \mathrm{F}_1(w, g_2 \cdot w) + \mathrm{F}_1(w, g_1 \cdot w)$ i.e. $\mathrm{F}_1(v, g_2 \cdot v) = \mathrm{F}_1(g_1 \cdot w, g_2 \cdot w)$, $\mathrm{F}_1(v, g_2 \cdot v) = \mathrm{F}_1(w, g_2 \cdot w)$, or $\mathrm{F}_1(v, g_2 \cdot v) = \mathrm{F}_1(w, g_1 \cdot w)$ since $\partial_2(x) = 0$. However by Theorem 3.1, we must have the case $\mathrm{F}_1(v, g_2 \cdot v) = \mathrm{F}_1(w, g_2 \cdot w)$ and $|g_2| = 2$.

Similarly for $\mathrm{F}_1(v, g_1 \cdot v)$ in $\partial_2(\mathrm{F}_2(v, g_1 \cdot v, g_2 \cdot v)) = \mathrm{F}_1(g_1 \cdot v, g_2 \cdot v) - \mathrm{F}_1(v, g_2 \cdot v) + \mathrm{F}_1(v, g_1 \cdot v)$, we get $\mathrm{F}_1(v, g_1 \cdot v) = \mathrm{F}_1(y, g_1 \cdot y)$ for some $y \in V(\mathrm{Cayley}(G,S))$ and $|g_1| = 2$ and lastly $\mathrm{F}_1(g_1 \cdot v, g_2 \cdot v) = \mathrm{F}_1(g_1 \cdot z, g_2 \cdot z)$ for some $z \in V(\mathrm{Cayley}(G,S))$ and $|g_1 g_2^{-1}| = 2$. So we see that the only way to have nontrivial $\widetilde{H_2}$ for a group with generating set of size two is for the group to be the Klein 4 group. $\square$

LEMMA 3.9. $\widetilde{H_n} = 0$ *in the neighborhood complex of the Cayley graph for an irredundant generating set of size* n *if* $n \geq 3$.

PROOF. Since there is a bijection between our basis for $\mathbb{C}^{|\mathrm{F}_n(\Delta)|}$ and $\mathrm{F}_n(\Delta)$, we can use the set $\mathrm{F}_n(\Delta)$ to show that $\ker(\partial_n) = 0$. Let x be a linear combination of n-simplexes in $\mathrm{F}_n(\Delta)$ such that $\partial_n(x) = 0$ and $x \neq 0$. We shall say that a n-simplex is in x if that n-simplex has nonzero coefficient in x.

Let $v \in V(\mathrm{Cayley}(G,S))$ then the neighborhood $N_D(v) = \mathrm{F}_n(v, g_1 \cdot v, ..., g_n \cdot v)$ is mapped by ∂_n to $\mathrm{F}_{n-1}(g_1 \cdot v, g_2 \cdot v, ..., g_n \cdot v) - \mathrm{F}_{n-1}(v, g_2 \cdot v, ..., g_n \cdot v) + \cdots + \mathrm{F}_{n-1}(v, g_1 \cdot v, ..., g_{n-1} \cdot v)$. Since $\partial_n(x) = 0$, if $\mathrm{F}_n(v, g_1 \cdot v, ..., g_n \cdot v)$ is in x then there must be another neighborhood $N_D(w)$ in x such that $\mathrm{F}_{n-1}(g_1 \cdot v, g_2 \cdot v, ..., g_n \cdot v) = \mathrm{F}_{n-1}(g_1 \cdot w, g_2 \cdot w, ..., g_n \cdot w)$. However, Corollary 3.5 states that neighborhoods cannot intersect in an $(n-1)$-dimensional simplices for $n \geq 3$. $\square$

With all the structure restrictions that we have shown for a neighborhood complex corresponding to the finite group, we can fully describe the homology when the irredundant generating set is of size two.

Theorem 3.10. *$dim\left(\widetilde{H_1}\right) = |F_1(\Delta)| - 2|G| + 1$ in the neighborhood complex of the Cayley graph for an irredundant generating set of size two except for the Klein 4 group.*

$$0 \longrightarrow \mathbb{C}^{|F_2(\Delta)|} \xrightarrow{\partial_2} \mathbb{C}^{|F_1(\Delta)|} \xrightarrow{\partial_1} \mathbb{C}^{|F_0(\Delta)|} \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0$$

Proof. Let G be any group except for the Klein 4 group. The neighborhood complex for any group is connected so $\widetilde{H_0} = 0$. Thus $\ker(\partial_0) = \operatorname{im}(\partial_1)$. By the rank-nullity theorem $\dim(\ker(\partial_0)) = |F_0(\Delta)| - \dim(\operatorname{im}(\partial_0)) = |G| - 1$. Similarly,

$$\begin{aligned}\dim(\ker(\partial_1)) &= |F_1(\Delta)| - \dim(\operatorname{im}(\partial_1))\\ &= |F_1(\Delta)| - \dim(\ker(\partial_0))\\ &= |F_1(\Delta)| - |G| + 1.\end{aligned}$$

According to Theorem 3.7, $\widetilde{H_2} = 0$. Thus $\ker(\partial_2) = 0$ and so $\dim(\operatorname{im}(\partial_2)) = |F_2(\Delta)| = |G|$. We conclude that $\dim\left(\widetilde{H_1}\right) = \dim(\ker(\partial_1)) - \dim(\operatorname{im}(\partial_2)) = |F_1(\Delta)| - 2|G| + 1$ □

Corollary 3.11. *Let $S = \{g_1, g_2\}$ be a generating set for the group G not isomorphic to the Klein four group. Using the notation from Corollary 3.3, the homology is as follows:*

- $\widetilde{H_2} = 0$
- $dim\left(\widetilde{H_1}\right) = \left(3 - \frac{1}{2}(\alpha + \beta)\right)|G| - 2|G| + 1$
- $\widetilde{H_0} = 0.$

These structural restrictions not only give us information in the case where the generating set is of size two but also gives us the following result about groups with generating set of any size.

THEOREM 3.12. *$\widetilde{H_i} = 0$ for $i \geq 3$.*

PROOF. Let γ be a linear combination of elements from $F_i(\Delta)$ such that $\partial_i(\gamma) = 0$. Recall that for all $\alpha \in F_i(\Delta)$,

$$\partial_i(\alpha) = \sum_{j \in \{0,\ldots,i\}} (-1)^j F_{i-1}(x_0, \ldots, \widehat{x_j}, \ldots x_i).$$

We can see for $\partial_i(\gamma)$ to equal zero we need every (i-1)-simplex in $\partial_i(\gamma)$ to appear in pairs (i.e. an even number of times). However, this would mean that γ must have two i-simplices for each pair which share a particular $(i-1)$-simplex. By Corollary 3.5, the pair of i-simplices must be in the same (i+1)-simplex. Let η_1 and η_2 be one such pair of i-simplices.

Besides the $(i-1)$-simplex that η_1 and η_2 share, η_1 has $\binom{i+1}{(i-1)+1} - 1 = i$ other $(i-1)$-simplices which must be shared with other i-simplices from the same (i+1)-simplex in order for $\partial_i(\gamma) = 0$. However, the $(i+1)$-simplex containing η_1 and η_2 has $\binom{(i+1)+1}{i+1} = i+2$ i-simplices. Thus γ must contain all of the i-simplices in the $(i+1)$-simplex. A similar argument is valid for every such pairing and so $\gamma \in \operatorname{im}(\partial_{i+1})$. ☐

Using elementary methods we can capture the homology. The group action on the homology induces a representation of the group and affords us further structure which we explore in the next section.

CHAPTER 4

Group Action on the Neighborhood Complexes

Since the Cayley graph is constructed from a group, there is a natural group action on the graph. Moreover, if the Cayley graph is built by left multiplication then right multiplication by a group element is a graph automorphism. The neighborhoods, and thereby the chain complex and the homology, are created using only the structure of the Cayley Graph. Thus the action on the Cayley graph induces an action on the chain complex and the homology.

Let G be a group generated by $\{g_1, .., g_n\}$. Then right multiplication by g_i on the group elements induces an action on the labeling of the vertex set for the Cayley graph. The induced permutation on the labeling gives an action, σ_i on $\mathbb{C}^{|F_i(\Delta)|}$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^{|F_n(\Delta)|} \arrow[r, "\partial_n"] \arrow[d, "\sigma_n"] & \dots \arrow[r, "\partial_2"] \arrow[d] & \mathbb{C}^{|F_1(\Delta)|} \arrow[r, "\partial_1"] \arrow[d, "\sigma_1"] & \mathbb{C}^{|F_0(\Delta)|} \arrow[r, "\partial_0"] \arrow[d, "\sigma_0"] & \mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}^{|F_n(\Delta)|} \arrow[r, "\partial_n"] & \dots \arrow[r, "\partial_2"] & \mathbb{C}^{|F_1(\Delta)|} \arrow[r, "\partial_1"] & \mathbb{C}^{|F_0(\Delta)|} \arrow[r, "\partial_0"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

In the natural basis σ_i can be represented as a matrix of $0, \pm 1$ based on the permutation. More concretely, let α be a i-simplex in $F_i(\Delta)$, e_α be the corresponding basis vector in $\mathbb{C}^{|F_i(\Delta)|}$ and ρ is the aforementioned permutation of the vertices. Then $\sigma_i(e_\alpha) = \text{sgn}(\alpha) e_{\rho(\alpha)}$ where $\text{sgn}(\alpha)$ is defined to be the parity of the permutation which restores the elements of α to ascending order.

A homomorphism from a group G to the general linear group $GL_n(\mathbb{C})$ over $\mathbb{C}$ of degree n is a matrix representation of G of degree n [5, p.1]. A representation is called *irreducible* if it does not leave any proper nontrivial subspaces fixed [5, p.6]. The vertices in the Cayley graph are the group elements and the action of G is group multiplication so the regular representation will appear quite often. A permutation representation is a representation permuting the basis vectors of $\mathbb{C}^n$. The regular representation is the permutation representation on cosets of the trivial group or, in other words, group multiplication.

Since we are working over $\mathbb{C}$, we know that all of our linear representations of G will be a direct sum of irreducible representations. We say that a linear representation "contains" a module if the module is a summand of the direct sum decomposition of the linear representation. We will also use the term constituents to the refer to the terms in the direct sum decomposition.

One minor result can be given for cyclic groups generated by a single element.

Theorem 4.1. *The module defined by the action of the cyclic group C_n on $\widetilde{H_1}$ for the neighborhood complex of Cayley$(C_n, \{g\})$ is the trivial module.*

Proof. We only have one generator so $|F_2(\Delta)| = 0$. Thus the chain complex is

$$0 \xrightarrow{\partial_2} \mathbb{C}^{|F_1(\Delta)|} \xrightarrow{\partial_1} \mathbb{C}^{|F_0(\Delta)|} \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0$$

So the image of ∂_2 is zero and the homology, $\widetilde{H_1}$, is given only by the elements in the kernel of ∂_1. The kernel is one dimensional and is generated by $id \cdot g + g \cdot g^2 + g^2 \cdot g^3 + ... + g^{n-1} \cdot id$ i.e. $[1, 1, ..., 1]$. The action of g on $id \cdot g + g \cdot g^2 + g^2 \cdot g^3 + ... + g^{n-1} \cdot id$ is $g \cdot g^2 + g^2 \cdot g^3 + ... + g^{n-1} \cdot id + id \cdot g = id \cdot g + g \cdot g^2 + g^2 \cdot g^3 + ... + g^{n-1} \cdot id$. Thus this action is trivial. □

We will now focus on the case where the group is generated by two elements. Before we begin our analysis we need to recall one more fact from representation theory. The irreducible constituents of the regular representation are all the irreducible representations for the group. Furthermore, the regular module contains k copies of each k-dimensional irreducible submodule [5, p.7].

As mentioned earlier, the action of G on the Cayley graph induces an action on the chain complex and therefore on $F_i(\Delta)$ for $0 \le i \le 2$. There exist bijections from $F_0(\Delta)$ and $F_2(\Delta)$ to G which are compatible with the action of G. Therefore, the action of G on $F_0(\Delta)$ and $F_2(\Delta)$ is group multiplication and so the representation will be the regular representation in both cases. $F_1(\Delta)$ holds some more interest.

The action of $G = \langle g_1, g_2 \rangle$ on $F_1(\Delta)$ has three orbits:

- $E_{g_1} \doteq \{\mathrm{F}_1(v, g_1 \cdot v) | v \in V(\mathrm{Cayley}(G, \{g_1, g_2\}))\}$
- $E_{g_2} \doteq \{\mathrm{F}_1(v, g_2 \cdot v) | v \in V(\mathrm{Cayley}(G, \{g_1, g_2\}))\}$
- $E_{g_1 g_2^{-1}} \doteq \{\mathrm{F}_1(g_1 \cdot v, g_2 \cdot v) | v \in V(\mathrm{Cayley}(G, \{g_1, g_2\}))\}$.

Here $\mathrm{F}_1(v, g_1 \cdot v)$ denotes the edge with initial point v and terminal point $g_1 \cdot v$.

To see why these three sets of edges are orbits, recall that we have a generating set of size two and thus all edges are of the form $\mathrm{F}_1(v, g_1 \cdot v)$, $\mathrm{F}_1(v, g_2 \cdot v)$, or $\mathrm{F}_1(g_1 \cdot v, g_2 \cdot v)$ for some $v \in V(\mathrm{Cayley}(G, \{g_1, g_2\}))$. Furthermore, consider the action of a group element, g, on $\mathrm{F}_1(w, g_1 \cdot w)$. $\mathrm{F}_1(w, g_1 \cdot w)^g = \mathrm{F}_1(wg, g_1 \cdot (wg))$ since w and g are both in G, $wg \in G$ and thus $\mathrm{F}_1(wg, g_1 \cdot (wg))$ is still of the form $\mathrm{F}_1(v, g_1 \cdot v)$ for some $v \in G = V(\mathrm{Cayley}(G, \{g_1, g_2\}))$ namely $v = wg$. A similar argument will hold for edges of type $\mathrm{F}_1(v, g_2 \cdot v)$ and $\mathrm{F}_1(g_1 \cdot v, g_2 \cdot v)$.

LEMMA 4.2. $\{F_1(v, g_i \cdot v), F_1(g_i \cdot v, v)\}$ *defines a block system for* E_{g_i} *if* $|g_i| = 2$.

PROOF. First note that $\mathrm{F}_1(g_i \cdot v, v) = \mathrm{F}_1((g_i \cdot v), g_i \cdot (g_i \cdot v)) \in E_{g_i}$ since $|g_i| = 2$.

We need to show that either

$\{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\}^{g_i} \cap \{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\} = \{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\}$

or

$\{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\}^{g_i} \cap \{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\} = \varnothing$ for $i \in \{1, 2\}$.

Suppose that the intersection is nonempty. Then there are four possibilities:

(1) $\mathrm{F}_1(vg_i, g_i \cdot vg_i) = \mathrm{F}_1(v, g_i \cdot v)$

(2) $\mathrm{F}_1(vg_i, g_i \cdot vg_i) = \mathrm{F}_1(g_i \cdot v, v)$

(3) $\mathrm{F}_1(g_i \cdot vg_i, vg_i) = \mathrm{F}_1(v, g_i \cdot v)$

(4) $\mathrm{F}_1(g_i \cdot vg_i, vg_i) = \mathrm{F}_1(g_i \cdot v, v)$

We will now explore where each possibility can lead.

(1) $\mathrm{F}_1(vg_i, g_i \cdot vg_i) = \mathrm{F}_1(v, g_i \cdot v) \implies vg_i = v \implies g_i$ is the identity which is impossible.

(2) $\mathrm{F}_1(vg_i, g_i \cdot vg_i) = \mathrm{F}_1(g_i \cdot v, v) \implies vg_i = g_i \cdot v$ and $g_i \cdot vg_i = v$.

(3) $\mathrm{F}_1(g_i \cdot vg_i, vg_i) = \mathrm{F}_1(v, g_i \cdot v) \implies g_i \cdot vg_i = v$ and $vg_i = g_i \cdot v$.

(4) $\mathrm{F}_1(g_i \cdot vg_i, vg_i) = \mathrm{F}_1(g_i \cdot v, v) \implies vg_i = v \implies g_i$ is the identity which is impossible.

Cases 1 and 4 lead to contradictions while cases 2 and 3 gave the same relation $vg_i = g_i \cdot v$. Thus $\{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\}^{g_i} \cap \{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\} = \{\mathrm{F}_1(v, g_i \cdot v), \mathrm{F}_1(g_i \cdot v, v)\}$ if $vg_i = g_i \cdot v$ and otherwise disjoint. $\square$

Lemma 4.3. *The stabilizer in G of $\{F_1(1, g_i \cdot 1), F_1(g_i \cdot 1, 1)\}$ is $\langle g_i \rangle$ if $|g_i| = 2$.*

Proof. Suppose $g \in \mathrm{Stab}_G(\{\mathrm{F}_1(1, g_i \cdot 1), \mathrm{F}_1(g_i \cdot 1, 1)\})$.

$$\begin{aligned}\{\mathrm{F}_1(1, g_i \cdot 1), \mathrm{F}_1(g_i \cdot 1, 1)\}^g &= \{\mathrm{F}_1(1g, g_i \cdot 1g), \mathrm{F}_1(g_i \cdot 1g, 1g)\}\\ &= \{\mathrm{F}_1(1, g_i \cdot 1), \mathrm{F}_1(g_i \cdot 1, 1)\}\end{aligned}$$

Thus we have 2 cases:

(1) $\mathrm{F}_1(1, g_i \cdot 1) = \mathrm{F}_1(1g, g_i \cdot 1g)$ and $\mathrm{F}_1(g_i \cdot 1, 1) = \mathrm{F}_1(g_i \cdot 1 \cdot g, 1 \cdot g)$

(2) $\mathrm{F}_1(1, g_i \cdot 1) = \mathrm{F}_1(g_i \cdot 1g, 1g)$ and $\mathrm{F}_1(g_i \cdot 1, 1) = \mathrm{F}_1(1g, g_i \cdot 1g)$

Case 1 implies $1 = 1g$ and thus g is the identity.

Case 2 implies $g_i \cdot 1 = 1g$ and thus $g = g_i$. (Notice that if $|g_i| \neq 2$ we would have a contradiction in this case: $g_i \cdot 1 = 1g \implies g = g_i$ and yet $1 = g_i \cdot 1g \implies g = g_i^{-1}$.)

Therefore $\mathrm{Stab}_G(\{\mathrm{F}_1(1, g_i \cdot 1), \mathrm{F}_1(g_i \cdot 1, 1)\}) = \langle g_i \rangle$. □

The two lemmata above can also be given for the case of $\left|g_i g_j^{-1}\right| = 2$. Proofs are omitted here since they follow the same process as those above.

LEMMA 4.4. *$\{F_1(g_i \cdot v, g_j \cdot v), F_1(g_j \cdot v, g_i \cdot v)\}$ defines a block system for $E_{g_i g_j^{-1}}$ if $\left|g_i g_j^{-1}\right| = 2$.*

LEMMA 4.5. *The stabilizer in G of $\{F_1(g_i \cdot 1, g_j \cdot 1), F_1(g_j \cdot 1, g_i \cdot 1)\}$ is $\left\langle g_i g_j^{-1} \right\rangle$ if $\left|g_i g_j^{-1}\right| = 2$.*

Given these lemmata we can now prove our main theorem.

THEOREM 4.6. *Let $\{g_1, g_2\}$ be an irredundant generating set for a group G. Define the set $\mathcal{S} = \{g_1, g_2, g_1 g_2^{-1}\}$. The representation given by the group action on the set of edges, $F_1(\Delta)$, decomposes into the direct sum of*

- *one copy of the regular representation for each element in $\mathcal{S}$ which is not order two*
- *a representation of degree $\frac{|G|}{2}$ that is a direct sum of constituents of the regular representation for each element in $\mathcal{S}$ which has order two.*

Moreover for $a \in \mathcal{S}$ of order two and φ an irreducible representation of G with character χ, the number of summands of φ which appear in the representation of degree $\frac{|G|}{2}$ is given *by the formula:*

$$(4.1)\qquad \left(\chi,\rho\uparrow^{G}\right)=\left(\chi\downarrow_{\langle a\rangle},\rho\right)=\frac{1}{|\langle a\rangle|}\sum_{x\in\langle a\rangle}\chi(x)\overline{\rho(x)}=\frac{1}{2}\left(\chi(1)\overline{\rho(1)}+\chi(a)\overline{\rho(a)}\right)=\frac{1}{2}\left(\chi(1)-\chi(a)\right)$$

where ρ is the sign character of $\langle a\rangle$.

Proof. Since E_{g_1}, E_{g_2}, and $E_{g_1g_2^{-1}}$ are the orbits of the action of G on $F_1(\Delta)$ they must be disjoint. Thus the action of G on the edge set can be viewed as three separate actions, one for each orbit.

If $|a|\neq 2$ for $a\in\mathcal{S}$ there exists a bijection between E_a and G. Thus the action of G on E_a will give rise to the regular representation of G.

Suppose $|a|=2$ for $a\in\mathcal{S}$ and denote the subgroup generated by a by H. By Lemma 4.2 and Lemma 4.3, E_a has a block system and $\mathrm{Stab}_G\left(\{\mathrm{F}_1\left(1,a\cdot 1\right),\mathrm{F}_1\left(a\cdot 1,1\right)\}\right)=H$ (or $\mathrm{Stab}_G\left(\{\mathrm{F}_1\left(g_1\cdot 1,g_2\cdot 1\right),\mathrm{F}_1\left(g_2\cdot 1,g_1\cdot 1\right)\}\right)=H$ if $a=g_1g_2^{-1}$). Therefore by the (Krasner, Kaloujnine) embedding theorem [6, p.45], the action of G on E_a defines an embedding of G into the wreath product, $H^{\varphi}\wr G^{\psi}$, where φ is the permutation representation of H given by the action of H on $\{\mathrm{F}_1\left(1,a\cdot 1\right),\mathrm{F}_1\left(a\cdot 1,1\right)\}$ and ψ is the permutation representation of G given by the action of G on the blocks. From the point of view of representation theory this embedding is the induced representation of $\varphi\uparrow^{G}$. Notice that the action of H on $\{\mathrm{F}_1\left(1,a\cdot 1\right),\mathrm{F}_1\left(a\cdot 1,1\right)\}$ (or $\{\mathrm{F}_1\left(g_1\cdot 1,g_2\cdot 1\right),\mathrm{F}_1\left(g_2\cdot 1,g_1\cdot 1\right)\}$ if $a=g_1g_2^{-1}$) is nontrivial and thus φ is the nontrivial representation of H. $\square$

Let us consider an example which uses the above theorem.

EXAMPLE 4.1. *Consider the symmetric group on four points.*

	1A	2A	3A	2B	4A
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	-1	1
χ_5	3	-1	0	1	-1

TABLE 4.1. Character Table for S_4.

Consider the generating set $\{g_1 = (1,2), g_2 = (1,2,3,4)\}$ *for* $G = S_4$. *Notice that* $|g_1| = 2$, $|g_2| \neq 2$ *and* $|g_1 g_2^{-1}| \neq 2$. *So by Theorem 4.6 we have two copies of the regular representation in the action on the set of edges given by the action on the orbits* $E_{g_2} \doteq \{F_1(v, g_2 \cdot v) | v \in V(\mathit{Cayley}(G, \{g_1, g_2\}))\}$ *and* $E_{g_1 g_2^{-1}} \doteq \{F_1(g_1 \cdot v, g_2 \cdot v) | v \in V(\mathit{Cayley}(G, \{g_1, g_2\}))\}$. *Thus we have at least:*

Copies	irreducible representation
2	χ_1
2	χ_2
4	χ_3
6	χ_4
6	χ_5

Theorem 4.6 also tells us that we will have a half of the regular representation given by the action on the orbit $E_{g_1} \doteq \{F_1(v, g_1 \cdot v) | v \in V(\mathit{Cayley}(G, \{g_1, g_2\}))\}$. *To see exactly which irreducible representations will appear in this half of the regular representation we use the formula 4.1.*

$$(\chi_1, \rho\uparrow^G) = \tfrac{1}{2}(\chi_1(1) - \chi_1(g_1)) = \tfrac{1}{2}(1-1) = 0$$

$$(\chi_2, \rho\uparrow^G) = \tfrac{1}{2}(\chi_2(1) - \chi_2(g_1)) = \tfrac{1}{2}(1-(-1)) = 1$$

$$(\chi_3, \rho\uparrow^G) = \tfrac{1}{2}(\chi_3(1) - \chi_3(g_1)) = \tfrac{1}{2}(2-0) = 1$$

$$(\chi_4, \rho\uparrow^G) = \tfrac{1}{2}(\chi_4(1) - \chi_4(g_1)) = \tfrac{1}{2}(3-(-1)) = 2$$

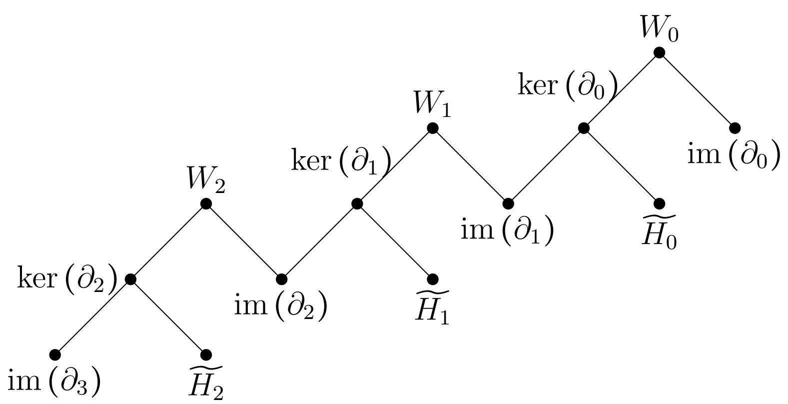
$$(\chi_5, \rho\uparrow^G) = \tfrac{1}{2}(\chi_5(1) - \chi_5(g_1)) = \tfrac{1}{2}(3-1) = 1$$

Note that g_1 *is in the conjugacy class labeled 2B, in the above table.*

Thus the irreducible representations given by the action of S_4 on the set of edges is as follows.

Copies of	*irreducible representation*
2	χ_1
3	χ_2
5	χ_3
8	χ_4
7	χ_5

By Theorem 4.6, we fully described the representation given by the action of G on the space of edges. In order to easily understand the representations given by the action on the other spaces, i.e. images, kernels, and homology, we will use the following structure.

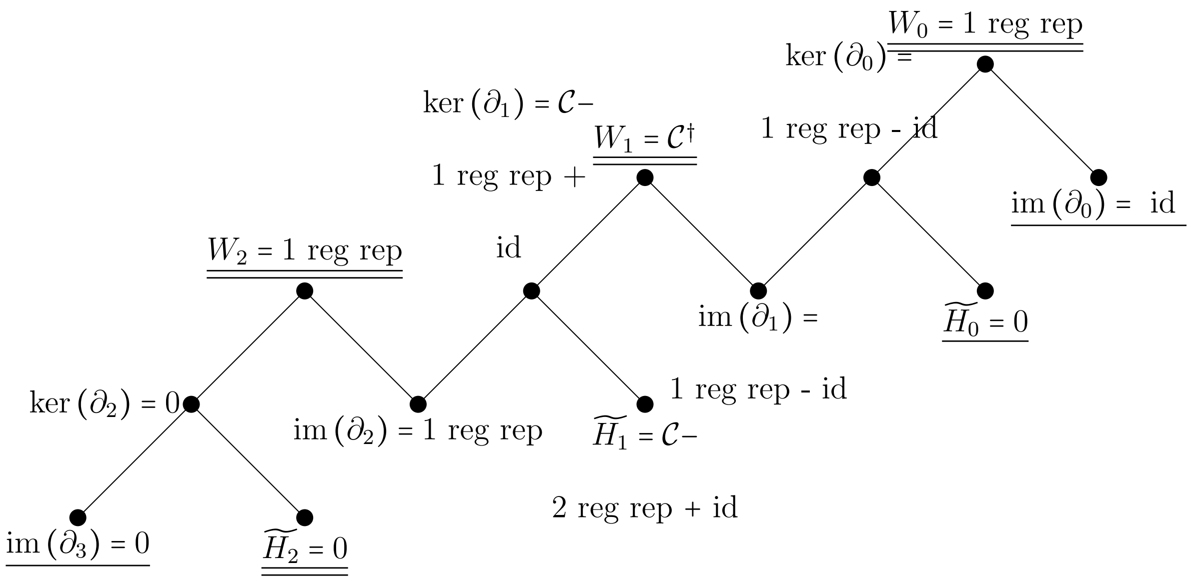


Meets are relations.

$\operatorname{im}(\partial_{i+1}) + \widetilde{H_i} = \ker(\partial_i)$ and

$\ker(\partial_i) + \operatorname{im}(\partial_i) = W_i$

With this means of organizing the information, we can state what the irreducible representations given by the action on the images, kernels, homology and whole space (i.e. set of edges). We have a few of the items in the figure by definition (single underline), a few more by theorems stated earlier (double underline), and the remaining entries can be deduced via the structure of the figure.



†See Theorem 4.6 for value of $\mathcal{C}$.

As the chart indicates, the irreducible representations corresponding to the action on the $\ker(\partial_1)$ and $\widetilde{H_1} = \ker(\partial_1)/\mathrm{im}(\partial_0)$ are known once the action on the set of edges, $F_1(\Delta)$, is understood. Thus we have the following corollary.

Corollary 4.7. *Let $\{g_1, g_2\}$ be an irredundant generating set for a group G. Let $\mathcal{C}$ be the collection of irreducible representations given by the action on the set of edges $F_1(\Delta)$. Then the irreducible representations given by the action of G on*

- *$\ker(\partial_1)$ is $\mathcal{C}$ minus one regular representation of G plus the trivial representation of G*
- *$\widetilde{H_1}$ is $\mathcal{C}$ minus two copies of the regular representation of G plus the trivial representation of G.*

Example 4.2. *Continuing the previous example. Consider the symmetric group on four points.*

Previous calculations showed the irreducible representations given by the action of S_4 on the set of edges is as follows.

Copies of	*irreducible representation*
2	χ_1
3	χ_2
5	χ_3
8	χ_4
7	χ_5

So by corollary 4.7 we can complete the following table of counts of irreducible representations.

irreducible representation	$im(\partial_2)$	$ker(\partial_1)$	$ker(\partial_1)/im(\partial_2)$	$F_1(\Delta)$
χ_1	*1*	*2*	*1*	*2*
χ_2	*1*	*2*	*1*	*3*
χ_3	*2*	*3*	*1*	*5*
χ_4	*3*	*5*	*2*	*8*
χ_5	*3*	*4*	*1*	*7*

□

Notice that Theorem 4.6 and Corollary 4.7 gives a complete description of the action of G where G is generated by irredundant generating set of size two. The next step is to extend these ideas to the case where the irredundant generating set is of size n. Recall Theorem 4.6 and notice that in the proof we never used the fact that there were only two generators. Rather we could replace $\mathcal{G} = \{g_1, g_2, g_1g_2^{-1}\}$ with $\mathcal{G} = \{g_1, g_2, ..., g_n\} \cup \left\{g_ig_j^{-1} | g_i, g_j \in S \text{ and } i < j\right\}$ and apply the same techniques to prove the following generalization.

THEOREM 4.8. *Let $S = \{g_1, g_2, ..., g_n\}$ be an irredundant generating set for a group G. Define the set $\mathcal{S} = S \cup \left\{g_ig_j^{-1} | g_i, g_j \in S \text{ and } i < j\right\}$. The representation given by the group action on the set of edges, $F_1(\Delta)$, consists of*

- *one copy of the regular representation for each element in $\mathcal{S}$ which is not order two*

- *a representation of degree $\frac{|G|}{2}$ that is a direct sum of constituents of the regular representation for each element in $\mathcal{S}$ which has order two.*

MMoreover for $a \in \mathcal{S}$ of order two and φ an irreducible representation of G with character χ, the number of summands of φ which appear in the representation of degree $\frac{|G|}{2}$ is given by the formula:

$$(\chi, \rho \uparrow^{G}) = (\chi \downarrow_{\langle a \rangle}, \rho) = \frac{1}{|\langle a \rangle|} \sum_{x \in \langle a \rangle} \chi(x)\overline{\rho(x)} = \frac{1}{2}\left(\chi(1)\overline{\rho(1)} + \chi(a)\overline{\rho(a)}\right) = \frac{1}{2}\left(\chi(1) - \chi(a)\right) \tag{4.2}$$

where ρ is the sign character of $\langle a \rangle$.

Another statement that we can make at this time towards the generalization to an irredundant generating set of size n is about the group action on $F_i(\Delta)$ where $2 \le i \le n$.

Theorem 4.9. *Let $S = \{g_1, g_2, \dots, g_n\}$ be an irredundant generating set for a group G. The representation given by the group action on $F_i(\Delta)$ for $2 \le i \le n$ is $\binom{n+1}{i+1}$ copies of the regular representation of G.*

Proof. Let $2 \le i \le n$. The result follows immediately from Corollary 3.5 since the corollary implies that no i-dimensional simplex will be shared by two $(i+1)$-dimensional simplices and thus there are $\binom{n+1}{i+1}$ orbits, recall Lemma 3.2, with n i-simplices in each orbit for the action of G on $F_i(\Delta)$. □

Corollary 4.10. *Let $S = \{g_1, g_2, \dots, g_n\}$ be an irredundant generating set for a group G. The representation given by the group action on $im(\partial_j)$ and $ker(\partial_j)$ for $3 \le j \le n$ is*

$$\left| \sum_{i=j}^{n} (-1)^i \binom{n+1}{i+1} \right|$$

copies of the regular representation of G.

Proof. By Theorem 4.9 we know the representation given by the group action on $F_i(\Delta)$ for $2 \leq i \leq n$ is a direct sum of $\binom{n+1}{i+1}$ copies of the regular representation of G.

Recall $\operatorname{im}(\partial_{n+1}) = 0$ by definition and $\widetilde{H}_i = 0$ for $i \geq 3$ by Theorem 3.12. Using this information we simply apply the relations $\operatorname{im}(\partial_{i+1}) + \widetilde{H}_i = \ker(\partial_i)$ and $\ker(\partial_i) + \operatorname{im}(\partial_i) = F_i(\Delta)$. $\square$

CHAPTER 5

Second Homology for Generating Sets of Size Greater than Two

In this chapter we focus on the second homology. Once we understand the representation given by the action on $\widetilde{H_2}$, we will have a full description of the action on all of our objects. (See second organizational structure in Appendix D.) This section gives results towards proving that the size of the second homology, $\widetilde{H_2}$, is determined by the number of certain pairs of generators appearing in the extended generating set. Recall that the extended generating set, $\mathcal{S} = S \cup \left\{ g_i g_j^{-1} | g_i, g_j \in S \text{ and } i < j \right\}$ where $S = \{g_1, g_2, .., g_n\}$ is the chosen irredundant generating set.

The first step to understanding the homology is to understand the structure of the image of ∂_3. In an attempt towards this goal we define a few relations that the elements of the image of ∂_3 must satisfy. Let ζ be a linear combination of triangles in $F_2(\Delta)$. We define $\text{Coeff}_\zeta(\mathrm{F}_2(x, g_i \cdot x, g_j \cdot x))$ to be the coefficient which appears with the triangle $\mathrm{F}_2(x, g_i \cdot x, g_j \cdot x)$ in ζ. Recall that the triangles in $F_2(\Delta)$ have an ordering. Also for any triangle in $F_2(\Delta)$ not appearing in ζ is considered to have coefficient zero with respect to Coeff_ζ. For example take $\zeta = -abc + 3adf + def$ then $\text{Coeff}_\zeta(abc) = -1$, $\text{Coeff}_\zeta(adf) = 3$, $\text{Coeff}_\zeta(def) = 1$ and $\text{Coeff}_\zeta(abd) = 0$.

Associated to each edge in $F_1(\Delta)$ we can define the following relations.

Relations:

(1) $|g_i| \neq 2:$ $\quad f_{\mathrm{F}_1(x, g_i\cdot x)}(\zeta) = \sum_{\substack{g\in S \\ g\neq g_i}} \operatorname{sgn}(\alpha)\ \operatorname{Coeff}_\zeta(\mathrm{F}_2\,(x, g_i\cdot x, g\cdot x)^\alpha)$

(2) $|g_i g_j| \neq 2:$ $\quad f_{\mathrm{F}_1(g_i\cdot x, g_j\cdot x)}(\zeta) = \sum_{\substack{g\in S \\ g\neq g_i}} \operatorname{sgn}(\alpha)\ \operatorname{Coeff}_\zeta(\mathrm{F}_2\,(g_i\cdot x, g_j\cdot x, g\cdot x)^\alpha)$

(3) $|g_i| = 2$ and thus $\mathrm{F}_1\,(x, g_i\cdot x) = \mathrm{F}_1\,(y, g_i\cdot y):$

(a) $f_{\mathrm{F}_1(x, g_i\cdot x)}(\zeta) = \sum_{\substack{g\in S \\ g\neq g_i}} \operatorname{sgn}(\alpha)\ \operatorname{Coeff}_\zeta(\mathrm{F}_2\,(x, g_i\cdot x, g\cdot x)^\alpha)$

(b) $f_{\mathrm{F}_1(y, g_i\cdot y)}(\zeta) = \sum_{\substack{g\in S \\ g\neq g_i}} \operatorname{sgn}(\alpha)\ \operatorname{Coeff}_\zeta(\mathrm{F}_2\,(y, g_i\cdot y, g\cdot y)^\alpha)$

(4) $|g_i g_j| = 2$ and thus $\mathrm{F}_1\,(g_i\cdot x, g_j\cdot x) = \mathrm{F}_1\,(g_i\cdot y, g_j\cdot y):$

(a) $f_{\mathrm{F}_1(g_i\cdot x, g_j\cdot x)}(\zeta) = \sum_{\substack{g\in S \\ g\neq g_i}} \operatorname{sgn}(\alpha)\ \operatorname{Coeff}_\zeta(\mathrm{F}_2\,(g_i\cdot x, g_j\cdot x, g\cdot x)^\alpha)$

(b) $f_{\mathrm{F}_1(g_i\cdot y, g_j\cdot y)}(\zeta) = \sum_{\substack{g\in S \\ g\neq g_i}} \operatorname{sgn}(\alpha)\ \operatorname{Coeff}_\zeta(\mathrm{F}_2\,(g_i\cdot y, g_j\cdot y, g\cdot y)^\alpha)$

where ζ is in the vector space $F_3(\Delta)$ and α is the sorting permutation for the triangle.

We will refer to the relations given by setting the above maps to zero many times later in this section.

Lemma 5.1. *The image of ∂_3 satisfies the above relations.*

Proof. We will show the relations hold by direct computations. There are two possibilities for elements in $F_3(\Delta)$ assuming we have a sufficient number of generators: $\mathrm{F}_3\,(x, g_i\cdot x, g_j\cdot x, g_k\cdot x)$ and $\mathrm{F}_3\,(g_i\cdot x, g_j\cdot x, g_k\cdot x, g_l\cdot x)$.

Case 1: $\mathrm{F}_3\left(x, g_i \cdot x, g_j \cdot x, g_k \cdot x\right)$

$$\zeta \doteq \partial_3(\mathrm{F}_3\left(x, g_i \cdot x, g_j \cdot x, g_k \cdot x\right)) = \mathrm{F}_2\left(g_i \cdot x, g_j \cdot x, g_k \cdot x\right) - \mathrm{F}_2\left(x, g_j \cdot x, g_k \cdot x\right)$$
$$+ \mathrm{F}_2\left(x, g_i \cdot x, g_k \cdot x\right) - \mathrm{F}_2\left(x, g_i \cdot x, g_j \cdot x\right)$$

The relations in the box above are given on edges and therefore we need only check the relations given by the edges in ζ: $\mathrm{F}_1\left(x, g_i \cdot x\right)$, $\mathrm{F}_1\left(x, g_j \cdot x\right)$, $\mathrm{F}_1\left(x, g_k \cdot x\right)$, $\mathrm{F}_1\left(g_i \cdot x, g_j \cdot x\right)$, $\mathrm{F}_1\left(g_i \cdot x, g_k \cdot x\right)$, and $\mathrm{F}_1\left(g_j \cdot x, g_k \cdot x\right)$.

(1a) $\mathrm{F}_1\left(x, g_i \cdot x\right):$

$$f_{\mathrm{F}_1(x, g_i \cdot x)}(\zeta) = \sum_{\substack{g \in S \\ g \neq g_i}} \mathrm{sgn}(\alpha)\ \mathrm{Coeff}_\zeta(\mathrm{F}_2\left(x, g_i \cdot x, g \cdot x\right)^\alpha)$$
$$= \mathrm{sgn}((1))\mathrm{Coeff}(\mathrm{F}_2\left(x, g_i \cdot x, g_k \cdot x\right)) + \mathrm{sgn}((1))\mathrm{Coeff}(\mathrm{F}_2\left(x, g_i \cdot x, g_j \cdot x\right))$$
$$= (1)(1) + (1)(-1) = 0$$

(1b) $\mathrm{F}_1\left(x, g_j \cdot x\right): f_{\mathrm{F}_1(x, g_j \cdot x)}(\zeta) = (1)(-1) + (-1)(-1) = 0$

(1c) $\mathrm{F}_1\left(x, g_k \cdot x\right): f_{\mathrm{F}_1(x, g_k \cdot x)}(\zeta) = (-1)(-1) + (-1)(1) = 0$

(1d) $\mathrm{F}_1\left(g_i \cdot x, g_j \cdot x\right): f_{\mathrm{F}_1(g_i \cdot x, g_j \cdot x)}(\zeta) = (1)(1) + (1)(-1) = 0$

(1e) $\mathrm{F}_1\left(g_i \cdot x, g_k \cdot x\right): f_{\mathrm{F}_1(g_i \cdot x, g_k \cdot x)}(\zeta) = (-1)(1) + (1)(1) = 0$

(1f) $\mathrm{F}_1\left(g_j \cdot x, g_k \cdot x\right): f_{\mathrm{F}_1(g_j \cdot x, g_k \cdot x)}(\zeta) = (1)(1) + (1)(-1) = 0$

Case 2: $\mathrm{F}_3\left(g_i \cdot x, g_j \cdot x, g_k \cdot x, g_l \cdot x\right)$

$$\zeta \doteq \partial_3(\mathrm{F}_3\left(g_i \cdot x, g_j \cdot x, g_k \cdot x, g_l \cdot x\right)) = \mathrm{F}_2\left(g_j \cdot x, g_k \cdot x, g_l \cdot x\right) - \mathrm{F}_2\left(g_i \cdot x, g_k \cdot x, g_l \cdot x\right)$$
$$+ \mathrm{F}_2\left(g_i \cdot x, g_j \cdot x, g_l \cdot x\right) - \mathrm{F}_2\left(g_i \cdot x, g_j \cdot x, g_k \cdot x\right)$$

The edges in ζ: $\mathrm{F}_1(g_i \cdot x, g_j \cdot x)$, $\mathrm{F}_1(g_i \cdot x, g_k \cdot x)$, $\mathrm{F}_1(g_i \cdot x, g_l \cdot x)$, $\mathrm{F}_1(g_j \cdot x, g_k \cdot x)$, $\mathrm{F}_1(g_j \cdot x, g_l \cdot x)$, and $\mathrm{F}_1(g_k \cdot x, g_l \cdot x)$.

(2a) $\mathrm{F}_1(g_i \cdot x, g_j \cdot x) : f_{\mathrm{F}_1(g_i \cdot x, g_j \cdot x)}(\zeta) = (1)(1) + (1)(-1) = 0$

(2b) $\mathrm{F}_1(g_i \cdot x, g_k \cdot x) : f_{\mathrm{F}_1(g_i \cdot x, g_k \cdot x)}(\zeta) = (1)(-1) + (-1)(-1) = 0$

(2c) $\mathrm{F}_1(g_i \cdot x, g_l \cdot x) : f_{\mathrm{F}_1(g_i \cdot x, g_l \cdot x)}(\zeta) = (-1)(-1) + (-1)(1) = 0$

(2d) $\mathrm{F}_1(g_j \cdot x, g_k \cdot x) : f_{\mathrm{F}_1(g_j \cdot x, g_k \cdot x)}(\zeta) = (1)(1) + (1)(-1) = 0$

(2e) $\mathrm{F}_1(g_j \cdot x, g_l \cdot x) : f_{\mathrm{F}_1(g_j \cdot x, g_l \cdot x)}(\zeta) = (-1)(1) + (1)(1) = 0$

(2f) $\mathrm{F}_1(g_k \cdot x, g_l \cdot x) : f_{\mathrm{F}_1(g_k \cdot x, g_l \cdot x)}(\zeta) = (1)(1) + (1)(-1) = 0$

Thus in both cases the relations hold and therefore any element in the image will satisfy the relations. □

There are two items that the careful reader will have noticed about the last proof. First, we never asked whether a generator has order two or not, even though the relations seem to be dependent on this property. However upon closer inspection the reader will notice that the property of the generator being order two only doubles the number of relations by giving relations for x and y. Therefore since in the proof, we are concerned with showing that the mapping of tetrahedrons based at x, for an arbitrary vertex x, satisfies the relations, we need not concern ourselves whether the generator is of order two. Then why have the second set of relations in (3) and (4) given by the order two generators? Computing examples indicates that the relations without (3b) and (4b) describe a space larger than the $\mathrm{im}(\partial_3)$. In experiments it seems that an element ζ in $F_2(\Delta)$ is in the image if and only if ζ satisfies the relations. Although the examples indicate that this claim is true, it has yet to be proven.

The second item that the careful reader will have noticed is that the definition of the relations required a sorting permutation and consequently an ordering on the labeling of the vertices. For the proof of Lemma 5.1, we took the ordering to be $x < g_i \cdot x < g_j \cdot x < g_k \cdot x < g_l \cdot x$.

However any ordering on the vertices will give the same result since for any change in sign in $f(\zeta)$ there is a corresponding change in ζ.

Now that we have the relations set up, we will continue towards our goal of understanding the homology $\widetilde{H_2}$. Let $S = \{g_1, g_2, .., g_n\}$ be an irredundant generating set for a group G and $\mathcal{S}$ be the extended generating set for S. We say an unordered pair of elements of $\mathcal{S}$, $\{a, b\}$, is a *square pair* if $|a| = |b| = |ab| = 2$. Two square pairs $\{a, b\}$ and $\{c, d\}$ are equivalent if $\{a, b, ab\}$ and $\{c, d, cd\}$ are equal as sets. Note that this definition is equivalent to saying that $\{a, b\}$ generate $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

THEOREM 5.2. *Let $S = \{g_1, g_2, .., g_n\}$ be an irredundant generating set for a group G and $\mathcal{S}$ be the extended generating set for S. If there exist an $a, b \in \mathcal{S}$ such that $\{a, b\}$ is a square pair then $\widetilde{H_2} \neq \{0\}$.*

PROOF. Suppose there exists g_i, $g_j \in S$ that form a square pair. Consider

$$\zeta \doteq \mathrm{F}_2(x, g_i \cdot x, g_j \cdot x) - \mathrm{F}_2(x, g_i \cdot x, g_i g_j \cdot x) + \mathrm{F}_2(x, g_j \cdot x, g_i g_j \cdot x) - \mathrm{F}_2(g_i \cdot x, g_j \cdot x, g_i g_j \cdot x).$$

First notice that every triangle in ζ is in $F_2(\Delta)$ since ζ can be rewritten as

$$\begin{aligned}\zeta =& \mathrm{F}_2(x, g_i \cdot x, g_j \cdot x) + \mathrm{F}_2((g_i \cdot x), g_i \cdot (g_i \cdot x), g_j \cdot (g_i \cdot x)) + \mathrm{F}_2((g_j \cdot x), g_i \cdot (g_j \cdot x), g_j \cdot (g_j \cdot x))\\ &+ \mathrm{F}_2((g_i g_j \cdot x), g_i \cdot (g_i g_j \cdot x), g_j \cdot (g_i g_j \cdot x)).\end{aligned}$$

$\zeta \notin \mathrm{im}(\partial_3)$ since the relation for $\mathrm{F}_1(x, g_i \cdot x) : f_{\mathrm{F}_1(x, g_i \cdot x)}(\zeta) = (1)(1) = 1 \neq 0$. Notice we only use $\mathrm{F}_2(x, g_i \cdot x, g_j \cdot x)$ from ζ since the other triangle with $\mathrm{F}_1(x, g_i \cdot x)$ is $\mathrm{F}_2(x, g_i \cdot x, g_i g_j \cdot x)$ and $g_i g_j \notin S$. Thus by Lemma 5.1, $\zeta \notin \mathrm{im}(\partial_3)$. However, the quick calculation below shows $\zeta \in \ker(\partial_2)$.

$$
\begin{aligned}
\partial_2(\zeta) = & \mathrm{F}_1\left(g_i \cdot x, g_j \cdot x\right) - \mathrm{F}_1\left(x, g_j \cdot x\right) + \mathrm{F}_1\left(x, g_i \cdot x\right) \\
& - \mathrm{F}_1\left(g_i \cdot x, g_i g_j \cdot x\right) + \mathrm{F}_1\left(x, g_i g_j \cdot x\right) - \mathrm{F}_1\left(x, g_i \cdot x\right) \\
& + \mathrm{F}_1\left(g_j \cdot x, g_i g_j \cdot x\right) - \mathrm{F}_1\left(x, g_i g_j \cdot x\right) + \mathrm{F}_1\left(x, g_j \cdot x\right) \\
& - \mathrm{F}_1\left(g_j \cdot x, g_i g_j \cdot x\right) + \mathrm{F}_1\left(g_i \cdot x, g_i g_j \cdot x\right) - \mathrm{F}_1\left(g_i \cdot x, g_j \cdot x\right) = 0
\end{aligned}
$$

Thus if there exist a square pair in $\mathcal{S}$ then $\widetilde{H_2} \neq \{0\}$. $\square$

Throughout the rest of this section we will be working towards proving the imprecise statement that "$\widetilde{H_2}$ is determined by the square pairs appearing in $\mathcal{S}$". We have shown that if there is a square pair in $\mathcal{S}$ then the second homology is nontrivial. Next we will show that the square pair gives a lower bound for the size of $\widetilde{H_2}$.

Suppose that $S = \{g_1, g_2, .., g_n\}$ is an irredundant generating set for a group G and that $\{g_i, g_j\}$ is a square pair. Since $\{g_i, g_j\}$ is a square pair, we have $\zeta(\mathrm{id}) \doteq \mathrm{F}_2\left(\mathrm{id}, g_i \cdot \mathrm{id}, g_j \cdot \mathrm{id}\right) - \mathrm{F}_2\left(\mathrm{id}, g_i \cdot \mathrm{id}, g_i g_j \cdot \mathrm{id}\right) + \mathrm{F}_2\left(\mathrm{id}, g_j \cdot \mathrm{id}, g_i g_j \cdot \mathrm{id}\right) - \mathrm{F}_2\left(g_i \cdot \mathrm{id}, g_j \cdot \mathrm{id}, g_i g_j \cdot \mathrm{id}\right)$ is in the homology by Theorem 5.2. Consider the action of $g_k \in S$ where $k \neq i, j$ on $\zeta(\mathrm{id})$.

$$
\begin{aligned}
\zeta(\mathrm{id}) \cdot g_k = & \mathrm{F}_2\left(g_k, g_i g_k, g_j g_k\right) - \mathrm{F}_2\left(g_k, g_i g_k, g_i g_j g_k\right) + \mathrm{F}_2\left(g_k, g_j g_k, g_i g_j g_k\right) - \mathrm{F}_2\left(g_i g_k, g_j g_k, g_i g_j g_k\right) \\
= & \mathrm{F}_2\left(g_k, g_i \cdot g_k, g_j \cdot g_k\right) - \mathrm{F}_2\left(g_k, g_i \cdot g_k, g_i g_j \cdot g_k\right) + \mathrm{F}_2\left(g_k, g_j \cdot g_k, g_i g_j \cdot g_k\right) \\
& - \mathrm{F}_2\left(g_i \cdot g_k, g_j \cdot g_k, g_i g_j \cdot g_k\right) \qquad (5.1) \\
= & \zeta(g_k)
\end{aligned}
$$

Next consider the action of g_i on $\zeta(\mathrm{id})$.

$$\begin{aligned}
\zeta(\mathrm{id}) \cdot g_i =& \mathrm{F}_2\left(g_i, \mathrm{id}, g_j g_i\right) - \mathrm{F}_2\left(g_i, \mathrm{id}, g_j\right) + \mathrm{F}_2\left(g_i, g_j g_i, g_j\right) - \mathrm{F}_2\left(\mathrm{id}, g_j g_i, g_j\right) \\
=& \mathrm{F}_2\left(\mathrm{id}, g_i \cdot \mathrm{id}, g_j \cdot \mathrm{id}\right) - \mathrm{F}_2\left(\mathrm{id}, g_i \cdot \mathrm{id}, g_i g_j \cdot \mathrm{id}\right) + \mathrm{F}_2\left(\mathrm{id}, g_j \cdot \mathrm{id}, g_i g_j \cdot \mathrm{id}\right) \\
& \quad - \mathrm{F}_2\left(g_i \cdot \mathrm{id}, g_j \cdot \mathrm{id}, g_i g_j \cdot \mathrm{id}\right) \\
=& \zeta(\mathrm{id})
\end{aligned} \tag{5.2}$$

Similarly for g_j. Thus the square pair given by g_i and g_j generate the stabilizer for $\zeta(\mathrm{id})$. Therefore, the action of G on Δ gives rise to a cyclic submodule generated by $\zeta(\mathrm{id})$. Notice that the module given by the action on $\zeta(\mathrm{id})$ corresponds to the trivial representation of $\langle g_i, g_j \rangle$ induced up to G. This line of reasoning gives the following result.

THEOREM 5.3. *Let $S = \{g_1, g_2, .., g_n\}$ be an irredundant generating set for a group G and $\mathcal{S}$ be the extended generating set for S. If $x, y \in \mathcal{S}$ form a square pair then dim$\left(\widetilde{H_2}\right) \geq \frac{|G|}{4}$.*

PROOF. This result follows directly from the calculations above and that $\widetilde{H_2}$ is a G-module. □

The same argument can be used for any number of square pairs as long as their pairwise intersection is trivial, i.e. the only element in the intersection of the groups generated by any two of the square pairs is the identity.

THEOREM 5.4. *Let $S = \{g_1, g_2, .., g_n\}$ be an irredundant generating set for a group G and $\mathcal{S}$ be the extended generating set for S. Suppose G has m distinct square pairs with generators in $\mathcal{S}$ such that the pairwise intersections of the square pairs are trivial. Then dim$\left(\widetilde{H_2}\right) \geq \frac{m|G|}{4}$.*

Proof. For each square pair the induced module for the action on $\zeta(\mathrm{id})$ will be in $\widetilde{H_2}$. However, we need to show that the module created by inducing the trivial module for each of the square pairs is of full size or more precisely that the module is a direct sum of the induced trivial modules.

Recall that in Lemma 5.1 we showed that the image of ∂_3 satisfies the relations on page 29. Therefore, if we can show that a linear combination of triangles does not satisfy the relations then that linear combination of triangles is not in the image of ∂_3.

Consider again $\zeta_{\{g_i,g_j\}}(\mathrm{id}) \doteq \mathrm{F}_2(\mathrm{id}, g_i\cdot\mathrm{id}, g_j\cdot\mathrm{id}) - \mathrm{F}_2(\mathrm{id}, g_i\cdot\mathrm{id}, g_ig_j\cdot\mathrm{id})$ $+\mathrm{F}_2(\mathrm{id}, g_j\cdot\mathrm{id}, g_ig_j\cdot\mathrm{id}) - \mathrm{F}_2(g_i\cdot\mathrm{id}, g_j\cdot\mathrm{id}, g_ig_j\cdot\mathrm{id})$. (Since there are now multiple square pairs we will denote which generators are used in $\zeta(\mathrm{id})$ by the subscript.) Notice that the edge $\mathrm{F}_1(\mathrm{id}, g_i\cdot\mathrm{id})$ appears in $\zeta_{\{g_i,g_j\}}(\mathrm{id})$ twice but only one of those triangles are of the form $\mathrm{F}_2(a, x\cdot a, y\cdot a)$ where x and y are in S since $g_ig_j\notin S$. Thus $\zeta_{\{g_i,g_j\}}(\mathrm{id})$ does not satisfy the relations given by $\mathrm{F}_1(\mathrm{id}, g_i\cdot\mathrm{id})$ and consequently $\zeta_{\{g_i,g_j\}}(\mathrm{id})\notin\mathrm{im}(\partial_3)$.

Since the pairwise intersections of the square pairs are trivial the edge $\mathrm{F}_1(\mathrm{id}, g_i\cdot\mathrm{id})$ cannot appear in any other $\zeta_{\{x,y\}}(b)$ for any generators in a square pair $\{x,y\}$ where $x\notin\{g_i,g_j\}, y\notin\{g_i,g_j\}$ or $b\in G$. Thus no linear combination of ζ's will be in the image. □

In the previous proof we only really used that the edge $\mathrm{F}_1(\mathrm{id}, g_i\cdot\mathrm{id})$ cannot appear in any other $\zeta_{\langle x,y\rangle}(b)$. Thus the same proof will work for the following theorem.

Theorem 5.5. *Let $S = \{g_1, g_2, .., g_n\}$ be an irredundant generating set for a group G. Suppose G has m distinct square pairs with the property that at least one generator of each square pair does not appear in any other square pair. Then dim$\left(\widetilde{H_2}\right)\geq\frac{m|G|}{4}$.*

We can think of a square pair as a square, as shown in Figure 5.1, then the assumption in Theorem 5.5 of at least one generator of each square pair does not appear in any other square

$$\begin{tikzcd} g_i \cdot v \arrow[r, "g_j"] & g_i g_j \cdot v \\ v \arrow[u, "g_i"] \arrow[r, "g_j"'] & g_j \cdot v \arrow[u, "g_i"'] \end{tikzcd}$$

FIGURE 5.1. $V_4 = \langle g_i, g_j \rangle$ square.

pair can be thought of as a "free edge" property. With this point of view the next natural question is what happens when we do not have a free edge? The smallest such example is C_2^3 with generating set $\{g_1, g_2, g_3\}$ such that $\{g_1, g_2\}$, $\{g_1, g_3\}$, and $\{g_2, g_3\}$ are all square pairs. If drawn as in Figure 5.1 these three square pairs would make a cube.

In order to state the next theorem we will need a new definition. We say that a square pair $\{x, y\}$ is a *k-square pair* if $x = ab$ and $y = cd$ where $\{a, b\}$ and $\{c, d\}$ are $(k-1)$-square pairs. In the case of C_2^3 with generating set $\{g_1, g_2, g_3\}$ such that $\{g_1, g_2\}$, $\{g_1, g_3\}$, and $\{g_2, g_3\}$ are all square pairs (can be thought of as 0-square pairs), we have one 1-square pair $\{g_1g_2, g_1g_3\}$. Recall that two square pairs $\{a, b\}$ and $\{c, d\}$ are equivalent if $\{a, b, ab\}$ and $\{c, d, cd\}$ are equal as sets. With this definition we see that we only have one 1-square pair since $\{g_1g_2, g_1g_3\} \sim \{g_1g_2, g_2g_3\} \sim \{g_1g_3, g_2g_3\}$.

THEOREM 5.6. *Take $S = \{g_1, g_2, g_3\}$ be an irredundant generating set for $G = C_2 \times C_2 \times C_2$ such that $A = \{g_1, g_2\}$, $B = \{g_1, g_3\}$, and $C = \{g_2, g_3\}$ are square pairs. Then there exists a 1-square pair, $D = \langle g_1g_2, g_1g_3 \rangle$. Furthermore, the trivial representations corresponding to subgroups generated by A, B, and C induced up to G and the nontrivial irreducible submodule of the trivial representation corresponding to subgroup generated by D induced up to G are contained in $\widetilde{H_2}$. In other words, $\widetilde{H_2}$ contains the direct sum of all of constituents of the induced trivial representations of the square pairs except one trivial module.*

PROOF. As we have observed above (see reasoning before Theorem 5.3), the trivial representation of the group generated by a square pair is isomorphic to a submodule of $\widetilde{H_2}$. What is left to be determined is if the induced representations from each of the square pairs interact in any way.

Let φ be the representation given by the action of G on $\widetilde{H_2}$.

Recall that the group generated by a square pair is isomorphic to the Klein four group and thus has index two with respect to G. Therefore the trivial character of $\langle A \rangle$ induced to G has two constituents: one is the trivial character of G, denoted by 1_G, and other is a nontrivial one dimensional character of G, denoted by χ_A. Similarly for B, C, and D we get the irreducible characters 1_G, χ_B, 1_G, χ_C, 1_G, and χ_D. The characters χ_A, χ_B, χ_C, and χ_D are all distinct since G is abelian and thus every element of G is in its own conjugacy class. Thus their corresponding modules are distinct and all of these characters are constituents of φ.

Now we restrict to the constituents of homogeneous component for the trivial G-module induced from A, B, C, and D. We will show this module is generated by the trivial modules induced from A, B, and C, i.e. the trivial module induced from D is contained in the direct sum of the trivial modules induced from A, B, and C.

Recall the definition of $\zeta_{\{g_i,g_j\}}(x)$:

$$\begin{aligned}\zeta_{\{g_i,g_j\}}(x) \doteq & \mathrm{F}_2\left(x, g_i \cdot x, g_j \cdot x\right) - \mathrm{F}_2\left(x, g_i \cdot x, g_i g_j \cdot x\right) + \mathrm{F}_2\left(x, g_j \cdot x, g_i g_j \cdot x\right) \\ & - \mathrm{F}_2\left(g_i \cdot x, g_j \cdot x, g_i g_j \cdot x\right).\end{aligned}$$

We will now show that the vector $\zeta_{\{g_1,g_2\}}(\mathrm{id}) + \zeta_{\{g_1,g_2\}}(g_3)$ spans a fixed subspace corresponding to the trivial module for G. As seen in equation 5.2, $\zeta_{\{g_1,g_2\}}(\mathrm{id})$ and $\zeta_{\{g_1,g_2\}}(g_3)$ are both fixed by g_1 and g_2 and thus $\zeta_{\{g_1,g_2\}}(\mathrm{id}) + \zeta_{\{g_1,g_2\}}(g_3)$ is fixed by g_1 and g_2. Equation

5.1 also shows that $\zeta_{\{g_1,g_2\}}(\mathrm{id}) \cdot g_3 = \zeta_{\{g_1,g_2\}}(g_3)$. The following quick calculation shows that $\zeta_{\{g_1,g_2\}}(g_3) \cdot g_3 = \zeta_{\{g_1,g_2\}}(\mathrm{id})$.

$$
\begin{aligned}
\zeta_{\{g_1,g_2\}}(g_3) \cdot g_3 =& \mathrm{F}_2\left(g_3 \cdot g_3, (g_1 \cdot g_3) \cdot g_3, (g_2 \cdot g_3) \cdot g_3\right) - \mathrm{F}_2\left(g_3 \cdot g_3, (g_1 \cdot g_3) \cdot g_3, (g_1 g_2 \cdot g_3) \cdot g_3\right) \\
&+ \mathrm{F}_2\left(g_3 \cdot g_3, (g_2 \cdot g_3) \cdot g_3, (g_1 g_2 \cdot g_3) \cdot g_3\right) \\
&- \mathrm{F}_2\left((g_1 \cdot g_3) \cdot g_3, (g_2 \cdot g_3) \cdot g_3, (g_1 g_2 \cdot g_3) \cdot g_3\right) \\
=& \mathrm{F}_2\left(g_3 \cdot g_3, g_1 \cdot (g_3 \cdot g_3), g_2 \cdot (g_3 \cdot g_3)\right) - \mathrm{F}_2\left(g_3 \cdot g_3, g_1 \cdot (g_3 \cdot g_3), g_1 g_2 \cdot (g_3 \cdot g_3)\right) \\
&+ \mathrm{F}_2\left(g_3 \cdot g_3, g_2 \cdot (g_3 \cdot g_3), g_1 g_2 \cdot (g_3 \cdot g_3)\right) \\
&- \mathrm{F}_2\left(g_1 \cdot (g_3 \cdot g_3), g_2 \cdot (g_3 \cdot g_3), g_1 g_2 \cdot (g_3 \cdot g_3)\right) \\
=& \mathrm{F}_2\left(\mathrm{id}, g_1 \cdot \mathrm{id}, g_2 \cdot \mathrm{id}\right) - \mathrm{F}_2\left(\mathrm{id}, g_1 \cdot \mathrm{id}, g_1 g_2 \cdot \mathrm{id}\right) \\
&+ \mathrm{F}_2\left(\mathrm{id}, g_2 \cdot \mathrm{id}, g_1 g_2 \cdot \mathrm{id}\right) - \mathrm{F}_2\left(g_1 \cdot \mathrm{id}, g_2 \cdot \mathrm{id}, g_1 g_2 \cdot \mathrm{id}\right) \\
=& \zeta_{\{g_1,g_2\}}(\mathrm{id})
\end{aligned}
$$

Therefore $\zeta_{\{g_1,g_2\}}(\mathrm{id}) + \zeta_{\{g_1,g_2\}}(g_3)$ is fixed by all three generators and so corresponds to a trivial module for G. Similarly $\zeta_{\{g_1,g_3\}}(\mathrm{id}) + \zeta_{\{g_1,g_3\}}(g_2)$, $\zeta_{\{g_2,g_3\}}(\mathrm{id}) + \zeta_{\{g_2,g_3\}}(g_1)$, $\zeta_{\{g_1g_2,g_2g_3\}}(\mathrm{id}) + \zeta_{\{g_1g_2,g_2g_3\}}(g_3)$ are fixed by G. So we have four copies of the trivial module. However, in the homology, one copy of the trivial module is redundant.

$$
\begin{aligned}
\zeta_{\{g_1,g_2\}}(\mathrm{id}) &+ \zeta_{\{g_1,g_2\}}(g_3) + \zeta_{\{g_1,g_3\}}(\mathrm{id}) + \zeta_{\{g_1,g_3\}}(g_2) + \zeta_{\{g_2,g_3\}}(\mathrm{id}) + \zeta_{\{g_2,g_3\}}(g_1) + \zeta_{\{g_1g_2,g_2g_3\}}(\mathrm{id}) \\
&+ \zeta_{\{g_1g_2,g_2g_3\}}(g_3) \\
=& \partial_2(-\mathrm{F}_3\left(\mathrm{id}, g_1 \cdot \mathrm{id}, g_2 \cdot \mathrm{id}, g_3 \cdot \mathrm{id}\right) + \mathrm{F}_3\left(g_3, g_1 \cdot g_3, g_2 \cdot g_3, g_3 \cdot g_3\right) \\
&- \mathrm{F}_3\left(g_2, g_1 \cdot g_2, g_2 \cdot g_2, g_3 \cdot g_2\right) + \mathrm{F}_3\left(g_1, g_1 \cdot g_1, g_2 \cdot g_1, g_3 \cdot g_1\right) \\
&+ \mathrm{F}_3\left(g_1g_2, g_1 \cdot g_1g_2, g_2 \cdot g_1g_2, g_3 \cdot g_1g_2\right) - \mathrm{F}_3\left(g_1g_3, g_1 \cdot g_1g_3, g_2 \cdot g_1g_3, g_3 \cdot g_1g_3\right) \\
&+ \mathrm{F}_3\left(g_2g_3, g_1 \cdot g_2g_3, g_2 \cdot g_2g_3, g_3 \cdot g_2g_3\right) \\
&- \mathrm{F}_3\left(g_1g_2g_3, g_1 \cdot g_1g_2g_3, g_2 \cdot g_1g_2g_3, g_3 \cdot g_1g_2g_3\right)) \in \operatorname{im}\left(\partial_2\right).
\end{aligned}
$$

$\square$

We have seen that in the C_2^3 case instead of the expected eight dimensional module, we get a seven dimensional module. This loss was determined to be the trivial module of the induced representation of the 1-square pair modulo the image. However, does this trend continue for C_2^n where $n > 3$?

EXAMPLE 5.1. *Let C_2^4 be generated by $\{g_1, g_2, g_3, g_4\}$ where $|g_i| = 2$.*

0-*square pairs with respect to the generating set:*

$\langle g_1, g_2\rangle, \langle g_1, g_3\rangle, \langle g_1, g_4\rangle, \quad \langle g_2, g_3\rangle, \langle g_2, g_4\rangle, \quad \langle g_3, g_4\rangle$

1-*square pairs:*

$\langle g_1g_2, g_1g_3\rangle, \langle g_1g_2, g_1g_4\rangle, \quad \langle g_1g_3, g_1g_4\rangle, \quad \langle g_2g_3, g_2g_4\rangle$

2-*square pair:*

$\langle g_2g_3, g_2g_4\rangle$

Notice that $\langle g_3g_4, g_3g_5\rangle$ is both a 1-*square pair and a* 2-*square pair.*

EXAMPLE 5.2. *Let C_2^5 be generated by $\{g_1, g_2, ..., g_5\}$ where $|g_i| = 2$.*

0-*square pairs with respect to the generating set:*

$\langle g_1, g_2\rangle, \langle g_1, g_3\rangle, \langle g_1, g_4\rangle, \langle g_1, g_5\rangle, \quad \langle g_2, g_3\rangle, \langle g_2, g_4\rangle, \langle g_2, g_5\rangle, \quad \langle g_3, g_4\rangle, \langle g_3, g_5\rangle, \quad \langle g_4, g_5\rangle$

1-*square pairs:*

$\langle g_1g_2, g_1g_3\rangle, \langle g_1g_2, g_1g_4\rangle, \langle g_1g_2, g_1g_5\rangle, \quad \langle g_1g_3, g_1g_4\rangle, \langle g_1g_3, g_1g_5\rangle, \quad \langle g_1g_4, g_1g_5\rangle,$

$\langle g_2g_3, g_2g_4\rangle, \langle g_2g_3, g_2g_5\rangle, \quad \langle g_2g_4, g_2g_5\rangle, \quad \langle g_3g_4, g_3g_5\rangle$

2-*square pairs:*

$\langle g_2g_3, g_2g_4\rangle, \langle g_2g_3, g_2g_5\rangle, \langle g_2g_4, g_2g_5\rangle, \quad \langle g_3g_4, g_3g_5\rangle$

3-*square pair:*

$\langle g_3g_4, g_3g_5\rangle$

Our proof for C_2^3 illustrated that the k-square pairs play a role. Our current conjecture for the group C_2^n with generating set $\{g_1, .., g_n\}$ such that $|g_i| = 2$ is that for each k-square pair reduces the size of the part of $\widetilde{H_2}$ which the square pairs generate. The following table gives the number of k-square pairs and the directly computed size of $\widetilde{H_2}$ for C_2^n where $4 \leq n \leq 7$ (See Appendix C for outputs).

	C_2^3	C_2^4	C_2^5	C_2^6	C_2^7
Rank of induced module from all V_4's	8	40	160	560	1792
Reduction due to 1-square pairs	1	$4 * 2^1$	$10 * 2^2$	$20 * 2^3$	$35 * 2^4$
Reduction due to 2-square pairs	0	1	$4 * 2^1$	$10 * 2^2$	$20 * 2^3$
Reduction due to 3-square pairs	0	0	1	$4 * 2^1$	$10 * 2^2$
Reduction due to 4-square pairs	0	0	0	1	$4 * 2^1$
Reduction due to 5-square pairs	0	0	0	0	1
Total Reduction mod image	1	9	49	209	769
Rank of induced module from all V_4's mod image	7	31	111	351	1023

Conjecture 5.1. Let $G = C_2^n$ with generating set $S = \{g_1, ..., g_n\}$ such that $|g_i| = 2$ for all i. Then the size of $\widetilde{H_2}$ is

$$\binom{n+1}{3}2^{(n-2)} - \sum_{i=1}^{n-2}\binom{(n+1)-i}{3}2^{(n-2)-i}.$$

We not only have a conjecture for the size of $\widetilde{H_2}$ but we also have a conjecture for which irreducible representations will appear in the action of C_2^n on $\widetilde{H_2}$. However in order to state that conjecture we will need some more notation.

Let

$$\tau_{\{g_i,g_j\},g_k}(id) \doteq (\zeta_{\{g_i,g_j\}}(id) - \zeta_{\{g_i,g_j\}}(id) \cdot g_k)$$

then we define

$$\tau_{\{g_i,g_j\},g_k}(id) * g_l \doteq -\tau_{\{g_i,g_j\},g_k}(id) + \tau_{\{g_i,g_j\},g_k}(id) \cdot g_k,$$

$$\tau_{\{g_i,g_j\},g_k}(id) * g_l g_m \doteq -(\tau_{\{g_i,g_j\},g_k}(id) * g_l) + (\tau_{\{g_i,g_j\},g_k}(id) * g_l) \cdot g_m,$$

and so on.

Conjecture 5.2. Let $G = C_2^n$ with generating set $S = \{g_1, ..., g_n\}$ such that $|g_i| = 2$ for all i. The representation of G given by the action of G on $\widetilde{H_2}$ is comprised of the trivial representations of the 0-square pairs and 1-square pairs induced up to G minus a few of the constituents. The constituents which must be removed are as follows:

Step 1: For each 1-square pair remove one copy of the trivial representation at the C_2^3 level. (Reduction size: $\binom{n}{3}2^{(n-3)}$)

Step k: ($k > 1$) For the k-square pairs remove one nontrivial representation of the form $\tau_{\{g_a,g_b\},g_c}(id) * g_{x_1}g_{x_2}...g_{x_k}$ at the C_2^{2+k} level. (Reduction size: $\binom{(n+1)-k}{3}2^{(n-2)-k}$)

Bibliography

[1] L. Lovász, "Kneser's conjecture, chromatic number, and homotopy," *Journal of Combinatorial Theory, Series A*, vol. 25, no. 3, pp. 319–324, 1978.

[2] J. Jonsson, *Simplicial Complexes of Graphs*. Springer, 2008.

[3] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*. Springer, first ed., 2005.

[4] H. Coxeter, *Regular Polytopes*. Dover Publications, third ed., 1973.

[5] K. Lux and H. Pahlings, *Representations of Groups*. Cambridge University Press, first ed., 2010.

[6] A. Hulpke, *Notes on Computational Group Theory*. 2010.

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.6.2*, 2013.

APPENDIX A

Code

Algorithm 1: CG_to_CC_Outputwriter

Data: Group, Generating_Sets, Distance_Set, Prime

Result: Cayley graph and Chain complex output file

for *Gen_Set* $\in$ *Generating_Sets* **do**

Cayley_Graph, Distance_Matrix := CayleyGraph(Gen_Set, Distance_Set); `#Uses GRAPE package via GAP`

Group_elements := List_Group_Elements(Gen_Set);

Nbhds := Create_Nbhds(Group_Elements,Distance_Matrix,Distance_Set); `#For each group element the function uses the Distance_Matrix to determine which elements are distance in Distance_Set`

Simplicial_Complex := Powerset(Nbhds);

Chain_Complex:=Create_Mats_for_Chain_Complex(Simplicial_Complex,Prime);

`#Creates the matrices by using the definition` $\sigma_i(e_\alpha) = \mathtt{sgn}(\alpha)e_{\rho(\alpha)}$. `See Section 3.`

end

Create and compile Latex file using the objects created above.†

†See Sample Output 1 in Appendix B.

Algorithm 2: Action_Analysis

Data: Group, Generating_Sets, Distance_Set, Prime

Result: Cayley graph and Chain complex output file

Gen_Set_for_Action:=Grab_Action_Set(Group); `#Defines a unique generating set for the action of the group.  This will allow us to compare the modules in the output even though the Gen_Set is different.`

for *Gen_Set* ∈ *Generating_Sets* **do**

  Cayley_Graph, Distance_Matrix := CayleyGraph(Gen_Set, Distance_Set);

  Group_elements := List_Group_Elements(Gen_Set);

  Nbhds := Create_Nbhds(Group_Elements,Distance_Matrix,Distance_Set);

  Simplicial_Complex := Powerset(Nbhds);

  Chain_Complex:=Create_Mats_for_Chain_Complex(Simplicial_Complex,Prime);

  New_Elem_List:=Act_by_Element(Group,Gen_Set_for_Action,Nbhds);

  Matrices_Between_CCs:=Matrices_Between_CCs(Simplicial_Complex,New_Elem_List);

  Matrices_Commute(Matrices_Between_CCs,Chain_Complex,Prime); `#Tests the matrices.  If they are constructed correctly they must commute.`

  Four_Action_output:=Four_Action_Analysis(Matrices_Between_CCs,Chain_Complex); `#Creates and runs a GAP script to investigate the action on the Images and Kernels of the boundary maps along with the action on the homology and whole vector space.`

  Create and compile Latex file using Four_Action_Analysis_output.†

end

†See Sample Output 2 in Appendix B.

APPENDIX B

Example Outputs

Following examples are calculated in a finite field of characteristic coprime to the order of G with all the necessary roots of unity for the irreducible characters below to be in bijection with the ordinary characters.

AlternatingGroup([1 .. 4])

Generating Set:	Directed:	F-Vector:	Homology of the chain complex:
['(1,2)(3,4)', '(1,2,3)']	True	[1, 12, 30, 12]	[0, 7, 0]
['(1,2)(3,4)', '(1,2,4)']	True	[1, 12, 30, 12]	[0, 7, 0]
['(2,4,3)', '(1,2,3)']	True	[1, 12, 30, 12]	[0, 7, 0]
['(1,2,3)', '(1,2,4)']	True	[1, 12, 36, 12]	[0, 13, 0]
['(1,2,4)', '(1,4,3)']	True	[1, 12, 30, 12]	[0, 7, 0]

Cayley Graph to Chain Complex Output File 1.

Group: SymmetricGroup([1 .. 4])
Generators of Cayley Graph: [(1,2), (1,2)(3,4), (1,2,3)]
Generators for Action: [(1,2), (1,2,3,4)] Prime: 5

Tables give the multiplicity of representation. Period stands for zero.

Step0

Irred Reps	Image	Kernel	Homology	Whole Space
1A	.	.	.	1
1B	1	1	.	1
2A	2	2	.	2
3A	3	3	.	3
3B	3	3	.	3

*Homology Block 0 dim.

Step1

Irred Reps	Image	Kernel	Homology	Whole Space
1A	2	2	.	2
1B	3	4	1	5
2A	5	5	.	7
3A	9	11	2	14
3B	8	8	.	11

Step2

Irred Reps	Image	Kernel	Homology	Whole Space
1A	1	2	1	4
1B	1	1	.	4
2A	2	3	1	8
3A	3	3	.	12
3B	3	4	1	12

Step3

Irred Reps	Image	Kernel	Homology	Whole Space
1A	.	.	.	1
1B	.	.	.	1
2A	.	.	.	2
3A	.	.	.	3
3B	.	.	.	3

*Image Block 0 dim. Kernel Block 0 dim. Homology Block 0 dim.

APPENDIX C

Square Pairs Conjecture

All the examples in this appendix are calculated in a finite field, given by the listed prime, of characteristic coprime to the order of G with all the necessary roots of unity for the irreducible characters below to be in bijection with the ordinary characters.

Group: C_2^3

Generators of Cayley Graph: [(1,2), (3,4), (5,6)]
Generators for Action: [(1,2), (3,4), (5,6)]
Prime:11

$\widetilde{H_2}$

Irred Reps	Image	Kernel	Homology	Whole Space
1A	1	4	3	4
1B	1	2	1	4
1C	1	1	.	4
1D	1	2	1	4
1E	1	1	.	4
1F	1	1	.	4
1G	1	2	1	4
1H	1	2	1	4
Sums	8	15	7	32

Group: C_2^4

Generators of Cayley Graph: [(1,2), (3,4), (5,6), (7,8)]
Generators for Action: [(1,2), (3,4), (5,6), (7,8)]
Prime:11

$\widetilde{H_2}$

Irred Reps	Image	Kernel	Homology	Whole Space
1A	4	10	6	10
1B	4	5	1	10
1C	4	5	1	10
1D	4	5	1	10
1E	4	7	3	10
1F	4	5	1	10
1G	4	7	3	10
1H	4	5	1	10
1I	4	5	1	10
1J	4	7	3	10
1K	4	5	1	10
1L	4	7	3	10
1M	4	5	1	10
1N	4	5	1	10
1O	4	5	1	10
1P	4	7	3	10
Sums	64	95	31	160

Group: C_2^5

Generators of Cayley Graph: [(1,2), (3,4), (5,6), (7,8), (9,10)]

Generators for Action: [(1,2), (3,4), (5,6), (7,8), (9,10)]

Prime:11

$$\widetilde{H_2}$$

Irred Reps	Image	Kernel	Homology	Whole Space
1A	10	20	10	20
1AA	10	16	6	20
1AB	10	12	2	20
1AC	10	12	2	20
1AD	10	13	3	20
1AE	10	13	3	20
1AF	10	16	6	20
1B	10	16	6	20
1C	10	12	2	20
1D	10	13	3	20
1E	10	16	6	20
1F	10	13	3	20
1G	10	13	3	20
1H	10	12	2	20
1I	10	13	3	20
1J	10	13	3	20
1K	10	13	3	20
1L	10	13	3	20
1M	10	12	2	20
1N	10	13	3	20
1O	10	12	2	20
1P	10	12	2	20
1Q	10	12	2	20
1R	10	16	6	20
1S	10	13	3	20
1T	10	13	3	20
1U	10	13	3	20
1V	10	12	2	20
1W	10	16	6	20
1X	10	12	2	20
1Y	10	13	3	20
1Z	10	13	3	20
Sums	320	431	111	640

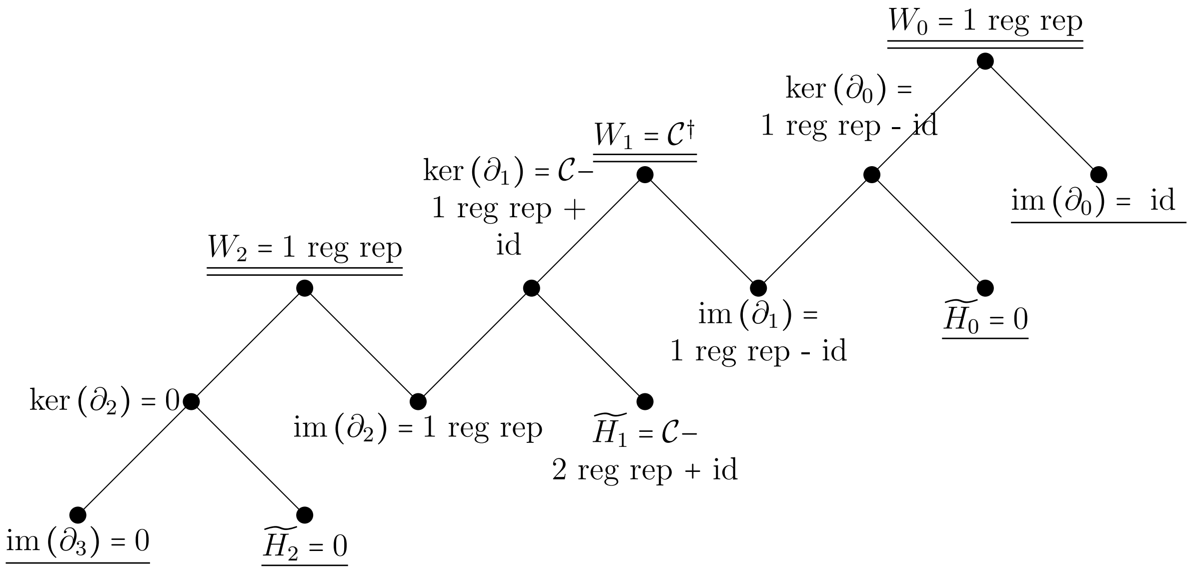
Group: C_2^6
Generators of Cayley Graph: [(1,2), (3,4), (5,6), (7,8), (9,10), (11,12)]
Generators for Action: [(1,2), (3,4), (5,6), (7,8), (9,10), (11,12)]
Prime:11

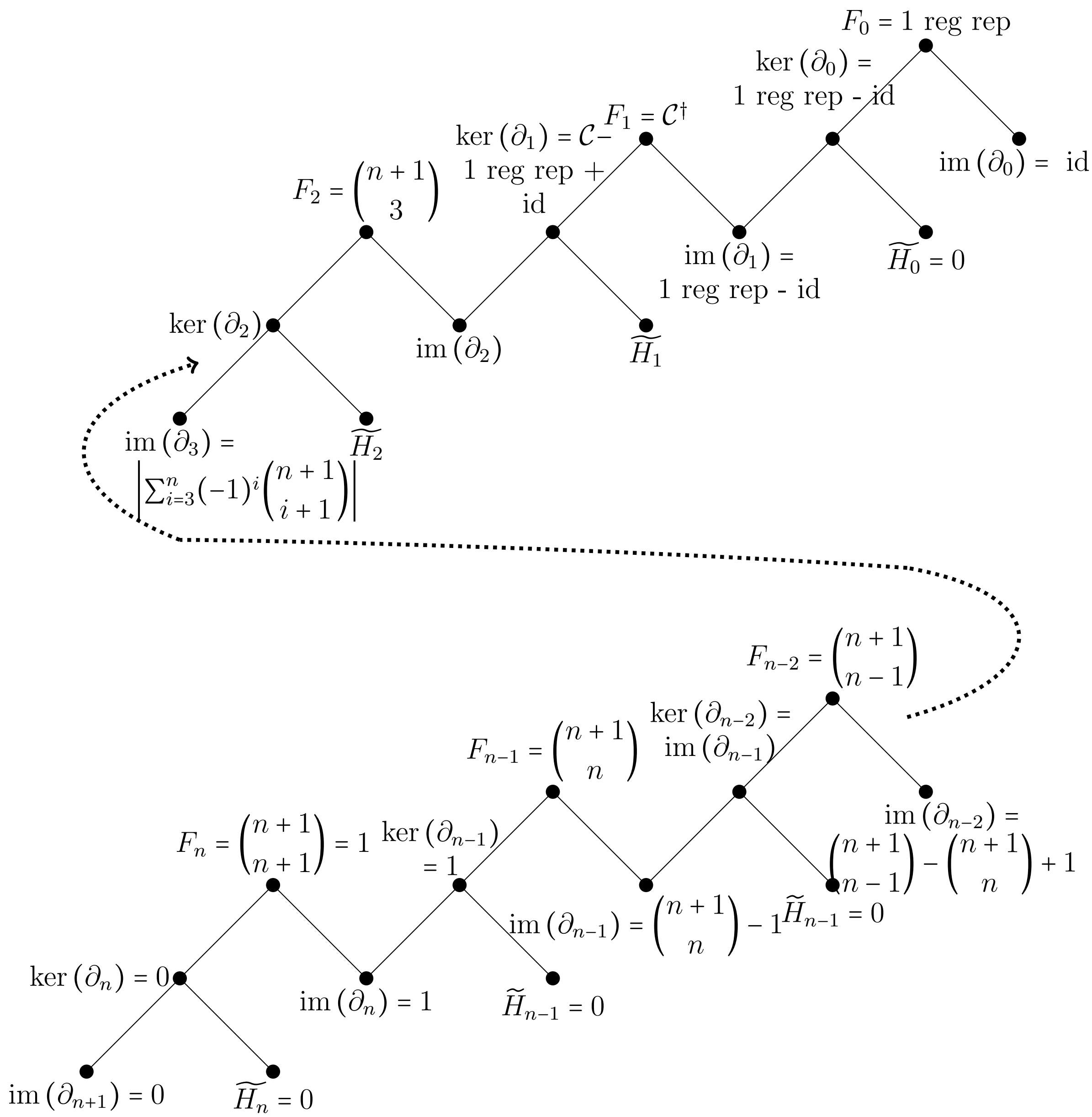
$$\widetilde{H_2}$$

Irred Reps	Image	Kernel	Homology	Whole Space
1A	20	35	15	35
1AA	20	24	4	35
1AB	20	26	6	35
1AC	20	24	4	35
1AD	20	24	4	35
1AE	20	26	6	35
1AF	20	24	4	35
1AG	20	24	4	35
1AH	20	24	4	35
1AI	20	26	6	35
1AJ	20	26	6	35
1AK	20	26	6	35
1AL	20	24	4	35
1AM	20	30	10	35
1AN	20	24	4	35
1AO	20	26	6	35
1AP	20	24	4	35
1AQ	20	26	6	35
1AR	20	24	4	35
1AS	20	26	6	35
1AT	20	30	10	35
1AU	20	24	4	35
1AV	20	24	4	35
1AW	20	24	4	35
1AX	20	24	4	35
1AY	20	24	4	35
1AZ	20	24	4	35
1B	20	26	6	35
1BA	20	26	6	35
1BB	20	30	10	35
1BC	20	24	4	35
1BD	20	26	6	35
1BE	20	24	4	35
1BF	20	26	6	35
⋮	⋮	⋮	⋮	⋮
1Z	20	24	4	35
Sums	1280	1631	351	2240

APPENDIX D

Organizational Structure





All values in the bottom of the figure are number of copies of the regular representation.

†See Theorem 4.6 for value of $\mathcal{C}$.