DISSERTATION

COMMUTATIVE ALGEBRA IN THE GRADED CATEGORY WITH APPLICATIONS TO EQUIVARIANT COHOMOLOGY RINGS

Submitted by

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ABSTRACT

COMMUTATIVE ALGEBRA IN THE GRADED CATEGORY WITH APPLICATIONS TO EQUIVARIANT COHOMOLOGY RINGS

Let G be either a finite group or a compact Lie group, and let k be a field of characteristic p. In [22], [23], Quillen proves the following conjecture of Atiyah and Swan: Let $H^*(G,k)$ be the group cohomology ring of G, then the Krull dimension of $H^*(G,k)$ is equal to the maximal rank over all elementary abelian p subgroups of G. Recall that an elementary abelian p-subgroup is defined as any group isomorphic to a product of $\mathbb{Z}/p$'s, and the rank is the number of factors. To prove the conjecture, Quillen uses an argument “by descent", showing the statement is true in the following more general setting: Suppose that G acts continuously on some topological space X (which satisfies some mild topological conditions), then the Krull dimension of the equivariant cohomology ring $H^*_G(X,k)$ is equal to the maximal rank of an elementary abelian p-subgroup A for which $X^A \neq \emptyset$. In the case where the topological space X is taken to be a point, the equivariant cohomology ring equals the group cohomology ring, thus proving Atiyah and Swan's conjecture.

In the same series of papers, Quillen sets the stage for a more general study of the commutative algebra of equivariant cohomology rings by showing that the prime spectrum of $H^*_G(X,k)$ is directly related to the elementary abelian p-subgroup structure of G, and the geometry of the G action on X. Since then, several studies have been done in this direction. For example, in [9] Duflot produces an isomorphism between the localization of $H^*_G(X,k)$ at a minimal prime, and that of the equivariant cohomology ring of the centralizer of the corresponding elementary abelian p-subgroup. In another paper [8], Duflot shows that the depth of $H^*_G(X,k)$ is bounded below by the rank of a central elementary abelian p-group which acts trivially on X.

More recently, Lynn [18] shows that the *degree* of the group cohomology ring has a summation decomposition which fits nicely into the geometric picture described above. She ends her paper by

posing the question of whether or not her decomposition can be generalized to the full equivariant cohomology ring. One of the main results of this dissertation is the development of a generalized degree formula, that answers this question in the affirmative. To derive this formula, it was necessary to begin with a study of the commutative algebra of graded objects in general. The dissertation is thus divided into two parts.

The intent of part 1 is to present a theory of localization and multiplicity for the graded category. Serre [25] gives the first full treatment of the theory of intersection multiplicities for varieties, and we wish to adapt this theory to the graded category. Our motivation stems from the fact that equivariant cohomology rings generally do not come with a standard grading (i.e. generated by a finite number of elements of degree 1,) and moreover, the grading they are endowed with often contains important geometric information. For example, the generators of an equivariant cohomology ring may arise as Chern classes of a particular vector bundle associated to a representation of the group G ([4] pg. 49.) In short, it was critically important that we understood the role of the grading when computing multiplicities, so we adapted the theory to make sure that the grading was accounted for.

Chapters 1 and 2 can be viewed as a completely self-contained account of this transition. While many of these results are known, we are unaware of one source which makes the details explicit. All rings are assumed to be commutative. We begin as generally as possible, over the graded category $\mathfrak{grmod}(A)$, where A is a full $\mathbb{Z}$-graded Noetherian ring. We define graded analogues of the typical measures from commutative algebra, which we denote by a *. For example, consider the *graded field* $F \cong k[x, x^{-1}]$, where k is a field (Example 1.10). Then, the ordinary Krull dimension of F is equal to 1, since F is a PID which is not a field. However, since we are focused on the grading, we are interested in the graded analogue, $*\dim(F)$, which computes the longest length of graded prime ideals in F. In this case, $*\dim(F) = 0$ since the only graded ideals are F and 0. This example drives much of the analysis which follows: define *composition series (and hence *length) using graded fields, in analogue to the ungraded case. Carrying out this style of analysis eventually leads us to a definition of *multiplicity.

We present for the $\mathbb{Z}$-graded category a fundamental theorem of *dimension theory for *local rings (Theorem 2.10.) Also, Theorem 2.26 shows that the *Koszul multiplicity agrees with the *Samuel multiplicity, mimicking the ungraded theory.

When it makes sense to compare the graded and ungraded measures, we do. For example, in the $\mathbb{Z}$-graded category, the *Krull dimension can only differ from the ungraded Krull dimension by 1 (Corollary 1.35,) and when taking the graded localization at a minimal prime, length and *length agree (Theorem 1.26.) In the positively graded case, there is no difference between the graded and ungraded versions of length and Krull dimension. However, localizing a positively graded ring yields a ring which is no longer positively graded, and we wanted to pin down precisely what happens in this case. Using various different ways to localize in the graded category, we show how the difference under length and Krull dimension is controlled.

Chapter 3 restricts the focus to positively graded rings R, where $R_0 = k$ is a field. This is the category where equivariant cohomology rings live, and we show that in this case, the traditional theory of multiplicities agrees with the graded theory developed in chapters 1 and 2, if one simply "forgets" the grading. We conclude chapter 3 with a discussion of the *degree* of a graded module, which we connect to the geometry of equivariant cohomology rings in the main theorem of part 2 of the dissertation. The degree is defined: $\deg(M) = \lim_{t\to 1}(1-t)^{*dim(M)}P_M(t)$, where $P_M(t)$ is the Poincare-series of M.

We prove two main theorems about the degree measure. First, we relate the degree to *multiplicity (Theorem 3.8). Let $M \neq 0$ be in $\mathfrak{grmod}(R)$. Suppose $x_1, \ldots, x_{D(M)} \in R$ are of degrees $d_1, \ldots, d_{D(M)}$, and they generate the graded ideal $\mathcal{I}$ which has the property that $*\ell(M/\mathcal{I}M) \leq \infty$ (later we make formal the idea that the x's form a graded system of parameters, $\bar{x}$, with $\mathcal{I}$ a graded ideal of definition.) Let $*e$ denote the *Samuel multiplicity, and let $*\chi$ denothe the *Koszul multiplicity. Then,

$$\deg(M) = \frac{*e_R(M, \mathcal{I}, D(M))}{d_1 \cdots d_{D(M)}} = \frac{*\chi^R(\bar{x}, M)}{d_1 \cdots d_{D(M)}}.$$

The next result on degree (Theorem 3.6) gives an "algebraic" summation decomposition of the degree by taking the graded localization of M at minimal primes: Let $M \in \mathfrak{grmod}(R)$, and $\mathcal{D}(M)$

be the set of minimal prime ideals $\mathfrak{p} \in Spec(R)$, necessarily graded, such that $*\dim_R(R/\mathfrak{p}) = *\dim_R(M)$. Then,

$$\deg(M) = \sum_{\mathfrak{p}\in\mathcal{D}(M)} *\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

Ultimately, this result is used in part 2 of this thesis to prove our main theorem on the degree of an equivariant cohomology ring.

Part 2 of the dissertation begins with an introduction to equivariant cohomology, and some examples from the cohomology of groups. In chapter 5, we review some of Quillen's foundational results which set the stage for a commutative algebraic study of equivariant cohomology rings. In particular, $H^*_G(X)$ is a graded Noetherian ring. All cohomology coefficients are taken to be in a field k of characteristic p for p prime.

The multiplicative structure of $H^*_G(X)$ is given by the cup-product, which is not a commutative product for p odd. This led Quillen to consider just the even degree part of the ring when p is odd (denoted $H_G(X)$,) which is commutative. Quillen then shows that $H^*_G(X)$ is a finitely generated graded module over $H_G(X)$. This puts us in position to apply the graded commutative algebra from part 1. We present two of Quillen's main theorems 5.6, 5.8, which establish a relationship between the geometry of the G-action on X, the subgroup structure of G, and the prime spectrum of $H_G(X)$.

In section 3 of chapter 5 we present a localization result of Duflot which we make use of to prove our main theorem on degree in chapter 6. Our main theorem (Theorem 6.8) is:

$$\deg(H^*_G(X)) = \sum_{[A,c]\in\mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} \deg(H^*_{C_G(A,c)}(c)).$$

To derive this, we apply the algebraic decomposition of degree stated at the end of part 1, to $H^*_G(X) \in \mathfrak{grmod}(H_G(X))$. Our proof establishes that each decomposition is equal term-by-term, and the result can be thought of as a geometric translation of the algebraic degree sum formula when applied to equivariant cohomology.

TABLE OF CONTENTS

Part I

Graded Commutative Algebra

Chapter 1

Introductory Results in the Graded Context

In this chapter we set notation, note some standard facts about graded rings and present some of the foundational ideas for $\mathbb{Z}$-graded algebraic objects, which are the main focus of this dissertation. Recall that a ring A is a *$\mathbb{Z}$-graded ring* if there exist abelian subgroups A_n of A such that $A = \oplus_{n\in\mathbb{Z}} A_n$ and $A_n \cdot A_m \subseteq A_{n+m}$, for any integers n and m. Elements of the subgroup A_n are called homogeneous elements of degree n. Note that $0 \in A_n$ for every n. For every $a \in A$, a may be written uniquely as $a = \Sigma_{n\in\mathbb{Z}} a_n$, where $a_n \in A_n$ and $a_j = 0$ for $|j|$ sufficiently large. The a_n are called homogeneous components of a.

If M is an A-module, then M is said to be a *$\mathbb{Z}$-graded module* if there exist abelian subgroups M_n of M such that $M = \oplus_{n\in\mathbb{Z}} M_n$ and $A_n \cdot M_m \subseteq M_{n+m}$, for any integers n and m. If M and N are graded A-modules, and $\psi : M \to N$ is an A-module homomorphism, then ψ is a *graded homomorphism of degree* d if for every integer n, $\psi(M_n) \subseteq N_{n+d}$.

Throughout part 1 of the dissertation, we use the convention that A is a $\mathbb{Z}$-graded ring. Later, we use the letters R and S for positively graded rings.

If M is a graded A-module, then the set of homogeneous elements of degree j is denoted by M_j. The shifted A-module M is denoted $M(d)$ and defined by:

$$M(d)_j \doteq M_{d+j},$$

for every $j \in \mathbb{Z}$.

Definition 1.1. ***The Category*** $\mathfrak{grmod}(-)$

Suppose A is a graded ring. The category $\mathfrak{grmod}(A)$ has objects finitely generated graded A-modules. The morphisms of $\mathfrak{grmod}(A)$ are the A-module homomorphisms which are graded of degree zero (i.e. degree-preserving).

1.1 Basic Results in grmod(A)

Let A be a graded ring. Recall that an ideal $\mathcal{I}$ of A is a graded ideal if and only if it is generated by homogeneous elements; this is equivalent to the condition that for every element of $\mathcal{I}$, all of its homogeneous components are in $\mathcal{I}$. To check whether a graded ideal $\mathcal{I}$ is prime we need only check that $\mathcal{I}$ is a proper ideal and for all homogeneous elements x, $y \in A$ with $xy \in \mathcal{I}$, we have $x \in \mathcal{I}$ or $y \in \mathcal{I}$.

If M is a graded A-module,

$$Ann_A(M) \doteq \{a \in A \mid am = 0 \text{ for every } m \in M\}$$

is a graded ideal of A. Note that, by definition,

$$Ann_A(M) = Ann_A(M(d)),$$

for every $d \in \mathbb{Z}$.

For an ideal $\mathcal{I}$ in a graded ring, $\mathcal{I}^*$ is defined as the largest, graded ideal contained in $\mathcal{I}$; i.e. $\mathcal{I}^*$ is the ideal generated by all homogeneous elements of $\mathcal{I}$. It's easily verified that if if $\mathfrak{p}$ is a prime ideal in A, $\mathfrak{p}^*$ is also a prime ideal.

Recall the following definitions from commutative algebra: An *associated prime* for an A-module M (forgetting any gradings) is an element $\mathfrak{p} \in Spec(A)$ such that $\mathfrak{p}$ is the annihilator of an element $m \in M$. The set of associated primes is denoted by $Ass_A(M)$. Equivalently, $\mathfrak{p} \in Ass_A(M)$ if and only if there exists an A-module homomorphism between $A/\mathfrak{p}$ and a sub-module of M. Next, the *support* of an A-module M, is the set $Supp_A(M) \doteq \{\mathfrak{p} \in Spec(A) : M_{\mathfrak{p}} \neq 0\}$. For M finitely generated over A, $\mathfrak{p} \in Supp_A(M)$ if and only if $Ann_A(M) \subseteq \mathfrak{p}$.

A prime ideal of A that contains $Ann_A(M)$, and is minimal amongst all primes containing $Ann_A(M)$ is called a *minimal prime* for M. If $M = A/\mathcal{I}$ for an ideal $\mathcal{I}$ of A, then a minimal prime for M is called a minimal prime over $\mathcal{I}$.

Note that every prime in $Ass_A(M)$ is in $Supp_A(M)$. We collect some standard results about $Ass_A(M)$ for the graded category below.

Proposition 1.2. *Let A be a graded ring with M a graded A-module.*

i) If $\mathfrak{p} \in Ass_A(M)$, then $\mathfrak{p}$ is a graded ideal of A and is the annihilator of a homogeneous element in A.

ii) Therefore, if $\mathcal{I}$ is a graded ideal in A, all primes in $Ass_A(A/\mathcal{I})$ are graded.

ii) If $\mathfrak{p}$ is a minimal prime for M, then $\mathfrak{p} \in Ass_A(M)$; thus, all minimal primes for M are graded.

It's useful to note the following. Let $M \in \mathfrak{grmod}(A)$, $M \neq 0$, and $\mathfrak{p} \in Ass_A(M)$ with $\mathfrak{p} = ann_A(m)$ for a homogeneous element $m \in M$. Suppose $m \in M_d$. There is a graded injective homomorphism $\phi : A/\mathfrak{p}(-d) \hookrightarrow M$, defined $\overline{a} \mapsto a \cdot m$, yielding a graded isomorphism of $A/\mathfrak{p}(-d)$ and a graded A-submodule of M.

1.1.1 Noetherian graded rings

When we say that a graded ring A is Noetherian, or a graded A-module M is a Noetherian A-module, we mean that it is Noetherian in the usual sense, forgetting the grading. In line with our efforts to set up the right environment for a transition of Serre's theory of multiplicities to the graded category, we look at properties of the graded associated primes of a module. In lemma 1.5, we show that for any $M \in \mathfrak{grmod}(A)$ a filtration of M exists by *graded* submodules, such that each successive quotient of the filtration is *graded* isomorphic to $A/\mathfrak{p}_i(d_i)$, for integers d_i, and primes $\mathfrak{p}_i$. Analogous to the ungraded case, we use this fact later (corollary 2.8) to derive a sum decomposition for graded multiplicity.

One can show [5] that the following conditions on A are equivalent:

- A is Noetherian.
- Every graded ideal in A is generated by a finite set of homogeneous elements.

- A_0 is Noetherian and A is a finitely generated A_0-algebra by a set of homogeneous elements.

This allows us also to note that if M is a finitely generated graded A-module, and A is Noetherian, then M is Noetherian, so that

- Every A-submodule N of M is finitely generated over A, and if N is graded, it is generated over A by a finite set of homogeneous elements.

- For every j, M_j is a Noetherian A_0-module and so every A_0-submodule of M_j is finitely generated: If one has an ascending chain $X_1 \subseteq X_2 \subseteq \cdots$ of A_0-submodules of M_j, then letting AX_i be the (graded) A-submodule generated by X_i, we must have $AX_i = AX_{i+1}$ for all i greater than or equal to some fixed N. But $X_i = AX_i \cap M_j$ for every i, so $X_i = X_{i+1}$ for $i \geq N$.

Definition 1.3. *The graded support of M, $*Supp_A(M)$, is the set of all graded prime ideals in the support of M. If $\mathcal{I}$ is a graded ideal in A, the graded variety of $\mathcal{I}$, $*V(\mathcal{I})$, is the set of all graded primes in A containing $\mathcal{I}$. Recall that if $\mathcal{J}$ is any ideal in A, graded or not, $V(\mathcal{J})$ is the set of prime ideals in A containing $\mathcal{J}$.*

Lemma 1.4. *If A is a graded Noetherian ring and $M \in \mathfrak{grmod}(A)$,*

a. $Supp_A(M) = V(Ann_A(M)) \doteq V(M)$*, so that* $*Supp_A(M) = *V(Ann_A(M)) \doteq *V(M)$.

b. If $\mathcal{I}$ is a graded ideal in A, $*V(M/\mathcal{I}M) = *V(M) \cap *V(\mathcal{I}) = *V(Ann_A(M) + \mathcal{I})$.

c. If $\mathcal{I}$ and $\tilde{\mathcal{I}}$ are two graded ideals in A then $*V(\mathcal{I}) = *V(\tilde{\mathcal{I}})$ if and only if $\sqrt{\mathcal{I}} = \sqrt{\tilde{\mathcal{I}}}$.

Proof. The proof of $a.$ can be found in [25]; also [25] tells us that $V(M/\mathcal{I}M) = V(M) \cap V(\mathcal{I}) = V(Ann_A(M) + \mathcal{I})$ and so $b.$ follows from this. For c., the forward implication follows since all minimal primes over $\mathcal{I}$ are graded, thus occur as minimal elements both in $V(\mathcal{I})$ and $*V(\mathcal{I})$, and $\sqrt{\mathcal{I}}$ is the intersection of the (finite number of) minimal primes over $\mathcal{I}$. Note that if $\mathcal{I}$ is graded, so is $\sqrt{\mathcal{I}}$. □

The filtration described in the following lemma provides a useful way of organizing information about how the sub-module structure of M is related to the associated primes of M. Later, it is used to give a summation decomposition of graded multiplicity over the minimal primes (Corollary 2.8 pg. 38) and degree (Theorem3.6 pg.56), both of which are invariants that are central to this dissertation.

Lemma 1.5. *If A is a Noetherian graded ring and $M \in \mathfrak{grmod}(A)$ is nonzero, there exists a finite filtration $M^\bullet$ of M by graded submodules ($M^0 = M; M^n = 0$), integers d_i and graded primes $\mathfrak{p}_i \in Spec(A)$ with graded isomorphisms of graded A-modules, $M^i/M^{i+1} \cong A/\mathfrak{p}_i(-d_i)$. Furthermore, given a finite list of graded primes $(\mathfrak{p}_i \mid 1 \leq i \leq n)$ in $Spec(A)$ (not necessarily distinct), and a graded filtration $M^\bullet$ of M by graded submodules as above, we must have*

$$Ass_A(M) \subseteq \{\mathfrak{p}_i \mid 1 \leq i \leq n\} \subseteq *Supp_A(M)$$

and these three sets must have the same minimal elements, the set of which consists of the minimal primes of M. Finally, if $\mathfrak{p}$ is a minimal prime for M, forgetting all gradings and using the fact that the ordinary localization $M_\mathfrak{p}$ is a finitely generated Artinian $A_\mathfrak{p}$-module, the number of times that $A/\mathfrak{p}$, possibly shifted, occurs as a graded A-module isomorphic to a subquotient of $M^\bullet$ is always equal to the length of $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module and is thus independent of the choice of the graded filtration $M^\bullet$.

Proof. We remind the reader of the proof of the first statement, adapted to the graded case: Using the Noetherian hypothesis, since $M \neq 0$, $Ass_A(M) \neq \emptyset$, so we may pick an element $\mathfrak{p}_1 \in Ass_A(M)$. Then $\mathfrak{p}_1$ is graded and there exists a homogeneous element $m_1 \in M$ such that $\mathfrak{p}_1 = ann_A(m_1)$. Suppose $\deg(m_1) = d_1$, then $A/\mathfrak{p}_1(-d_1)$ is graded isomorphic to a graded A-submodule of M which we call M^1.

If $M^1 = M$, we are done. If not, we take the A-module M/M^1, notice that it is nonzero, and produce an associated prime $\mathfrak{p}_2 \in Ass_A(M/M^1)$. Since M/M^1 is a graded A-module $\mathfrak{p}_2$ is also graded. Suppose $\mathfrak{p}_2 = Ann_A(\overline{m}_2)$ where $m_2 \notin M^1$ is a homogeneous element in M and

$\deg(m_2) = d_2$; $\overline{m}_2$ is the coset of m_2 in M/M^1. Thus there is a graded submodule $M^1 \subseteq M^2$ such that M^2/M^1 is graded isomorphic to $A/\mathfrak{p}_2(-d_2)$. Continue this process until the chain of graded submodules (M^i) necessarily ends by the Noetherian hypothesis.

For the last two statements, we refer to [25]. □

1.1.2 *Simple modules, *maximal ideals and *composition series

As usual, let A be a $\mathbb{Z}$-graded ring. A motivating example of the differences between the graded and ungraded categories is example 1.10 of a graded field. In this section we use graded fields to define the graded analogue of length, and show (lemma 1.15) for positively graded rings, length and *length are equal.

Definition 1.6. *A *simple A-module is a nonzero graded A-module with no nonzero proper graded submodules. A *composition series for a graded module $M \in \mathfrak{grmod}(A)$ is a chain of graded A-submodules of M, $0 = M^0 \subset \cdots \subset M^n = M$ such that each successive quotient M^i/M^{i-1} is isomorphic as a graded A-module to a *simple module. The length of the *composition series $0 = M^0 \subset \cdots \subset M^n = M$ is defined to be n.*

The only simple A-modules in the ungraded case are A-modules of the form $A/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of A (recall all rings are commutative). Thus we are led to define graded fields.

Theorem 1.7. *[11] Let F be a graded ring. The following are equivalent:*

1. *Every nonzero homogeneous element in F is invertible.*

2. *F_0 is a field and either $F = F_0$, or there exists a $d > 0$ and an $x \in F_d$ such that $F \cong F_0[x, x^{-1}]$ as a graded ring. In fact, in this last case, $d > 0$ is the smallest positive degree with $F_d \neq 0$.*

3. *The only graded ideals in F are F and 0.*

*A ring satisfying any of these three equivalent conditions is called a **graded field**.*

Lemma 1.8. *Suppose that M is a finitely generated graded module over a graded field $F = F_0[t, t^{-1}]$, where t has positive degree d and F_0 is a field. Then*

a) M is a free graded F-module, of finite rank, on a set of homogeneous generators.

b) M_0 is a finite-dimensional vector space over F_0 of F_0-dimension less than or equal to the rank of M over F_0.

Proof. Assume $M \neq 0$. Say M is finitely generated over F by homogeneous elements $e_1, \ldots, e_r$, where $r \geq 1$ is the minimal number for a homogeneous generating set for M as an F-module.Then M is free on the $e_j s$: certainly this set spans M over F. Suppose that there is a relation $\sum_j \alpha_j e_j = 0$, with $\alpha_j \in F$. We may assume that all the $\alpha_j s$ are homogeneous. If $\alpha_r \neq 0$, then it is invertible in F, so $\sum_{j=1}^{r-1} \alpha_r^{-1} \alpha_j e_j + e_r = 0$, implying that r is not minimal. Therefore $\alpha_r = 0$; and continuing the process, $\alpha_j = 0$ for every j.

Set $d_j = \deg e_j$. Now, note that $X \doteq \{t^{-d_j/d} e_j \mid 1 \leq j \leq r \text{ and } d \text{ divides } d_j\}$ is a basis for M_0 over F_0; of course, if d does not divide any d_j, then $M_0 = 0$. To see this, note that X is linearly independent over F_0, since the $e_j s$ are linearly independent over F. If $x \in M_0$, then $x = \sum_j \alpha_j e_j$, where α_j is a homogeneous element of F and $\deg \alpha_j + d_j = 0, \forall j$. Now, if $\alpha_j \neq 0$, d divides its degree, by definition of F. Thus, d divides d_j for every j such that $\alpha_j \neq 0$. If d divides d_j, then $\alpha_j = \beta_j t^{-d_j/d}$, where $\beta_j \in F_0$. Thus x is in the F_0-span of X. □

Definition 1.9. *If A is a graded ring, a graded ideal $\mathcal{N}$ is *maximal if and only if $\mathcal{N} \neq A$ and $\mathcal{N}$ is a maximal element in the set of all proper graded ideals of A.*

Example 1.10. *If F is a graded field with a nonzero positive degree element, then F is *simple as a module over itself, but it is not simple as such. To see this write $F = F_0[t, t^{-1}]$, with $\deg(t) = d > 0$, and F_0 a field. So F is certainly *simple, but if $\mathcal{J}$ is the ungraded ideal generated by $t + 1$, $\mathcal{J}$ is a nonzero proper F-submodule of F, so F is not simple. Furthermore, F has a unique *maximal ideal, the zero ideal, but has as least as many ungraded nonzero maximal ideals as the nonzero elements of F. While F has a *composition series, it has no composition series.*

For a graded A-module M, we use the notation $\ell_A(M)$ to denote the length of M as an A-module, forgetting all gradings.

Similarly to the ungraded case, M is a *simple A-module if and only if there exists a *maximal ideal $\mathcal{N}$ of A, an integer d and a graded A-module isomorphism $M \cong (A/\mathcal{N})(d)$: if M is *simple, let x be any nonzero homogeneous element of M, say $\deg(x) = -d$. Then, the submodule of M generated by x is nonzero and graded, so must be all of M. The homomorphism $A(d) \to M$ of graded A-modules defined by $a \mapsto ax$ is thus surjective; its kernel is a graded ideal in $A(d)$ of the form $\mathcal{N}(d)$ for some graded ideal $\mathcal{N}$ of A; since M is *simple, $\mathcal{N}$ must be *maximal. The converse is left to the reader.

Other facts parallel to the ungraded case include: 1) for every proper graded ideal $\mathcal{I}$ in A, there exists a *maximal ideal $\mathcal{N}$ containing $\mathcal{I}$; 2) if $\mathcal{N}$ is a proper graded ideal of A, then $\mathcal{N}$ is *maximal if and only if $A/\mathcal{N}$ is a graded field. Thus, every *maximal ideal in A is a graded prime ideal. Furthermore, if $\mathcal{N}$ is *maximal in A, then $\mathcal{N}_0$ is a maximal ideal in A_0.

The fundamental theorem about *composition series mirrors that in the ungraded case. The proof of the following is nearly identical to the ungraded case ([11],Theorem 2.13), with only minor adjustments made to account for the grading, and we leave this effort to the reader.

Theorem 1.11. *Suppose for $M \in \mathfrak{grmod}(A)$ that a *composition series of length n for M exists. Then, every chain of graded submodules of M has length $\leq n$, and can be refined to a *composition series of length n. Every *composition series for M has length n.*

Definition 1.12. *If M has a *composition series as an A-module, the *length of $M \in \mathfrak{grmod}(A)$ is defined to be the length of a *composition series for M. We use the notation $*\ell_A(M)$ for this number.*

Some properties of $*\ell_A$ are:

- If $0 \to M \to N \to P \to 0$ is an exact sequence in $\mathfrak{grmod}(A)$, then N has a *composition series if and only if both M and P do; and in this case, $*\ell_A(N) = *\ell_A(M) + *\ell_A(P)$.

- If $d \in \mathbb{Z}$, then $*\ell_A(M(d)) = *\ell_A(M)$.

Definition 1.13. *$M \in \mathfrak{grmod}(A)$ is said to be a *Artinian module if M satisfies DCC on all chains of graded A-submodules of M.*

Unlike the Noetherian case, an A-module M can be *Artinian without being Artinian: an example is given by $A = M$, where A is a graded field with a nonzero positive degree element.

Similarly to the ungraded case, we have

Lemma 1.14. *Suppose that A is a graded Noetherian ring and $M \in \mathfrak{grmod}(A)$. Then the following are equivalent:*

*a) M is *Artinian.*

b) $*\ell_A(M) < \infty$.

*c) $*V(M)$ consists of a finite number of *maximal ideals.*

Proof. The proof of the equivalence of a) and b) in the ungraded case, as in [3], adapts in a straightforward way to the graded case. Note that the proof of "b) implies a)" does not require A to be Noetherian.

To see how b) implies c), assume that M has a *composition series

$$0 = M^0 \subset M^1 \subset \cdots \subset M^{n-1} \subset M^n = M;$$

the *simplicity of the subquotients means that there are *maximal graded ideals $\mathfrak{m}_i$ of A and integers d_i such that $M^i/M^{i-1} \cong (A/\mathfrak{m}_i)(d_i)$ as graded A-modules. Thus, $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq Ann_A(M)$. If $\mathfrak{p}$ is a prime minimal over $Ann_A(M)$, then we have seen that $\mathfrak{p}$ is graded. Since $\mathfrak{m}_1 \cdots \mathfrak{m}_n \subseteq \mathfrak{p}$, we must have $\mathfrak{m}_i \subseteq \mathfrak{p}$ for at least one i. But $\mathfrak{m}_i$ is *maximal, so $\mathfrak{m}_i = \mathfrak{p}$. Therefore $*V(M) \subseteq \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

For c) implies b), since $\sqrt{Ann_A(M)}$ is the intersection of the primes minimal over $Ann_A(M)$, and there are a finite number of these, all graded, the hypothesis implies that this finite list of primes consists entirely of *maximal ideals; say these ideals are $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Thus, there is an N such that $(\mathfrak{m}_1 \cdots \mathfrak{m}_n)^N \subseteq Ann_A(M)$ and there is a sequence $\tilde{\mathfrak{m}}_1, \ldots, \tilde{\mathfrak{m}}_{nN}$ of *maximal ideals in

A, not necessarily distinct, whose product is contained in $Ann_A(M)$. Analogously to the ungraded case, one can then construct a *composition series for M. $\square$

A graded ring S is positively graded if and only if $S_i = 0$ for $i < 0$. The graded ideal S_+ of S is defined as $\oplus_{i>0} S_i$. Note that if $M \in \mathfrak{grmod}(S)$, since S is positively graded, there exists an integer e such that $M_i = 0$ for all $i < e$. Also, for a proper, graded ideal $\mathfrak{m}$ of S, the following are equivalent:

- $\mathfrak{m}$ is *maximal in S.
- $\mathfrak{m} = \mathfrak{m}_0 \oplus S_+$, and $\mathfrak{m}_0$ (the degree zero elements of $\mathfrak{m}$) is a maximal ideal in S_0.
- $S/\mathfrak{m}$ is a graded field, concentrated in degree zero.
- $\mathfrak{m}$ is a maximal ideal in S.

For positively graded rings, there is no difference between *length and length:

Lemma 1.15. *Suppose that S is a positively graded Noetherian ring, and $M \in \mathfrak{grmod}(S)$ is such that $*\ell_S(M) < \infty$. Then, $*\ell_S(M) = \ell_S(M)$.*

Proof. Since $*V(M)$ consists of a finite number of *maximal ideals, there is a sequence of graded S-modules

$$0 = M^0 \subset M^1 \subset \cdots \subset M^{n-1} \subset M^n = M,$$

*maximal graded ideals $\mathfrak{m}_i$ of S and integers d_i such that $M^i/M^{i-1} \cong (S/\mathfrak{m}_i)(d_i)$ as graded S-modules. By the remark above, $S/\mathfrak{m}_i$ is concentrated in degree 0 and each $\mathfrak{m}_i$ is a maximal ideal in S. So, forgetting gradings everywhere, the given *composition series is a composition series. $\square$

Even in the cases where *length and length coincide, we'll usually just talk about *length, emphasizing constructions using graded modules only. For example,

Lemma 1.16. *Suppose S is a positively graded ring and $X \in \mathfrak{grmod}(S)$.*

a) If $*\ell_S(X) < \infty$, there exists an integer J such that if $j > J$, then $X_j = 0$.

b) If S_i is finitely generated as an S_0-module for every i, then X_j is a finitely generated S_0-module, for every j.

c) Suppose S_0 is Artinian, S_i is finitely generated as an S_0-module for every i, and there exists an integer J such that if $j > J$, then $X_j = 0$. Then, $\ell_{S_0}(X_j) < \infty$ for every j, and $*\ell_S(X) = \ell_{S_0}(X) < \infty$, where $\ell_{S_0}(X) \doteq \sum_j \ell_{S_0}(X_j)$ is the (total) S_0-length of X.

Proof. For every $t \in \mathbb{Z}$ define $X_{\geq t} \doteq \oplus_{s \geq t} X_s$. Since S is positively graded, $X_{\geq t}$ is a graded S-submodule of X. Since X is finitely generated over S, and S is positively graded, there exists a $t_0 \in \mathbb{Z}$ such that $X_{\geq t_0} = X$. So we have a descending chain of graded S-submodules of X

$$\cdots \subseteq X_{\geq t_0+k} \subseteq X_{\geq t_0+k-1} \subseteq \cdots \subseteq X_{\geq t_0+1} \subseteq X_{\geq t_0} = X.(*)$$

For a), if $*\ell_S(X) < \infty$, X is *Artinian, so this chain stabilizes. By definition, this means that there exists an $J \geq t_0$ such that $X_j = 0$ for $j > J$.

For b), let t_0 be defined as in the first paragraph above; assume that $X_{t_0} \neq 0$. Then, one can prove, by induction on j, that each X_j is finitely generated over S_0 as follows. If $j = t_0$, then since X is finitely generated as an S-module, say by $x_1, \ldots x_N$, if $\beta_{t_0} = \{x_i \mid \deg(x_i) = t_0\}$, X_{t_0} must be generated by β_{t_0} as an S_0-module. Assume that $j > t_0$ and X_u is finitely generated over S_0 for $u < j$. Then, $X_{<j} = \oplus_{u=t_0}^{j-1} X_u = \oplus_{u<j} X_j$, is finitely generated over S_0. Choose a finite set $\beta_{<j}$ of homogeneous elements that generate $X_{<j}$ over S_0. Choose finite generating sets α_u for each S_u over S_0. Let $\beta_j = \{x_i \mid \deg(x_i) = j\}$. The claim is that the finite set $B_j \doteq \{ae \mid a \in \alpha_u, e \in \beta_{<j} \text{ and } u + \deg(e) = j\} \cup \beta_j$ spans X_j over S_0: if $x \in X_j$, then $x = \sum_i a_i x_i$, with a_i homogeneous in S for every i, and if $a_i x_i \neq 0$, $\deg(a_i) + \deg(x_i) = j$; from now on we'll just talk about the indices i such that $a_i x_i \neq 0$. If $\deg(a_i) = 0$, then $x_i \in \beta_j \subseteq B_j$ and $a_i \in S_0$. If $\deg(a_i) > 0$, then $\deg(x_i)$ is strictly less than j so that x_i is in the S_0-span of $\beta_{<j}$; certainly, a_i is in the S_0-span of $\alpha_{\deg(a_i)}$, so $a_i x_i$ is in the S_0-span of B_j.

For c), given J such that $X_J = 0$ for $j > J$, and choosing $t_0 \leq J$ such that $X_j = 0$ for $j < t_0$, the chain (*) terminates at the left in 0, and has successive quotients isomorphic to a graded

S-module X_j (concentrated in degree j), where the S-module structure is determined by $rx = 0$ if $r \in S_+$. Since b) says that each X_j is finitely generated over S_0, and S_0 is Artinian, the chain (*) may be refined to a *composition series of X, of length equal to $\sum_j \ell_{S_0}(X_j)$.

□

Lemma 1.17. *Let A be a graded Noetherian ring which is a finitely generated graded algebra over a field $k \subseteq A_0$, $M \in \mathfrak{grmod}(A)$, V a graded finite dimensional vector space over k, and say that* $\operatorname{vdim}_k(V) = d$. *If $a \in A$, $m \otimes v \in M \otimes_k V$, then give $M \otimes_k V$ an A-module structure by $a \cdot (m \otimes v) \doteq (a \cdot m) \otimes v$. Then*

$$*\ell_A(M \otimes_k V) = *\ell_A(M) \cdot d.$$

Proof. Since V is finite dimensional, there exists an n such that $j > n$ implies $V_j = 0$. Define a graded filtration of $M \otimes_k V$ by graded A-modules: $\mathcal{F}^i \doteq M \otimes_k (V_0 \oplus \cdots \oplus V_{n-i})$ for $0 \leq i \leq n$, and $\mathcal{F}^{n+1} \doteq 0$. Consider that $\mathcal{F}^i/\mathcal{F}^{i+1} \cong M \otimes_k V_{n-i}$, and the additive property of length allows $*\ell_A(M \otimes_k V) = \sum_{i=0}^n *\ell_A(\mathcal{F}^i/\mathcal{F}^{i+1}) = \sum_{i=0}^n *\ell_A(M \otimes_k V_{n-i})$.

By hypothesis, each graded component V_j of V is a finite dimensional graded vector space concentrated in degree j. Thus, there is a graded isomorphism for each j, $V_j \cong k^{f(j)}(-j)$ where $f(j)$ is a function giving the vector space dimension of V_j. Since $M \otimes_k V_j \cong M \otimes_k k^{f(j)} \cong \oplus_1^{f(j)} M(-j)$, we have that $*\ell_A(M \otimes_A V_j) = *\ell_A(M) \cdot f(j)$. By hypothesis, $\sum_{j=0}^n f(j) = d$, the total vector space dimension of V, and finally $*\ell_A(M \otimes_k V) = \sum_{i=0}^n *\ell_A(M \otimes_k V_{n-i}) = \sum_{i=0}^n *\ell_A(M) \cdot f(n-i) = *\ell_A(M) \sum_{i=0}^n f(n-i) = *\ell_A(M) \cdot d$. □

1.2 Graded Localization

Localizing in the graded category can be done in a few ways. We may localize as usual, forgetting the graded structures, we may localize at sets consisting of homogeneous elements, or as in Grothendieck [13], consider the degree zero part of this last localized module. In this section we make the relevant definitions, and compare the different methods. Localization is a vital tool which we employ when studying the commutative algebra of equivariant cohomology rings in part 2 of this dissertation.

Definition 1.18. *Let T be a multiplicatively closed subset (MCS) consisting entirely of homogeneous elements of A. We'll call this a "GMCS". Since T is an MCS we may construct the localization $T^{-1}M$ as usual. By definition, $T^{-1}M$ is graded by: $(T^{-1}M)_i \doteq \{\frac{m}{t} \in T^{-1}M \mid m \text{ is homogeneous and } \deg m - \deg t = i\}$.With this grading, $T^{-1}M$ becomes a graded $T^{-1}A$-module. In the case where $\mathfrak{p} \in Spec(A)$, and T is the set of homogeneous elements of $A - \mathfrak{p}$, we use the notation $M_{[\mathfrak{p}]}$ to denote the localization $T^{-1}M$, graded as above.*

For a GMCS T, we'll assume from now on that $1 \in T$ and $0 \notin T$.

The following list of lemmas collect some facts about graded localizations.

Lemma 1.19. *Let $\mathfrak{p} \in Spec(A)$. The set of homogeneous elements in $A - \mathfrak{p}$ is equal to the set of homogeneous elements in $A - \mathfrak{p}^*$. Therefore, $M_{[\mathfrak{p}]} = M_{[\mathfrak{p}^*]}$*

Lemma 1.20. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A, with $\mathfrak{q}$ graded. Then, $(A/\mathfrak{q})_{[\mathfrak{p}]} \neq 0$ if and only if $\mathfrak{q} \subseteq \mathfrak{p}^*$. If $\mathfrak{p}$ is a minimal prime of A, then $(A/\mathfrak{q})_{[\mathfrak{p}]} \neq 0$ if and only if $\mathfrak{q} = \mathfrak{p}$.*

Lemma 1.21. *If $M \in \mathfrak{grmod}(A)$, and T is a GMCS in A, then*

a) $T^{-1}M \in \mathfrak{grmod}(T^{-1}A)$.

b) If A is a Noetherian ring then $T^{-1}A$ is a Noetherian ring and $T^{-1}M \in \mathfrak{grmod}(T^{-1}A)$.

c) There is a one-one, inclusion-preserving correspondence between the prime ideals in A that are disjoint from T, and the prime ideals in $T^{-1}A$ given by $\mathfrak{p} \mapsto T^{-1}\mathfrak{p}$; moreover this correspondence restricts to a one-one correspondence between the graded prime ideals in A disjoint from T and the graded prime ideals in $T^{-1}A$, and further restricts to a one-one correspondence between the ideals (all graded) in $Ass_A(M)$ that are disjoint from T, and the ideals (also all graded) in $Ass_{T^{-1}A}T^{-1}M$.

Lemma 1.22. *Let $M \in \mathfrak{grmod}(A)$, T a GMCS in A, and let d be any integer. Then there is a graded isomorphism of graded $T^{-1}A$-modules $T^{-1}(M(d)) \cong (T^{-1}M)(d)$.*

Of course, the degree zero part of $T^{-1}M$ is of interest, so we start with setting notation: Let T be a GMCS in A. As usual, for every j, $(T^{-1}M)_j$ denotes the degree j part of the graded localization. If $\mathfrak{p} \in Spec(A)$, then we denote the degree 0 part of $M_{[\mathfrak{p}]}$ by $M_{(\mathfrak{p})}$. Then, $M_{(\mathfrak{p})}$ is an $A_{(\mathfrak{p})}$-module.

Lemma 1.23. *Suppose that $M \in \mathfrak{grmod}(A)$.*

a) $\mathfrak{p} \in Supp_A(M)$ *if and only if* $M_{[\mathfrak{p}^*]} \neq 0$ *if and only if* $\mathfrak{p}^* \in *V(M)$. *Therefore,*

$$*V(M) = *Supp_A(M) = \{\mathfrak{q} \in *V(A) \mid M_{[\mathfrak{q}]} \neq 0\}.$$

b) If $0 \to M \to N \to P \to 0$ *is a short exact sequence in* $\mathfrak{grmod}A$*, then* $*V(N) = *V(M) \cup *V(P)$.

Proof. Since $V(N) = V(M) \cup V(P)$, b) follows.

For a), it's straightforward to see that the ungraded object $M_{\mathfrak{p}} \neq 0$ implies that $M_{[\mathfrak{p}^*]} \neq 0$. If $M_{[\mathfrak{p}^*]} \neq 0$, and $Ann_A(M)$ is not contained in $\mathfrak{p}^*$, then since both are graded ideals, there exists a homogeneous element $r \in Ann_A(M)$ such that $r \notin \mathfrak{p}^*$. But then, $m/t = 0/r = 0$ for every $m \in M$ and homogeneous $t \notin \mathfrak{p}^*$. Finally, suppose that $Ann_A(M) \subseteq \mathfrak{p}^*$, yet $M_{\mathfrak{p}} = 0$. If $x_1, \ldots, x_j$ are homogeneous elements of M generating M as an A-module, since $x_i/1 = 0$ for every i, there exist $s_i \notin \mathfrak{p}$ such that $s_i x_i = 0$ for each i. We may assume that each s_i is homogeneous, since x_i is. Since $s_i \notin \mathfrak{p}$, $s_i \notin \mathfrak{p}^*$, so that $s = s_1 s_2 \cdots s_j \notin \mathfrak{p}^*$ and is homogeneous. Furthermore, $sm = 0$ for every $m \in M$, so $s \in \mathfrak{p}^*$, a contradiction. $\square$

Lemma 1.24. *For $\mathfrak{p}$ a graded prime in A, T a GMCS, $(T^{-1}\mathfrak{p})_0 = T^{-1}\mathfrak{p} \cap (T^{-1}A)_0$, and if $\mathfrak{p} \cap T = \emptyset$ then $(T^{-1}\mathfrak{p})_0$ is a prime ideal in $(T^{-1}A)_0$.*

Example 1.25. *If A is a graded ring, $\mathfrak{p}$ is a graded prime ideal in A, and T is the GMCS consisting of the homogeneous elements of $A - \mathfrak{p}$, then $T^{-1}\mathfrak{p} \doteq \mathfrak{p}_{[\mathfrak{p}]}$ is a *maximal ideal in $T^{-1}A \doteq A_{[\mathfrak{p}]}$ and $\mathfrak{p}_{(\mathfrak{p})}$ is a maximal ideal in $A_{(\mathfrak{p})}$.*

Now, if M is a graded A-module and $\mathfrak{p}$ is a graded prime ideal, we know that the standard localization $M_{\mathfrak{p}}$ isn't usually graded as we allow inhomogeneous elements of A not in $\mathfrak{p}$ to be inverted. If $\mathfrak{p}$ is a minimal prime ideal, it must be graded, as we have seen, and from ungraded commutative algebra, we know that $M_{\mathfrak{p}}$ has finite length as an $A_{\mathfrak{p}}$-module. But we can also consider the graded localization $M_{[\mathfrak{p}]}$ and the comparison between length and *length:

Theorem 1.26. *Suppose that A is a Noetherian graded ring. Let $M \in \mathfrak{grmod}(A)$, and $\mathfrak{p}$ be a prime minimal over the graded ideal $Ann_A(M)$. Then, a *composition series exists for the graded $A_{[\mathfrak{p}]}$-module $M_{[\mathfrak{p}]}$. Moreover,*

$$*\ell_{A_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. We will produce a *composition series for $M_{[\mathfrak{p}]}$, as an $A_{[\mathfrak{p}]}$-module and calculate its length.

Construct a graded filtration $M^\bullet$ as in Lemma 1.5, and then localize this filtration using the graded localization. We now have a filtration of $M_{[\mathfrak{p}]}$ by graded $A_{[\mathfrak{p}]}$-submodules which looks like $0 = (M^{r+1})_{[\mathfrak{p}]} \subseteq (M^r)_{[\mathfrak{p}]} \subseteq \cdots \subseteq (M)_{[\mathfrak{p}]}$. By exactness of localization and the condition on successive quotients of $M^\bullet$ we have that $(M^i/M^{i+1})_{[\mathfrak{p}]} \cong (A/\mathfrak{p}_i(-d_i))_{[\mathfrak{p}]}$ is a graded isomorphism of $A_{[\mathfrak{p}]}$-modules, for appropriate integers d_i, where the graded primes $\mathfrak{p}_i$ are chosen as in 1.5.

There is a graded isomorphism $((A/\mathfrak{p}_i)(-d_i))_{[\mathfrak{p}]} \cong (A/\mathfrak{p}_i)_{[\mathfrak{p}]}(-d_i)$, and $(A/\mathfrak{p}_i)_{[\mathfrak{p}]}(-d_i) \neq 0$ if and only if $\mathfrak{p} = \mathfrak{p}_i$ (by minimality of $\mathfrak{p}$).

In the case that $\mathfrak{p} \neq \mathfrak{p}_i$, $(A/\mathfrak{p}_i)_{[\mathfrak{p}]} = 0$ and we have $(M^{i+1})_{[\mathfrak{p}]} = (M^i)_{[\mathfrak{p}]}$. Now throw away all such submodules $(M^i)_{[\mathfrak{p}]}$ which are equal to the submodule $(M^{i+1})_{[\mathfrak{p}]}$ to get a reduced filtration $((\overline{M}^j)_{[\mathfrak{p}]})$ of $M_{[\mathfrak{p}]}$, where for each j, $(\overline{M}^{j+1})_{[\mathfrak{p}]} \subset (\overline{M}^j)_{[\mathfrak{p}]}$ is a strict inclusion, $(\overline{M}^{s+1})_{[\mathfrak{p}]} = 0$ for some $s \leq r$, and the zeroth term is equal to $M_{[\mathfrak{p}]}$. The claim is that this reduced filtration forms a *composition series for $M_{[\mathfrak{p}]}$ of *length equal to the number of times that $A/\mathfrak{p}$, shifted, appeared as a successive quotient in the original filtration $M^\bullet$.

For each j the successive quotient $(\overline{M}^j)_{[\mathfrak{p}]}/(\overline{M}^{j+1})_{[\mathfrak{p}]}$ is graded isomorphic to $(A/\mathfrak{p})_{[\mathfrak{p}]}(-d_j)$, as an $A_{[\mathfrak{p}]}$-module. But $(A/\mathfrak{p})_{[\mathfrak{p}]}$ is a graded field, since $A_{[\mathfrak{p}]}$ has a unique graded prime ideal $\mathfrak{p}_{[\mathfrak{p}]}$; thus, $(\overline{M}^j)_{[\mathfrak{p}]}/(\overline{M}^{j+1})_{[\mathfrak{p}]} \cong (A/\mathfrak{p})_{[\mathfrak{p}]}(-d_j)$ is a *simple $A_{[\mathfrak{p}]}$-module for each j.

Going back to the original filtration $M^\bullet$ and forgetting the grading everywhere, recall that the number of times that $A/\mathfrak{p}$ appears as a successive quotient in any finite filtration of M which has successive quotients isomorphic to $A/\mathfrak{q}$ for some prime $\mathfrak{q}$, graded or not, is always the same, and is equal to $\ell_{A_\mathfrak{p}}(M_\mathfrak{p})$.

□

1.2.1 *Local rings

Definition 1.27. *If A is a graded ring, then A is *local if and only if there is one and only one *maximal ideal of A.*

Some examples of *local rings are immediate. For example, a graded field is always *local, with unique *maximal ideal 0. This shows that generally, a *maximal ideal of a graded ring A may not be a maximal ideal of A. If $\mathfrak{p}$ is a graded prime in A, then $A_{[\mathfrak{p}]}$ is a *local ring with unique *maximal ideal $\mathfrak{p}_{[\mathfrak{p}]}$.

Also, as one might expect, if A is a *local graded ring, with unique *maximal ideal $\mathcal{N}$, then

- For every proper ideal $\mathcal{I}$ (graded or not) of A, $\mathcal{I}^* \subseteq \mathcal{N}$.
- Every homogeneous element of $A - \mathcal{N}$ is invertible: i.e., for every $x \in A - \mathcal{N}$ with $\deg x = d$, there exists a $y \in A - \mathcal{N}$ of degree $-d$ such that $xy = 1 \in A_0$.
- $A/\mathcal{N}$ is a graded field; also, for every $y \in \mathcal{N}_j$ and every $x \in \mathcal{A}_{-j}$, $1 - xy \in A_0$ is a unit in A_0.

Lemma 1.28. *(Graded Nakayama's lemma) Suppose $(A, \mathcal{N})$ is a *local ring and M is a finitely generated graded A-module with N a graded A-submodule of M. If $\mathfrak{q}$ is a proper graded ideal in M, then $N + \mathfrak{q}M = M$ implies that $M = N$.*

Proof. (Slight variation of proof in [3].) We may assume $N = 0$ by passing to M/N. Say $M \neq 0$; choose a homogeneous generating set $x_1, \ldots, x_r$ for M over A with a minimal number $r \geq 1$ of nonzero homogeneous elements. Suppose that $\mathfrak{q}M = M$; then there are homogeneous elements

$\alpha_j \in \mathfrak{q} \subseteq \mathcal{N}$ such that $x_r = \alpha_1 x_1 + \cdots + \alpha_r x_r$; we must have $\deg \alpha_j + \deg x_j = \deg x_r$ for every j such that $\alpha_j x_j \neq 0$. By minimality, $\alpha_r x_r \neq 0$ and so $\deg \alpha_r = 0$. Using the remarks above, $1 - \alpha_r$ is an invertible element of A_0. Thus, we may write x_r as an A-linear combination of $x_1, \ldots, x_{r-1}$, contradicting the minimality of r. □

Proposition 1.29. *If A is *local and Noetherian with unique *maximal ideal $\mathcal{N}$, and M is a nonzero finitely generated graded A-module with $\mathcal{N}$ a minimal prime over $Ann_A(M)$, then M is a *Artinian A-module, and for each $j \in \mathbb{Z}$, M_j is an Artinian A_0-module and $\ell_{A_0} M_j \leq *\ell_A M$. If, in addition, there is a homogeneous element of degree 1 (or , equivalently, -1) in $A - \mathcal{N}$, $\ell_{A_0} M_j = *\ell_A M$ for every j.*

Proof. M is *Artinian, since $*V(M) = \{\mathcal{N}\}$ by hypothesis. In fact, in this case, M has a *composition series with the property that each successive quotient is annihilated by $\mathcal{N}$ and is also free of rank one over the graded field $A/\mathcal{N}$. Taking the degree j part of each module in this *composition series, we get a chain of A_0-submodules of M_j and the dimension of each successive quotient over the field $K \doteq (A/\mathcal{N})_0 \doteq A_0/\mathcal{N}_0$ is either zero or 1. Thus, since $\mathcal{N}_0$ also annihilates each successive quotient in this "degree j" filtration, we see that we can make appropriate deletions in the "degree j" part of the *composition series for M to yield a composition series for M_j over A_0 of length less than or equal to $*\ell_A(M)$.

For the last statement, supposing that there is a homogeneous element of degree 1 in $A - \mathcal{N}$, then there are nonzero elements of every degree in the graded A-module $(A/\mathcal{N})(d)$, for every $d \in \mathbb{Z}$; to see this, note that $A/\mathcal{N}$ is a graded field, equal to $K[T, T^{-1}]$, where T has least positive degree in $A/\mathcal{N}$, namely degree 1. So, each successive quotient in the *composition series for M is nonzero in every degree. After taking the "degree j" part of this *composition series, each quotient must be of rank 1 over K. Thus the equality holds.

□

Corollary 1.30. *Suppose that A is a Noetherian graded ring. Let $M \in \mathfrak{grmod}(A)$, and $\mathfrak{p}$ be a prime minimal over $Ann_A(M)$, necessarily graded. Then $A_{[\mathfrak{p}]}$ is a *local ring, $A_{(\mathfrak{p})}$ is a local ring,*

*$M_{[\mathfrak{p}]}$ is an *Artinian $A_{[\mathfrak{p}]}$-module and $M_{(\mathfrak{p})}$ is an Artinian $A_{(\mathfrak{p})}$-module. Also,*

$$\ell_{A_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) \leq *\ell_{A_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

In addition, if there is a homogeneous element of degree 1 in $A - \mathfrak{p}$, then

$$\ell_{A_{(\mathfrak{p})}}(M_{(\mathfrak{p})}) = *\ell_{A_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

1.3 Krull Dimension in grmod(A)

The height of a prime ideal $\mathfrak{p}$ of A, graded or not, is the longest length n (which always exists, using the Noetherian hypothesis) of a chain of primes $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$; this height is denoted $\mathrm{ht}(\mathfrak{p})$. Similarly, we define the **graded height** of a graded prime ideal $\mathfrak{p}$ in the ring A, as the longest length m (which always exists, using the Noetherian hypothesis) of a chain of graded primes $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m = \mathfrak{p}$. We denote the graded height of the graded prime $\mathfrak{p}$ by ${}^*\mathrm{ht}(\mathfrak{p})$.

For every graded prime $\mathfrak{p}$, $\mathrm{ht}(\mathfrak{p}) \geq {}^*\mathrm{ht}(\mathfrak{p})$.

Forgetting the grading on A and M, one defines the Krull dimension of a graded A-module M as usual; here this is denoted by $\dim_A(M)$. As usual, $\dim(A) \doteq \dim_A(A)$.

Definition 1.31. *The **graded Krull dimension** of a graded A-module M, denoted ${}^*\dim_A(M)$, is the greatest D such that there exists a strictly increasing chain*

$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_D$$

of graded prime ideals in A such that $Ann_A(M) \subseteq \mathfrak{p}_0$. If no such greatest D exists, M has infinite graded Krull dimension. For the zero module, we define ${}^*\dim(0) = -\infty$.

Of course, ${}^*\dim(A) \doteq {}^*\dim_A(A)$.

For any graded A-module M,

- ${}^*\dim_A(M) \leq \dim_A(M)$.

Also, since $Ann_A(M) = Ann_A(M(n))$, for every $n \in \mathbb{Z}$,

- $\dim_A(M) = \dim_A(M(n))$, for every $n \in \mathbb{Z}$.
- ${}^*\dim_A(M) = {}^*\dim_A(M(n))$, for every $n \in \mathbb{Z}$.

Example 1.32. *Let F be a graded field of the form $F \cong F_0[t, t^{-1}]$, where $\deg(t) > 0$. The only graded prime in K is 0, so that ${}^*\dim(F) = 0$. On the other hand, $\dim(F) = 1$.*

Example 1.33. *If A is *local with unique *maximal ideal $\mathcal{N}$, and $\mathcal{N}$ is also a minimal prime ideal in A, then A is a *Artinian ring with ${}^*\dim(A) = 0$, and A_0 is an Artinian ring of Krull dimension zero with unique nilpotent maximal ideal $\mathcal{N}_0$.*

Lemma 1.34. *Suppose that $\mathfrak{p} \in$ Spec(A). ($\mathfrak{p}$ may or may not be graded.) We know that $\mathfrak{p}$ has finite height; say $\mathrm{ht}(\mathfrak{p}) = d$.*

i) If $\mathfrak{q} \in Spec(A)$ and $\mathfrak{p}^* \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ then either $\mathfrak{q} = \mathfrak{p}$ or $\mathfrak{q} = \mathfrak{p}^*$.

ii) There exists a chain of primes $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_d = \mathfrak{p}$, such that $\mathfrak{q}_0, \cdots, \mathfrak{q}_{d-1}$ are all graded.

iii) If $\mathfrak{p}$ is graded then there exists a chain of graded prime ideals such that

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = \mathfrak{p},$$

so that $\mathrm{ht}(\mathfrak{p}) = {}^*\mathrm{ht}(\mathfrak{p})$.

iv) If $\mathfrak{p}$ is not graded ($\mathfrak{p}^*$ is a proper subset of $\mathfrak{p}$), then

$$\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + 1 = {}^*\mathrm{ht}(\mathfrak{p}^*) + 1.$$

Proof. i) If $\mathfrak{p}$ is graded then $\mathfrak{p}^* = \mathfrak{p}$ and thus $\mathfrak{q} = \mathfrak{p}^* = \mathfrak{p}$. So, suppose instead that $\mathfrak{p}$ is ungraded, that is, $\mathfrak{p}^*$ is a proper subset of $\mathfrak{p}$. Replacing A by $A/\mathfrak{p}^*$, we may assume that A is a domain and $\mathfrak{p}^* = 0$.

Take the GMCS T to be the set of all nonzero homogeneous elements of A. Then, $T^{-1}A$ is a graded field, and has Krull dimension equal to 0 or 1. However, $0 \subseteq T^{-1}\mathfrak{q} \subseteq T^{-1}\mathfrak{p}$ is a chain of primes in $T^{-1}A$ and so either $\mathfrak{q} = 0$ or $\mathfrak{q} = \mathfrak{p}$.

The proofs for parts (ii)-(iv) may be found in [5].

□

Corollary 1.35. *If A is a graded Noetherian ring, then*

$$^*\dim(A) \leq \dim(A) \leq {}^*\dim(A) + 1;$$

therefore if $M \in \mathfrak{grmod}(A)$,

$$^*\dim_A(M) \leq \dim_A(M) \leq {}^*\dim_A(M) + 1.$$

Proof. If A has finite Krull dimension, the first inequality is always true; also, there is a maximal ideal $\mathfrak{m}$ of A, not necessarily graded, such that $\mathrm{ht}(\mathfrak{m}) = \dim(A)$. But then $\dim(A) = \mathrm{ht}(\mathfrak{m}) \leq {}^*\mathrm{ht}(\mathfrak{m}^*) + 1 \leq {}^*\dim(A) + 1$. If A does not have finite Krull dimension, then for every positive integer e there is a prime ideal $\mathfrak{p}$ of height larger than e. But then ${}^*\mathrm{ht}(\mathfrak{p}^*)$ is larger than $e - 1$, so ${}^*\dim(A)$ is infinite as well. □

Krull dimension for modules over positively graded rings

Definition 1.36. *If S is a positively graded ring,*

$$Proj(S) \doteq \{\mathfrak{p} \in \mathit{Spec}(S) \mid \mathfrak{p} \textit{ is graded and } S_+ \not\subseteq \mathfrak{p}\}.$$

Note that if $\mathfrak{p} \in Proj(S)$, then the set of homogeneous elements of $S - \mathfrak{p}$ has at least one nonzero element of strictly positive degree. Also, $\mathcal{N}$ is *maximal ideal in S if and only if $\mathcal{N} = \mathcal{N}_0 \oplus S_+$, with $\mathcal{N}_0$ a maximal ideal in S_0. Thus, $Proj(S)$ contains no *maximal ideals.

For positively graded rings, there is no difference between ${}^*\dim$ and $\dim$:

Lemma 1.37. *Let the ring S be a positively graded Noetherian ring of finite Krull dimension and $M \in \mathfrak{grmod}(S)$. Then,*

i) $\dim(S) = {}^*\dim(S)$*; therefore,*

ii) $\dim_S(M) = {}^*\dim_S(M)$.

For graded localizations of positively graded rings, the following is well-known:

Theorem 1.38. *Suppose that S is a positively graded Noetherian ring of finite Krull dimension, and $\mathfrak{p}$ is a graded prime ideal of S. Then, if $S_+ \subseteq \mathfrak{p}$, $\dim(S_{[\mathfrak{p}]}) = {}^*\dim(S_{[\mathfrak{p}]})$, and if $S_+ \not\subseteq \mathfrak{p}$, $\dim(S_{[\mathfrak{p}]}) = {}^*\dim(S_{[\mathfrak{p}]}) + 1$.*

Proof. Since $S_{[p]}$ is a localization of S, it is Noetherian. Ignoring the grading and recalling the standard order-preserving correspondence between the set of all primes of S disjoint from T and the prime ideals of of $T^{-1}S$, for any MCS or GMCS T in S. So $\infty > \dim(S) \geq \dim(S_{[\mathfrak{p}]})$.

We have already seen, then, that ${}^*\dim(S_{[\mathfrak{p}]}) \leq \dim(S_{[\mathfrak{p}]}) \leq {}^*\dim(S_{[\mathfrak{p}]}) + 1$.

Now let T be the GMCS consiting of all homogeneous elements of S not in $\mathfrak{p}$.

In the case where $S_+ \subseteq \mathfrak{p}$, we must have $\mathfrak{p} = (\mathfrak{p} \cap S_0) \oplus S_+$. For any element $t \in T$, this forces $\deg t = 0$. Thus, $S_{[\mathfrak{p}]}$ is a positively graded Noetherian ring of finite Krull dimension, so $\dim(S_{[\mathfrak{p}]}) = {}^*\dim(S_{[\mathfrak{p}]})$.

Now, $S_{[\mathfrak{p}]}/\mathfrak{p}_{[\mathfrak{p}]} = (S/\mathfrak{p})_{[\mathfrak{p}]}$ is a graded field, and it does have a positive degree element since $S_+ \not\subseteq \mathfrak{p}$: Choose any homogeneous $t \in S_+$, $t \notin \mathfrak{p}$. Then $t \in T$, and has positive degree, thus $(t+\mathfrak{p})/1$ is a nonzero, positive degree element of $(S/\mathfrak{p})_{[\mathfrak{p}]}$. Forgetting the grading, this domain has dimension 1. Thus, there must exist a prime $\mathfrak{q}$, necessarily ungraded, of $S_{[\mathfrak{p}]}$ such that

$$\mathfrak{p}_{[\mathfrak{p}]} \subset \mathfrak{q}.$$

Therefore

$$\dim(S_{[\mathfrak{p}]}) \geq \mathrm{ht}(\mathfrak{p}_{[\mathfrak{p}]}) + 1 = {}^*\mathrm{ht}(\mathfrak{p}_{[\mathfrak{p}]}) + 1 = {}^*\dim(S_{[\mathfrak{p}]}) + 1,$$

yielding the conclusion. $\square$

The following lemma establishes a relationship between primes in the localized ring and primes in the degree 0 part of the localization, the ideas are implicit in [13].

Lemma 1.39. *Suppose that S is a Noetherian positively graded ring, and T is any GMCS that contains at least one element of positive degree. If $\mathfrak{q}$ is a prime ideal in $(T^{-1}S)_0$, then there exists a unique graded prime $\mathfrak{p} \in Proj(S)$, disjoint from T, such that $\mathfrak{q} = (T^{-1}\mathfrak{p})_0$.*

Proof. Uniqueness is left to the reader. To establish existence, let $\mathfrak{q} \in Spec(T^{-1}S)_0$. Define for $i \geq 0$,

$$\mathfrak{p}_i \doteq \{x \in S_i \mid \exists j > 0, t \in T_j \text{ s.t. } \frac{x^j}{t^i} \in \mathfrak{q}\},$$

so that, since $\mathfrak{q}$ is prime,

$$\mathfrak{p}_0 = \{r \in S_0 \mid \frac{r}{1} \in \mathfrak{q}\}.$$

Define $\mathfrak{p} \doteq \oplus_{i\geq 0}\mathfrak{p}_i$, we will show that $\mathfrak{p}$ satisfies the required conditions.

First, each $\mathfrak{p}_i$ is an abelian group with respect to $+$. For if $x, y \in \mathfrak{p}_i$, $i \geq 0$, then there exists a $k_1, k_2 > 0$ and $s \in T_{k_1}, t \in T_{k_2}$ such that $\frac{x^{k_1}}{s^i}$ and $\frac{y^{k_2}}{t^i}$ are in $\mathfrak{q}$. Then, $(x+y)^{k_1+k_2} = \sum_{\alpha+\beta=k_1+k_2} c_{(\alpha,\beta)}x^\alpha y^\beta$, for the binomial coefficient $c_{(\alpha,\beta)} \in S_0$. Now, either $\alpha \geq k_1$ or $\beta \geq k_2$. If $\alpha \geq k_1$, then $\frac{x^\alpha y^\beta}{s^i t^i} = \frac{x^{k_1}}{s^i} \cdot \frac{x^{\alpha-k_1}y^\beta}{t^i}$. This is a product of an element in $\mathfrak{q}$ with an element in $(T^{-1}S)_0$, so it must be in $\mathfrak{q}$. A similar computation handles the case that $\beta \geq k_2$. Therefore, $\frac{(x+y)^{k_1+k_2}}{(st)^i} \in \mathfrak{q}$, and so $x + y \in \mathfrak{p}_i$.

To show that that $\mathfrak{p}$ is an ideal in S, one needs only to show that $S_i\mathfrak{p}_j \subseteq \mathfrak{p}_{i+j}$ for every i, j. Suppose $s \in S_i$ and $x \in \mathfrak{p}_j$. There exists $k > 0, t \in T_k$ with $\frac{x^k}{t^j} \in \mathfrak{q}$. Then, $\frac{(sx)^k}{t^{j+i}} = \frac{s^k}{t^i} \cdot \frac{x^k}{t^j}$, the product of an element in $(T^{-1}S)_0$ with an element in $\mathfrak{q}$, and therefore $sx \in \mathfrak{p}_{i+j}$ so that $\mathfrak{p}$ is a graded ideal in S.

Furthermore, $\mathfrak{p} \cap T = \emptyset$: if not, choose a $t \in \mathfrak{p}_i \cap T$. So, there exists a $k > 0$ and an $s \in T_k$ such that $\frac{t^k}{s^i} \in \mathfrak{q}$. However, the product $\frac{s^i}{t^k} \cdot \frac{t^k}{s^i}$ must also be in $\mathfrak{q}$, which contradicts that $1 \notin \mathfrak{q}$. Since T has at least one nonzero element of positive degree, and $\mathfrak{p} \cap T = \emptyset$, $S_+ \not\subseteq \mathfrak{p}$.

To verify that $\mathfrak{p}$ is prime, suppose that $f \in S_n$, $g \in S_m$, and $fg \in \mathfrak{p}_{n+m}$. There exists a $k > 0$ and $t \in T_k$ such that $\frac{(fg)^k}{t^{m+n}} \in \mathfrak{q}$. Now, $\frac{(fg)^k}{t^{m+n}} = \frac{f^k}{t^n} \cdot \frac{g^k}{t^m} \in \mathfrak{q}$, and by primality of $\mathfrak{q}$, together with the definition of $\mathfrak{p}$, either $f \in \mathfrak{p}_n$ or $g \in \mathfrak{p}_m$.

We have established that $\mathfrak{p} \in Proj(S)$, and it only remains to show that $\mathfrak{q} = (T^{-1}\mathfrak{p})_0$. Suppose that $\xi \in \mathfrak{q}$, so ξ may be written as $\frac{x}{t}$, with $x \in S_i$, $t \in T_i$. If $i > 0$, then $x^i/t^i = \xi^i \in \mathfrak{q}$, so $x \in \mathfrak{p}_i$ by definition, and $\xi = \frac{x}{t} \in (T^{-1}\mathfrak{p})_0$. If $i = 0$, then $\frac{t}{1}\xi = \frac{x}{1} \in \mathfrak{q}$, so $x \in \mathfrak{p}_0$ and $\xi = \frac{x}{t} \in (T^{-1}\mathfrak{p})_0$.

On the other hand, suppose that $\frac{x}{t} \in (T^{-1}\mathfrak{p})_0$, $x \in \mathfrak{p}_i$, $t \in T_i$. By definition, there exists a $k > 0$, and an $s \in T_k$ such that $\frac{x^k}{s^i} \in \mathfrak{q}$. Then, $\frac{s^i}{t^k} \cdot \frac{x^k}{s^i} \in \mathfrak{q}$ since $\frac{s^i}{t^k} \in (T^{-1}S)_0$ and $\frac{x^k}{s^i} \in \mathfrak{q}$. Of course, $\frac{s^i}{t^k} \cdot \frac{x^k}{s^i} = \left(\frac{x}{t}\right)^k$, and by primality of $\mathfrak{q}$, $\frac{x}{t} \in \mathfrak{q}$. $\square$

Thus we have

Theorem 1.40. *Suppose S is a positively graded ring, and T is a GMCS in S containing at least one element of positive degree. Then, there exists a one-to-one inclusion-preserving correspondence*

$$\{\mathfrak{p} \in \textit{Proj}(S) \mid \mathfrak{p} \cap T = \emptyset\} \leftrightarrow \{\mathfrak{q} \in Spec(T^{-1}S)_0\};$$

this correspondence takes $\mathfrak{p}$ to $(T^{-1}\mathfrak{p})_0$.

Using the correspondence of the above theorem, we have, as expected,

Corollary 1.41. *Suppose S is a positively graded Noetherian ring of finite Krull dimension. Let $\mathfrak{p} \in Proj(S)$. Then $S_{(\mathfrak{p})}$ is a local Noetherian ring of finite Krull dimension and*

$$\dim(S_{(\mathfrak{p})}) = {}^*\mathrm{ht}(\mathfrak{p}) = {}^*\mathrm{ht}(\mathfrak{p}_{[\mathfrak{p}]}) = {}^*\dim(S_{[\mathfrak{p}]}) = \dim(S_{[\mathfrak{p}]}) - 1.$$

Finally, we point out that, in many cases, *local graded rings are graded localizations of positively graded rings at graded prime ideals.

Suppose that $(A, \mathcal{N})$ is a *local ring. Then, there exists a homogeneous element of strictly positive degree in $A - \mathcal{N}$ if and only if there exists a homogeneous element of strictly negative degree in $A - \mathcal{N}$: if $s \in A - \mathcal{N}$ is homogeneous of degree $d > 0$ then, since every homogenous element of $A - \mathcal{N}$ is invertible, there exists a $t \in A - \mathcal{N}$ that is homogeneous and $st = 1 \in A_0$. Necessarily, the degree of t is $-d$. Since the argument is reversible, we have the conclusion.

Thus, we have two alternatives:

- There are homogeneous elements of $A - \mathcal{N}$ of strictly positive and strictly negative degrees; or

- $\mathcal{N}_d = A_d$ for all $d \neq 0$, and $\mathcal{N}_0$ is the unique maximal ideal of A_0. In this case, it's required that $A_d A_{-d} \subseteq \mathcal{N}_0$ for all d, else $\mathcal{N}$ can't be a graded ideal in A. If A is already positively graded, that condition is certainly satisfied, and since the set of homogeneous elements of A not in $\mathcal{N}$ is the set $A_0 - \mathcal{N}_0$, the elements of which are already invertible in A, A is certainly a graded localization of a positively graded ring, namely itself. If A is not positively graded, though, in this case, one might not be able to obtain A as a localization of a positively graded object. But we will not generally consider these rings.

Anyway, in the case of the first of the two alternatives, let

$$S(A) = \oplus_{d \geq 0} A_d$$

be the "positive part" of A; this too is a graded ring, and it is certainly positively graded. Considering the graded abelian subgroup $S(\mathcal{N}) = \oplus_{d \geq 0} \mathcal{N}_d$ of $S(A)$, we see that it is a graded prime ideal in $S(A)$. We claim that $S(A)_{[S(\mathcal{N})]}$ is isomorphic as a graded ring to A, with $\mathcal{N}$ corresponding to $S(\mathcal{N})_{[S(\mathcal{N})]}$, under the well-defined injective homomorphism of graded rings defined by $a/b \mapsto ab^{-1}$, if $a \in S(A)_d$ and b is a homogeneous element of degree $e \geq 0$ in $A - \mathcal{N}$. To see that the homomorphism is surjective, suppose that x is a homogeneous element of degree j in A. If $j \geq 0$, $x/1 \mapsto x$, and $x/1 \in S(A)_{[S(\mathcal{N})]}$. If $j < 0$, the assumption of the first alternative says that there is a homogeneous element $t \in A - \mathcal{N}$ with $\deg(t) = k > 0$. Then, there is a positive integer l such that $lk + j > 0$ so that $t^l x / t^l \in S(A)_{[S(\mathcal{N})]}$ and $t^l x / t^l \mapsto x$.

Poincaré series and dimension for positively graded rings

The ring $\mathbb{Z}[[t]][t^{-1}]$ is denoted by $\mathbb{Z}((t))$; thus an element of $\mathbb{Z}((t))$ is a formal Laurent series $f(t)$ with integer coeffiicients; there always exists an $n \in \mathbb{Z}$ with $t^n f(t) \in \mathbb{Z}[[t]]$.

In order to define the Poincaré series, we must assume that S is a positively graded Noetherian ring, with S_0 Artinian. So, for every i, S_i is a finitely generated S_0-module; if $M \in \mathfrak{grmod}(S)$, then $M_j = 0$ for $j << 0$ and M_j has finite length over S_0.

Definition 1.42. *Suppose S is a positively graded ring, with S_0 Artinian, and M is a finitely generated graded S-module. Then the **Poincaré series** of M is the formal Laurent series with integer coefficients*

$$P_M(t) = \sum_{i \in \mathbb{Z}} \ell_{S_0}(M_i) t^i,$$

where $\ell_{S_0}(M_i)$ is the length of the finitely generated module M_i over the Artinian ring S_0.

Whenever we write down a Poincaré series, we make the assumption that S is positively graded, Noetherian and S_0 is Artinian, but we won't always restate this.

Sometimes the Poincaré series is called the Hilbert series, or the Hilbert-Poincaré series.

Theorem 1.43. *(The Hilbert-Serre Theorem) [3] Let S be a positively graded Noetherian ring with S_0 Artinian, $M \in \mathfrak{grmod}(S)$. Suppose that S is generated as a S_0-algebra by elements $x_1, \ldots, x_n$ of positive degrees $d_1, \ldots, d_n$. Then,*

$$P_M(t) = \frac{q(t)}{\prod_{i=1}^{n}(1 - t^{d_i})},$$

where $q(t) \in \mathbb{Z}[t, t^{-1}]$.

Furthermore, if M has no elements of negative degree, $q(t) \in \mathbb{Z}[t]$.

Example 1.44.

- *Find the Poincare series of the ring $k[x_0, \cdots, x_r]$, where k is a field and the degree of each x_i is 1. By definition, $k[x_0, x_1, \cdots, x_r]_d$ is the set of all monomials in $r + 1$ variables whose degree is d. An arbitrary element looks like $x_0^{a_0} x_1^{a_1} \cdots x_r^{a_r}$ where $a_0 + a_1 + \cdots + a_r = d$. We recognize that calculating the the dimension of the degree d*

component is equivalent to the counting problem, "How many ways can d chips be placed in $r+1$ buckets?" One can readily check the answer is

$$\binom{r+d}{r}.$$

The function $1/(1-t)$ has power series equal to $1+t+t^2+t^3+\cdots$ and recall the power series for $1/(1-t)^{r+1}$ is given by $\sum_{d=0}^{\infty}\binom{d+r}{r}t^d$. For a quick check of this fact, one can take successive derivatives of $1/(1-t)$ and observe the pattern.

Thus, the d^{th} coefficient of the function $1/(1-t)^{r+1}$, written in power series form, is identical to the dimension of the d^{th} component of the graded polynomial ring in r variables over K.

- *Find the Poincare series of the ring $k[x_0,\cdots,x_r]$, where k is a field and each x_i has $\deg(x_i)=d_i>0$. A similar computation reveals,*

$$P(k[x_0,\cdots,x_r]) = \frac{1}{\Pi_i(1-t^{d_i})}\ .$$

- *Let $R=k[x_1,\ldots,x_n]$ where the degree of each x_i equals 1, and let f be a degree d homogeneous polynomial in R. For the reader familiar with algebraic geometry, the ring $R/(f)$ is the coordinate ring of a degree d hypersurface in projective space. Then, there is a short exact sequence in $\mathfrak{grmod}(R)$, $0\to R(-d)\xrightarrow{\cdot f} R\to R/(f)\to 0$, using the additivity of the Poincare series and that $P_{R(-d)}(t)=t^dP_R(t)$, we have*

$$\begin{aligned}
P_{R/(f)}(t) &= P_R(t) - P_{R(-d)}(t)\\
&= \frac{1}{(1-t)^n} - \frac{t^d}{(1-t)^n}\\
&= \frac{(1-t)(1+t+\cdots+t^{d-1})}{(1-t)^n}\\
&= \frac{1+t+\cdots+t^{d-1}}{(1-t)^{n-1}}.
\end{aligned}$$

Notice that the Krull dimension of $R/(f)$ is $n-1$ and that this is also the order of the pole at $t=1$ of the Poincare series - this is not a coincidence as we shall see later.

Suppose now that we have a non-standard grading on R, say that the degree of x_i is equal to d_i. We compute

$$\begin{aligned}
P_{R/(f)}(t) &= P_R(t) - P_{R(-d)}(t) \\
&= \frac{1}{(1-t^{d_1})\cdots(1-t^{d_n})} - \frac{t^d}{(1-t^{d_1})\cdots(1-t^{d_n})} \\
&= \frac{(1-t)(1+t+\cdots+t^{d-1})}{(1-t)^n(1+t+\cdots+t^{d_1-1})\cdots(1+t+\cdots+t^{d_n-1})} \\
&= \left(\frac{1}{1-t}\right)^{n-1}\left[\frac{1+t+\cdots+t^{d-1}}{(1+t+\cdots+t^{d_1-1})\cdots(1+t+\cdots+t^{d_n-1})}\right].
\end{aligned}$$

More elementary facts about Poincaré series are collected below. The graded rings S, $\hat{S}$ are as in the Hilbert-Serre Theorem.

- Suppose that the graded abelian group M is simultaneously in $\mathfrak{grmod}(S)$ and $\mathfrak{grmod}(\hat{S})$ and $S_0 = \hat{S}_0$. Then whether we consider M as an S-module or as a $\hat{S}$-module, its Poincaré series does not change. For example, if $y_1, \ldots, y_s \in S_+$ are homogeneous, and $\hat{S} = S_0\langle y_1, \ldots, y_s\rangle$ is (by definition) the graded subring of S generated by S_0 and $y_1, \ldots, y_s$: if M is also a finitely generated $\hat{S}$-module, whether we consider M as an S-module, or as an $\hat{S}$-module, its Poincaré series is the same.

- If $M \in \mathfrak{grmod}(S)$, then so is $M(n)$, for every $n \in \mathbb{Z}$, and

$$P_{M(n)}(t) = t^{-n} P_M(t).$$

- If $0 \to P \to M \to N \to 0$ is a short exact sequence in $\mathfrak{grmod}(S)$, then

$$P_M(t) = P_P(t) + P_N(t).$$

- If $M, N \in \mathfrak{grmod}(S)$, then $P_{M \otimes_{S_0} N}(t) = P_M(t)P_N(t)$, if $M \otimes_{S_0} N$ is given the usual grading.

Definition 1.45. *Let M be in $\mathfrak{grmod}(S)$, $M \neq 0$.*

- *If $M \in \mathfrak{grmod}(S)$, $d_1(M)$ is the least j such that there exist positive integers $f_1, \ldots, f_j$ with*

$$(\prod_{i=1}^{j}(1 - t^{f_i}))P_M(t) \in \mathbb{Z}[t, t^{-1}].$$

  By definition, $d_1(M) = 0$ if and only if $P_M(t)$ is in $\mathbb{Z}[t, t^{-1}]$.

- *$s_1(M)$ is the least s such that there exist homogeneous elements $y_1, \ldots, y_s \in S_+$ with M finitely generated over $S_0\langle y_1, \ldots, y_s\rangle \subseteq S$. By definition, $s_1(M) = 0$ if and only if M is a finitely generated graded S_0-module.*

Note that if $M \neq 0$, $d_1(M)$ exists by the Hilbert-Serre theorem and $s_1(M)$ exists: for a finite set of homogeneous generators for S_+, the number of elements in that set is an upper bound for $s_1(M)$. It's also important to point out that $d_1(M)$ is the order of the pole of $P_M(t)$ at $t = 1$.

We "define", for convenience, $d_1(0) = s_1(0) = -\infty$.

Note that if $n \in \mathbb{Z}$, then $d_1(M(n)) = d_1(M)$, since $P_{M(n)}(t) = t^{-n}P_M(t)$. Also, $s_1(M(n)) = s_1(M)$ by definition and ${}^*\dim_S(M(n)) = {}^*\dim_S(M)$.

The following theorem and proposition could be considered "folklore", but the paper of Smoke cited is, as far as we know, the first appearance of these statements in the literature; the last equality of the theorem was proved earlier in this dissertation.

Theorem 1.46. ***Smoke's Dimension Theorem*** *(Theorem 5.5 of [24])*

Suppose that S is a positively graded ring of finite Krull dimension, with S_0 Artinian. Let $M \in \mathfrak{grmod}(S)$. If $d_1(M), s_1(M)$ are defined as above, we have

$$d_1(M) = s_1(M) = {}^*\dim_S(M) < \infty\,.$$

Under the hypotheses of the theorem, we've already seen that ${}^*\dim_S(M) = \dim_S(M)$*, so all of these numbers equal* $\dim_S(M)$ *as well.*

1.3.1 Graded systems of parameters

Returning to the more general case of A a $\mathbb{Z}$-graded ring, we define analogously to Serre, a graded system of parameters.

Definition 1.47. *Let* A *be a positively graded Noetherian ring of finite Krull dimension, or a *local Noetherian ring with unique *maximal ideal* $\mathcal{N}$*. Define* $\mathfrak{m}$ *to be the graded ideal* A_+ *in the first case, and the ideal* $\mathcal{N}$ *in the second. Suppose* $M \neq 0$ *is in* $\mathfrak{grmod}(A)$*. A sequence* $y_1, \ldots, y_D$ *of homogeneous elements of* $\mathfrak{m}$*, such that*

- *the graded* A*-module* $M/(y_1, \ldots, y_D)M$ *has finite *length over* A *and*
- $D = {}^*\dim_A(M)$

is called a graded system of parameters (GSOP) for the A*-module* M*.*

Note that by definition, a GSOP for M is also a GSOP for $M(n)$, for every $n \in \mathbb{Z}$ (and vice versa).

In the positively graded case, an alternative definition of a GSOP is given by:

Lemma 1.48. *Suppose that* S *is a positively graded Noetherian ring of finite Krull dimension, with* S_0 *Artinian, and* $y_1, \ldots, y_u$ *are homogeneous elements of* S_+*. Let* $M \in \mathfrak{grmod}(S)$*. Then,* $M/(y_1, \ldots, y_u)M$ *has finite *length over* S *if and only if* M *is a finitely generated graded* $S_0\langle y_1, \ldots, y_u\rangle$*-module.*

Proof. Let $t_0 \in \mathbb{Z}$ be chosen such that $M_j = 0$ for $j < t_0$. Suppose that $X \doteq M/(y_1, \ldots, y_u)M$ has finite *length over S. We've seen that there exists an integer $t_0 \leq t_1$ such that $X_j = 0$ if $j > t_1$. Using Lemma 2.9, M_j is finitely generated over S_0, so for every j such that $t_0 \leq j \leq t_1$ we may choose a finite set E_j of generators, possibly empty, for M_j over S_0. Then, we prove that M is generated by the finite set $E \doteq \cup_{j=t_0}^{t_1} E_j$ over $S_0\langle y_1, \ldots, y_u\rangle$; to do this we show, using induction on $\deg(z)$, that a homogeneous element z of M is in the submodule of M generated by E over $S_0\langle y_1, \ldots, y_u\rangle$. To start the induction, note that if $\deg(z) \leq t_1$, the claim is certainly true. Let $s > t_1$ and suppose that the inductive hypothesis holds for every homogeneous w of degree strictly less than s. Let z be a homogeneous element of M of degree s. Since $s > t_1, (M/(y_1, \ldots, y_u)M))s = 0$, so $s \in (y_1, \ldots, y_u)M$. Write $z = \sum_{\alpha=1}^{u} y_\alpha m_\alpha$. Since $\deg(y_\alpha) + \deg(m_\alpha) = s$ for every α such that $y_\alpha m_\alpha \neq 0$, and $\deg(y_\alpha) > 0$ for every such α, we must have $\deg(m_\alpha) < s$ for every α with $y_\alpha m_\alpha \neq 0$. Thus by induction, m_α is a linear combination of elements of E, with coefficients in $S_0\langle y_1, \ldots, y_u\rangle$. Clearly, then, so is z. Note that this part of the proof never used that S_0 is Artinian.

Conversely, suppose M is generated by a finite set E of nonzero homogeneous elements as a graded $S_0\langle y_1, \ldots, y_u\rangle$-module. Set $\mathcal{I} \doteq (y_1, \ldots, y_u)$. Let $t = \max\{\deg(e) \mid e \in E\}$. Then, for $j > t$, $(M/\mathcal{I}M)_j = 0$: If $x \in M_j$, $j > t$, write $x = \sum_{e \in E} f_e e$, where $f_e \in S_0\langle y_1, \ldots, y_u\rangle$ is homogeneous. If $\deg(f_e) \neq 0$, and $f_e e \neq 0$, then $f_e e \in \mathcal{I}M$. Therefore, x is equivalent to $\sum_{f_e \neq 0, \deg(f_e)=0} f_e e$ mod $\mathcal{I}M$. However, for every summand in this last sum, we must have $\deg(e) = \deg(x) > t$ if $f_e e \neq 0$, a contradiction. Thus, x is equivalent to 0 mod $\mathcal{I}M$. Lemma 2.9 tells us that since S_0 is Artinian, $*\ell_S(M/\mathcal{I}M) < \infty$. $\square$

The following proposition is another part of the "folklore" knowledge, but the citation is the first that we know of in the literature.

Proposition 1.49. *(Theorem 6.2 of [24]) Suppose that S is a positively graded Noetherian ring of finite Krull dimension, with S_0 Artinian, and $M \neq 0$ is in $\mathfrak{grmod}(S)$. Let $D(M) \doteq d_1(M) = s_1(M) = {}^*\dim_S(M) = \dim_S(M)$, so Theorem 1.46 and Lemma 1.48 tell us that a GSOP exists for M. Moreover, $D(M)$ is the length of any GSOP and if $y_1, \ldots, y_{D(M)} \in S_+$ is a GSOP for M, $y_1, \ldots, y_{D(M)}$ are algebraically independent over S_0.*

Chapter 2

Multiplicities for graded modules

In this chapter we define *Samuel multiplicity and *Koszul multiplicity for modules in $\mathfrak{grmod}(A)$, where A is a $\mathbb{Z}$-graded ring. All of this work is done analogously to Serre, and completes our transition of Serre's theory of multiplicities to the graded category.

The *Samuel multiplicity is explored using the tools of the graded category which we have developed thus far: *length, *dimension, graded localization, etc. The *Koszul multiplicity is defined using tools from homological algebra. In each case, to adapt the theory from the ungraded case, we have the added complication of our objects being bi-graded - the internal grading that the module inherits from $\mathfrak{grmod}(A)$, and an external grading coming from either the associated graded module in the case of Samuel multiplicities, or the complex grading for Koszul multiplicities. By carefully keeping track of the bi-grading, we verify that all morphisms respect both gradings, and as one might expect, the bi-grading does not cause any problems. We show, as in the ungraded case, the two multiplicities agree.

Finally, the graded multiplicity theory agrees with the ungraded theory by simply forgetting the grading, when we work over positively graded rings. This is to be expected, for we have shown that *length and length agree in the positively graded case. We treat positively graded rings in more detail in chapter 3.

2.1 *Samuel polynomials and *multiplicities

We begin by outlining the procedure for defining the Hilbert and Samuel polynomials in the ungraded case (see [25] for full discussion/proofs).

Suppose that H is a positively graded ring with H_0 Artinian, and that H is generated as an H_0-algebra by a finite number of homogeneous elements $x_1, \ldots, x_u$ in H_1. Such a ring H is then called a "standard" graded ring. For any finitely generated, positively graded H-module M, M_n is a finitely generated H_0-module for every n. Since H_0 is Artinian, the Hilbert function,

$n \mapsto \ell_{H_0}(M_n)$, is defined for all integers $n \geq 0$. Using induction on the number of generators for H as an H_0-algebra, and the additivity of length over exact sequences, one may prove that the Hilbert function is polynomial-like; in other words there is a unique polynomial f with rational coefficients such that $f(n) = \ell_{H_0}(M_n)$ for all n sufficiently large. The polynomial describing the function $n \mapsto \ell_{H_0}(M_n)$ is called the Hilbert polynomial of M (over H).

Recall the delta notation from the theory of polynomial-like functions: if f is a function with an integer domain, then Δf is the function defined by $\Delta f(n) \doteq f(n+1) - f(n)$. Then, we know that f is polynomial-like if and only if Δf is polynomial-like. We may iterate the operator "Δ" on integer domain functions, obtaining operators Δ^r, for $r \geq 0$.

As previously, we will use upper indices for filtrations of A-modules, whether graded or ungraded. Sometimes we'll use increasing filtrations and sometimes decreasing though. We'll use notations like $M^\bullet$ or often $\mathcal{F}(M)$ for filtrations of M by A-modules.

To define a Samuel polynomial, for this and the next two paragraphs, suppose that A is an ungraded Noetherian ring, M an ungraded finitely generated A-module, and $\mathcal{I}$ is an ideal of A such that $M/\mathcal{I}M$ has finite length over A; this last is true if and only if $V(\mathcal{I} + Ann_A(M))$ consists of a finite number of maximal ideals in A.

Definition 2.1. *A filtration $\mathcal{F}(M)$ with $\mathcal{F}^{i+1}(M) \subseteq \mathcal{F}^i(M)$ for every $i \geq 0$, is called $\mathcal{I}$-bonne (or in English, $\mathcal{I}$-good) if $\mathcal{I}\mathcal{F}^n(M) \subseteq \mathcal{F}^{n+1}(M)$, for every $n \geq 0$, and with equality for $n >> 0$.*

Example 2.2. *If $\mathcal{I}$ is an ideal in A, the $\mathcal{I}$-adic filtration $\cdots \subseteq \mathcal{I}^{j+1}M \subseteq \mathcal{I}^j M \subseteq \cdots \subseteq \mathcal{I}M \subseteq M$ is $\mathcal{I}$-bonne.*

Summarizing the discussion in [25], given an ideal $\mathcal{I}$ with $\ell_A(M/\mathcal{I}M) < \infty$ and an $\mathcal{I}$-bonne filtration $\mathcal{F}(M)$, $\ell_A(M/\mathcal{F}^n(M))$, is well-defined. Now, $V(M/\mathcal{I}M) = V(Ann_A(M) + \mathcal{I})$ consists of a finite number of maximal ideals; without loss of generality we may assume that $Ann_A(M) = 0$ and $V(M/\mathcal{I}M) = V(\mathcal{I})$ consists of a finite number of maximal ideals, so that $A/\mathcal{I}$ is an Artinian ring. The positively graded associated graded module $gr(M) = \oplus_{n\geq 0}\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M)$ is finitely generated over the positively graded associated graded ring $gr(A) = \oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1}$. Furthermore

$gr(A)$ is generated over $gr(A)_0 = A/\mathcal{I}$, an Artinian ring, by elements of degree one, and the Hilbert polynomial for $gr(M)$ as a $gr(A)$-module exists.

Then, $n \mapsto \ell_A(M/\mathcal{F}^{n+1}(M)) - \ell_A(M/\mathcal{F}^n(M)) = \ell_A(\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M))$ is polynomial-like, and the general theory of polynomial-like functions tells us that the Samuel function $n \mapsto \ell_A(M/\mathcal{F}^n(M))$ is also polynomial-like. The polynomial describing this function is called the Samuel polynomial $p(M, \mathcal{F}, n)$ of the A-module M with respect to the filtration $\mathcal{F}$ and the ideal $\mathcal{I}$.

We make new, similar definitions in the graded category, now assuming A is a graded Noetherian ring and $M \in \mathfrak{grmod}(A)$.

To define the *Hilbert polynomial, start with M and H as above except require that they are bigraded objects: Suppose that H is a bigraded ring such that $H_{i,j} = 0$ for $i < 0$, $H_{0,*} \doteq \oplus_{j\in\mathbb{Z}} H_{0,j}$ is a graded ring that is *Artinian and H is generated as an bigraded algebra over the graded ring $H_{0,*}$ by a finite number of elements in $H_{1,*} \doteq \oplus_{j\in\mathbb{Z}} H_{1,j}$. M is taken to be a bigraded H-module such that $M_{i,j} = 0$ for $j < 0$ and M is generated as an H-module by a finite number of bihomogeneous elements. Then, for each $k \geq 0$, $M_{k,*} \doteq \oplus_{j\in\mathbb{Z}} M_{k,j}$ is a finitely generated graded $H_{0,*}$-module, so $*\ell_{H_{0,*}}(M_{k,*})$ is well-defined for every $k \geq 0$. Furthermore, the function $k \mapsto *\ell_{H_{0,*}}(M_{k,*})$ is polynomial like. To see this, following the argument in [25] for the ungraded case, use induction on the number of bihomogeneous generators (taken from $H_{1,*}$) for H as an $H_{0,*}$-algebra, and additivity of $*\ell$ over exact sequences of graded modules. The exact sequence used in Theorem II.B.3.2 of [25] becomes an exact sequence of graded modules, with middle map multiplication by a generator of bidegree $(1, d)$:

$$0 \to N_{n,*} \to M_{n,*}(-d) \to M_{n+1,*} \to R_{n+1,*} \to 0;$$

there is a shift for the second graded degree in the second term, and the rest of proof is the same otherwise with length replaced by *length. Furthermore, using the argument of Theorem II.B.3.2 of [25] for the ungraded case, we see that if H is generated as a bigraded algebra over $H_{0,*}$ by r elements of bidegree $(1, -)$, then the *Hilbert polynomial has degree less than or equal to $r - 1$.

In the same spirit, we define a *Samuel function by making appropriate changes to consider the grading, as follows.

Suppose A is a graded ring, $\mathcal{I}$ is a graded ideal in A and $\mathcal{F}(M)$ is a graded $\mathcal{I}$-bonne filtration of M: the meaning of $\mathcal{I}$-bonne is the same as in the ungraded case; in this case the $\mathcal{I}$-adic filtration is a graded, $\mathcal{I}$-bonne filtration.

Note that if $\mathcal{I}$ is a graded ideal in A, $\mathcal{F}(M)$ is a graded $\mathcal{I}$-bonne filtration of M, and $d \in \mathbb{Z}$ is a fixed integer, we may shift degrees by d throughout the filtration yielding an $\mathcal{I}$-bonne filtration $\mathcal{F}(d)$ of $M(d)$: $\mathcal{F}(d)^n(M(d)) \doteq (\mathcal{F}^n(M))(d)$. To see that this filtration is also $\mathcal{I}$-bonne, just compute that $\mathcal{I}(\mathcal{F}(d)^n(M(d))) = (\mathcal{I}\mathcal{F}^n(M))(d)$ as follows. Suppose that $x \in (\mathcal{I}\mathcal{F}^n(M))(d)_j = (\mathcal{I}\mathcal{F}^n(M))_{d+j}$, so that $x = \sum_t \alpha_t m_t$, where $\alpha_t \in \mathcal{I}, m_t \in \mathcal{F}^n(M)$ are all homogeneous and $\deg(\alpha_t) + \deg(m_t) = d + j$ whenever $\alpha_t m_t \neq 0$. Thus, $\deg(m_t) = d + (j - \deg(\alpha_t))$ for all such t, so that $m_t \in (\mathcal{F}(d)^n)(M(d))_{j-\deg(\alpha_t)}$, $\alpha_t m_t \in \mathcal{I}(\mathcal{F}(d)^n(M(d)))_j$ for every t and $x \in \mathcal{I}(\mathcal{F}(d)^n(M(d)))_j$. The converse is similarly proved. In particular, the d-shift of the $\mathcal{I}$-adic filtration on M is the $\mathcal{I}$-adic filtration on $M(d)$.

Adding a new definition,

Definition 2.3. *A graded ideal $\mathcal{I}$ of A such that $*\ell_A(M/\mathcal{I}M) < \infty$ is called a graded ideal of definition for M (a GIOD for M).*

(This is a little different from Serre's definition [25] of an ideal of definition in the ungraded case.) We've seen that $\mathcal{I}$ is a graded ideal of definition for M if and only if all graded primes containing $\mathcal{I} + Ann_A(M)$ are *maximal. Given a GIOD $\mathcal{I}$ for M, and a graded $\mathcal{I}$-bonne filtration $\mathcal{F}(M)$, $*\ell_A(M/\mathcal{F}^n(M))$, is then well-defined. Passing without loss of generality to the case $Ann_A(M) = 0$ as in the ungraded case, we see that $A/\mathcal{I}$ is a *Artinian ring and that the associated bigraded module $gr(M) = \oplus_{n\geq 0}\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M)$, where $gr(M)_{n,j} \doteq \mathcal{F}^n(M)_j/\mathcal{F}^{n+1}(M)_j$, is finitely generated over the associated bigraded ring $gr(A) = \oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1}$ (where $gr(A)_{n,j} \doteq \mathcal{I}^n_j/\mathcal{I}^{n+1}_j$). Note that $gr(A)$ is generated by elements of bidegree $(1, -)$, as an algebra over the *Artinian graded ring $A/\mathcal{I}$ and thus the *Hilbert polynomial for $gr(M)$ as a $gr(A)$-module exists.

Definition 2.4. *Suppose that $\mathcal{I}$ is a GIOD for $M \in \mathfrak{grmod}A$ and $\mathcal{F}$ is an $\mathcal{I}$-bonne filtration of M. The *Samuel function with respect to $\mathcal{F}$ and $\mathcal{I}$ is defined on the nonnegative integers by $n \mapsto *\ell_A(M/\mathcal{F}^n(M))$.*

Since $*\ell_A(M/\mathcal{F}^{n+1}(M)) - *\ell_A(M/\mathcal{F}^n(M)) = *\ell_A(\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M))$, the Δ operator applied to the *Samuel function is polynomial-like, so

Lemma 2.5. *If $M \in \mathfrak{grmod}(A)$ and $\mathcal{I}$ is a GIOD for M, the *Samuel function for the graded $\mathcal{I}$-bonne filtration $\mathcal{F}(M)$ is polynomial-like.*

To set notation, the polynomial that calculates $*\ell_A(M/\mathcal{F}^n(M))$ for $n >> 0$ will be called $*p(M, \mathcal{F}, n)$, and if $\mathcal{F}$ is the $\mathcal{I}$-adic filtration on M, we will instead write $*p(M, \mathcal{I}, n)$.

Lemma 2.6. *Suppose that $M \in \mathfrak{grmod}(A)$ and $\mathcal{F}(M)$ is a graded $\mathcal{I}$-bonne filtration of M for some GIOD $\mathcal{I}$ for M. Then*

a) For every $d \in \mathbb{Z}$, $\mathcal{I}$ is a GIOD for $M(d)$, $\mathcal{F}(d)(M(d))$ is an $\mathcal{I}$-bonne filtration of the graded A-module $M(d)$ and $*p(M(d), \mathcal{F}(d), n) = *p(M, \mathcal{F}, n)$.

b) $*p(M, \mathcal{I}, n) = *p(M, \mathcal{F}, n) + R(n)$, where R is a polynomial with nonnegative leading coefficient and degree strictly less than that of the degree of $*p(M, \mathcal{I}, n)$.

c) If $(Ann_A(M) + \mathcal{I})/Ann_A(M)$ is generated by r homogeneous elements, then the degree of $*p(M, \mathcal{I}, n)$ is less than or equal to r, and $\Delta^r(*p)$ is a constant less than or equal to $*\ell_A(M/\mathcal{I}M)$.

d) If $0 \to N \to M \to P \to 0$ is a short exact sequence in $\mathfrak{grmod}(A)$, and $\mathcal{I}$ is a GIOD for M, then $\mathcal{I}$ is a GIOD for both N and P and

$$*p(M, \mathcal{I}, n) + R(n) = *p(N, \mathcal{I}, n) + *p(P, \mathcal{I}, n),$$

where R is a polynomial with nonnegative leading coefficient and degree strictly less than that of $*p(N, \mathcal{I}, n)$.

e) If $\mathcal{I}$ and $\hat{\mathcal{I}}$ are two GIODs for M such that $*V(\mathcal{I} + Ann_A(M)) = *V(\hat{\mathcal{I}} + Ann_A(M))$, then the degree of $*p(M, \mathcal{I}, n)$ equals the degree of $*p(M, \hat{\mathcal{I}}, n)$.

Proof. We've already noted that $\mathcal{I}(\mathcal{F}(d)^n(M(d))) = (\mathcal{I}\mathcal{F}^n(M))(d)$; so that $\mathcal{F}(d)(M(d))$ is an $\mathcal{I}$-bonne filtration of $M(d)$. The *Samuel polynomials are identical since $M(d)/\mathcal{F}(d)^n(M(d)) = M(d)/(\mathcal{F}^n(M)(d)) = (M/(\mathcal{F}^n(M))(d)$, for every n. The proofs of b)-e) follow exactly the proofs in Section II.B.4 of Lemma 3 and Propositions 10 and 11 of [25], adapted with clear notational changes to the graded case, and are not given here. □

Since we will be interested in the leading coefficient of *Samuel polynomials, b) above tells us that we may as well just consider $\mathcal{I}$-adic filtrations and suppress all talk about $\mathcal{I}$-bonne filtrations; the need to consider general $\mathcal{I}$-bonne filtrations $\mathcal{F}$ is indicated in the proof of d), even though we haven't given it, since the proof of d) uses the Artin-Rees lemmas, which also holds in the graded context.

Definition 2.7. *Suppose that $M \in \mathfrak{grmod}(A)$, $\mathcal{I}$ is a GIOD for M and $d \in \mathbb{Z}$, $d \geq \deg(*p(M, \mathcal{I}, n))$. The* ****Samuel multiplicity*** *of M with respect to $\mathcal{I}$ is defined as*

$$*e(M, \mathcal{I}, d) \doteq \Delta^d(*p(M, \mathcal{I}, n)).$$

By properties of the finite difference operator Δ, we see that $*e(M, \mathcal{I}, d) = 0$ whenever $d > \deg(*p(M, \mathcal{I}, n))$. When $d = \deg(*p(M, \mathcal{I}, n))$, $*e(M, \mathcal{I}, d)$ is a positive integer. Further, one may compute that

$$*p(M, \mathcal{I}, n) = \frac{*e(M, \mathcal{I}, d)}{d!} n^d + \text{lower order terms.}$$

Using Lemma 2.6d), we see that if

$$0 \to N \to M \to P \to 0$$

is a short exact sequence in $\mathfrak{grmod}(A)$, $\mathcal{I}$ is a GIOD for M and $d \geq \deg(*p(M, \mathcal{I}, n))$, then both $*e(N, \mathcal{I}, d)$ and $*e(P, \mathcal{I}, d)$ exist and

$$*e(M,\mathcal{I},d) = *e(N,\mathcal{I},d) + *e(P,\mathcal{I},d).$$

Therefore, using Lemma 2.6a) as well, we have

Corollary 2.8. *Suppose that $M \in \mathfrak{grmod}(A)$, $\mathcal{I}$ is a GIOD for M and $M^\bullet$ is a graded filtration of M such that $0 = M^0 \subset M^1 \subset \cdots M^{N-1} \subset M^N = M$, and, for each $N \geq i \geq 1$, there are graded prime ideals $\mathfrak{p}_i$ in A, integers d_i and graded isomorphisms of A-modules $(A/\mathfrak{p}_i)(d_i) \cong M^i/M^{i-1}$. Then,*

i) $\mathcal{I}$ is a GIOD for $A/\mathfrak{p}_i$ and $*p(A/\mathfrak{p}_i,\mathcal{I},n)$ exists, for $1 \leq i \leq N$.

ii) If $D \doteq \max\{\deg(*p(A/\mathfrak{p}_i,\mathcal{I},n)) \doteq d_i \mid 1 \leq i \leq N\}$ and $\mathcal{D}(M^\bullet) \doteq \{\mathfrak{p}_j \mid d_j = D\}$,

$$*e(M,\mathcal{I},D) = \sum_{\mathfrak{p}\in\mathcal{D}(M^\bullet)} n_{\mathfrak{p}}(M^\bullet)(*e(A/\mathfrak{p},\mathcal{I},D)),$$

where $n_{\mathfrak{p}}(M^\bullet)$ is equal to the number of times $A/\mathfrak{p}$, possibly shifted, occurs as an A-module isomorphic to a subquotient of the filtration $M^\bullet$. Furthermore, all of the integers on both sides of the equation are strictly positive.

Finally, we point out

Remark 2.9. *If S is a positively graded Noetherian ring with S_0 Artinian, $M \in \mathfrak{grmod}(S)$ and $\mathcal{I}$ is a GIOD for M, then Lemma 1.15 tells us that, when we forget the grading, $\mathcal{I}$ has the property that $\ell_S(M/\mathcal{I}^nM) = *\ell_S(M/\mathcal{I}^nM) < \infty$. Therefore, $*p(M,\mathcal{I},n) = p(M,\mathcal{I},n)$, where $p(M,\mathcal{I},n)$ is computed as in [25] after forgetting the grading. So, if $d \geq \deg(*p(M,\mathcal{I},n)) = \deg(p(M,\mathcal{I},n))$, $*e(M,\mathcal{I},d)$ is the exact same multiplicity $e(M,\mathcal{I},d)$ defined in [25] in the ungraded case, after forgetting the grading.*

*In the case that $(A,\mathcal{N})$ is a *local Noetherian ring such that $A-\mathcal{N}$ has a homogeneous element of degree 1, then lemma 1.16 tells us that for every j, and every $X \in \mathfrak{grmod}(A)$ such that $*\ell_A(X) < \infty$, $\ell_{A_0}(X_j) = *\ell_A(X)$ for every j. For any graded ideal $\mathcal{J}$ for M, it turns out in this case that $(\mathcal{J}M)_0 = \mathcal{J}_0M_0$: the containment "$\supseteq$" is clear; if $T \in A-\mathcal{N}$ has degree 1, every* *element of* $(\mathcal{J}M)_0$ *has the form* $\sum_j a_j x_j$ *where* $a_j \in \mathcal{J}$ *and* $x_j \in M$ *and* $\deg(a_j) + \deg(x_j) = 0$ *whenever* $a_j x_j \neq 0$. *However,* T *is invertible in* A, *and* $\sum_j a_j x_j = \sum_j (a_j T^{-\deg(a_j)})(T^{\deg(a_j)} x_j) \in \mathcal{J}_0 M_0$. *Therefore, if* $\mathcal{I}$ *is a GIOD for* M, *we see that* $\mathcal{I}_0$ *has the property that* $\ell_{A_0}(M_0/\mathcal{I}_0^n M_0) = *\ell_A(M/\mathcal{I}^n M) < \infty$ *for every* n, *thus* $*p(M, \mathcal{I}, n) = p(M_0, \mathcal{I}_0, n)$, *where* $p(M_0, \mathcal{I}_0, n)$ *is the ordinary Samuel polynomial constructed in the ungraded case for the* A_0*-module* M_0. *Therefore, in this case, for* $d \geq \deg(*p(M, \mathcal{I}, n)) = \deg(p(M, \mathcal{I}, n))$, $*e(M, \mathcal{I}, d)$ *is equal to the multiplicity* $e(M_0, \mathcal{I}_0, d)$ *defined in the ungraded case.*

2.1.1 *Dimension, *Samuel polynomials and GSOPs for *local rings

In this section, A is a *local Noetherian graded ring with unique *maximal graded ideal $\mathcal{N}$. Here we present an analogue in the graded category to the fundamental theorem of dimension theory for local rings. This theorem shows the relationship between *Krull dimension, graded systems of parameters, and the degree of the *Samuel polynomial.

When working over $\mathfrak{grmod}(R)$, for R positively graded and R_0 a field, we combine the fundamental dimension theorem for *local rings to Smoke's dimension theorem (1.46). In this case, the order of the pole of the Poincare series at $t = 1$, equals the measures from the fundamental *local dimension theorem, which in turn equal the ungraded Krull dimension. This is summarized in corollary 2.11. Returning to the theory of multiplicities, we conclude the section with a sum decomposition of the *Samuel multiplicity by minimal primes (corollary 2.13).

Suppose that $\mathcal{I}$ is a GIOD for M, thus the only graded primes containing $\mathcal{I} + Ann_A(M)$ are *maximal. But $\mathcal{N}$ is the only graded *maximal ideal in A, thus $\mathcal{I}$ is a GIOD for M if and only if $\mathcal{N}$ is the only graded prime ideal of A containing $\mathcal{I} + Ann_A(M)$. The previous section shows that the degree of the *Samuel polynomial of M with respect to the $\mathcal{I}$-adic filtration does not depend on the choice of $\mathcal{I}$. We call this degree $*d(M)$. Of course, $\mathcal{N}$ is always a GIOD for M.

If $M \in \mathfrak{grmod}(A)$, $M \neq 0$, $*s(M)$ is defined to be the least s such that there exist homogeneous elements $w_1, \ldots, w_s \in \mathcal{N}$ such that the graded A-module $M/(w_1, \ldots, w_s)M$ has finite *length over A. Note that $*s(M) = 0$ if and only if $*\ell_A(M) < \infty$.

The fundamental theorem for *local dimension theory is:

Theorem 2.10. *If* $(A, \mathcal{N})$ *is a *local Noetherian ring and* $M \in \mathfrak{grmod}(A)$*, then*

$$* \dim_A(M) = *d(M) = *s(M).$$

Proof. The proof of this mimics the proof of the analogous theorem in the ungraded, local case given in [25] in Section III.B.2, Theorem 1, but we give a sketch anyway. First, if x is a homogeneous element of $\mathcal{N}$, let ${}_xM$ be the graded A-module consisting of all elements m of M such that $xm = 0$. If $\deg(x) = d$, then there are short exact sequences in $\mathfrak{grmod}(A)$

$$0 \to_x M(-d) \to M(-d) \xrightarrow{\cdot x} xM \to 0,$$

$$0 \to xM \to M \to M/xM \to 0.$$

If $\mathcal{I}$ is a GIOD for M, it is also a GIOD for every module in the exact sequences above. Furthermore, the short exact sequences and Lemma 2.6 say that $*p({}_xM, \mathcal{I}, n) - *p(M/xM, \mathcal{I}, n)$ is a polynomial of degree strictly less than $*d(M)$. It's straightforward to see that $*s(M) \leq *s(M/xM)+1$.

We may as well assume that the GIOD we are using to calculate $*d(M)$ is $\mathcal{N}$.

Next, set $\mathcal{D}(M)$ to be the (finite) set of all $\mathfrak{p}$ in $*V(M)$ with the property that ${}^*\dim_A(M) = {}^*\dim_A(A/\mathfrak{p}) = {}^*\dim(A/\mathfrak{p})$; it's important to note that $\mathcal{D}(M)$ could also be defined as the set of all primes in $V(M)$ with $\dim_A(M) = \dim_A(A/\mathfrak{p})$ since the minimal elements in the sets $*V(M)$ and $V(M)$ are exactly the same. If a homogeneous element x is not in any prime of $\mathcal{D}(M)$, then ${}^*\dim_A(M/xM) < {}^*\dim_A(M)$; this is true for exactly the same reason as in the ungraded case: $*V(M/xM) = *V((x) + Ann_A(M))$.

Finally, one proceeds to the proof by first arguing that ${}^*\dim_A(M) \leq *d(M)$, then $*d(M) \leq *s(M)$, and lastly, $*s(M) \leq {}^*\dim_A(M)$.

For the first inequality one uses induction on $*d(M)$. Note that $*d(M) = 0$ means that there is a q such that $*\ell_A(M/\mathcal{N}^iM) = *\ell_A(M/\mathcal{N}^{i+1}M)$ for all $i \geq q$. But this forces $\mathcal{N}^qM = \mathcal{N}^{q+1}M$ and graded Nakayama's lemma says that $\mathcal{N}^q M = 0$, so that $*V(M)$ has exactly one ideal, $\mathcal{N}$ in it. By definition, ${}^*\dim_A(M) = 0$. Supposing that $*d(M) \geq 1$, as in [25], we reduce to the case $M = A/\mathfrak{p}$ for some graded prime ideal $\mathfrak{p}$ properly contained in $\mathcal{N}$. Taking a chain of graded prime ideals $\mathfrak{p} \doteq \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ in A, we may suppose that $n \geq 1$, and thus may choose a homogeneous element x in $\mathfrak{p}_1$ that is not in $\mathfrak{p}$. Since $x \notin \mathfrak{p}$, but $x \in \mathfrak{p}_1$, the chain $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ corresponds to a chain of primes in $*V(M/xM)$. Since $M = A/\mathfrak{p}$, and $x \notin \mathfrak{p}$, ${}_xM = 0$, so that $*p(M/xM, \mathcal{N}, n)$ has degree strictly less than $*d(M)$, and by induction, ${}^*\dim_A(M/xM) \leq *d(M/xM)$. Thus, $n - 1 \leq {}^*\dim_A(M/xM) \leq *d(M) - 1$ and $n \leq *d(M)$. This forces ${}^*\dim_A(M) \leq *d(M)$.

For the second inequality, if $x_1, \ldots, x_k$ is a list of homogeneous elements of $\mathcal{N}$ that generate a GIOD $\mathcal{I}$ for M, we must have that $*V(\mathcal{I} + Ann_A(M))$ contains only $\mathcal{N}$, so that $*p(M, \mathcal{I}, n)$ and $*p(M, \mathcal{N}, n)$ have the same degree $*d(M)$. But, Lemma 2.6 says that $*p(M, \mathcal{I}, n)$ has degree less than or equal to k. Thus, $*d(M) \leq *s(M)$.

For the third inequality, use induction on ${}^*\dim_A(M)$, which we may assume to be at least 1, since ${}^*\dim_A(M) = 0$ if and only if $\mathcal{N}$ is the only prime in $*V(M)$, so that M has finite *length and $*s(M) = 0$ by definition. If ${}^*\dim_A(M) \geq 1$, none of the primes in $\mathcal{D}(M)$ are *maximal, so there is a homogeneous element $x \in \mathcal{N}$ such that x is not in any of the primes in $\mathcal{D}(M)$. We've noted above that $*s(M) \leq *s(M/xM) + 1$ and ${}^*\dim_A(M) \geq {}^*\dim_A(M/xM) + 1$. These inequalities plus the induction hypotheses give us the result. $\square$

If R is a positively graded ring with $R_0 = k$ a field, then (R, R_+) is a *local ring, so we may apply the fundamental theorem for *local dimension. On the other hand, recall Smoke's dimension theorem (theorem 1.46). For any $M \in \mathfrak{grmod}(R)$ the hypotheses for Smoke's dimension theorem are satisfied, and we may therefore combine the two dimension theorems. For the reader's convenience we reprint the definitions of $d_1(M)$ and $s_1(M)$ from Smoke's theorem:

- If $M \in \mathfrak{grmod}(R)$, $d_1(M)$ is the least j such that there exist positive integers $f_1, \ldots, f_j$ with $(\prod_{i=1}^{j}(1 - t^{f_i}))P_M(t) \in \mathbb{Z}[t, t^{-1}]$.

- $s_1(M)$ is the least s such that there exist homogeneous elements $y_1, \ldots, y_s \in R_+$ with M finitely generated over $R_0\langle y_1, \ldots, y_s\rangle \subseteq R$.

Combining the two dimension theorems yields:

Corollary 2.11. *If R is a positively graded Noetherian ring with R_0 a field and $M \in \mathfrak{grmod}(R)$, then*

$$* \dim_R(M) = *d(M) = *s(M) = s_1(M) = d_1(M) = \dim_R(M).$$

Going back to the definition of a GSOP for the A-module M, as a corollary to Theorem 2.10. we have

Corollary 2.12. *If $(A, \mathcal{N})$ is a *local graded Noetherian ring and $M \in \mathfrak{grmod}(A)$, then a GSOP exists for M, and the length of every GSOP is equal to $* \dim_A(M) = *d(M) = *s(M)$. Moreover, if $A - \mathcal{N}$ has a homogeneous element T of degree 1, necessary invertible, and $d(M_0)$ is the degree of the ordinary Samuel polynomial $p(M_0, \mathcal{N}_0, n)$, then $d(M_0) = *d(M)$, and if $x_1, \ldots, x_D$ is a GSOP for M, where $D = {}^*\dim_A(M) = *d(M) = d(M_0)$, then $x_1T^{-e_1}, \ldots, x_DT^{-e_D}$ is an ordinary system of parameters for M_0 as an A_0-module, if $e_i = \deg(x_i)$.*

Proof. The first statement is clear using 2.10; for the second use Remark 2.9 to see that $*d(M) = d(M_0)$; if $x_1, \ldots, x_D$ generate a GIOD $\mathcal{I}$ for M, then $\mathcal{I}_0$ is generated by $x_1T^{-e_1}, \ldots, x_DT^{-e_D}$. Therefore, the ungraded dimension theorem ensures that $D = d(M_0) = \dim_{A_0}(M_0)$, so
$x_1T^{-e_1}, \ldots, x_DT^{-e_D}$ is an ordinary system of parameters for M_0. ☐

We also have a corollary to Corollary 2.8; here $\mathcal{D}(M)$ is defined as the set of minimal primes of maximal dimension (as in the proof of Theorem 2.10)

Corollary 2.13. *Suppose that $(A, \mathcal{N})$ is a *local graded Noetherian ring, $M \in \mathfrak{grmod}(A)$, $\mathcal{I}$ is a GIOD for M and $M^\bullet$ is a graded filtration of M such that $0 = M^0 \subset M^1 \subset \cdots M^{N-1} \subset M^N = M$, and, for each $N \geq i \geq 1$, there are graded prime ideals $\mathfrak{p}_i$ in A, integers d_i and graded isomorphisms of A-modules $(A/\mathfrak{p}_i)(d_i) \cong M^i/M^{i-1}$. Then, if $D \doteq {}^*\dim_A(M)$,*

$$*e(M, \mathcal{I}, D) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} *\ell_{A_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]})(*e(A/\mathfrak{p}, \mathcal{I}, D)).$$

Proof. Lemma 1.5 tells us that, for every minimal prime $\mathfrak{p}$ for M, there is at least one subquotient of the filtration isomorphic to the graded A-module $A/\mathfrak{p}$, possibly shifted. Therefore, adding the *local hypothesis,

- $\mathcal{D}(M^\bullet) = \mathcal{D}(M)$ since we now know that, for every shift d, the degree of $*p((A/\mathfrak{p})(d), \mathcal{I}, n) = *p(A/\mathfrak{p}, \mathcal{I}, n)$ is independent of the choice of $\mathcal{I}$ and is equal to $^*\dim_A(A/\mathfrak{p})$. Theorem 2.10 also tells us that the D in this corollary is exactly the same D as in Corollary 2.8.

- Moreover, for every prime $\mathfrak{p}$ in $\mathcal{D}(M)$, $A/\mathfrak{p}$, possibly shifted, occurs exactly $*\ell_{A_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \ell_{A_\mathfrak{p}}(M_\mathfrak{p})$ times (using Theorem 1.26 and Corollary 2.8) as a subquotient of the filtration $M^\bullet$, so that $n_\mathfrak{p}(M^\bullet) = *\ell_{A_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \ell_{A_\mathfrak{p}}(M_\mathfrak{p})$.

□

2.2 Koszul complexes in grmod(A) and *Koszul multiplicities

As usual, A is a Noetherian $\mathbb{Z}$-graded ring and $M \in \mathfrak{grmod}(A)$.

The definition of a complex of modules in $\mathfrak{grmod}(A)$ is as usual: this is a sequence $(\mathbf{M}, \partial)$

$$\cdots \to M_j \xrightarrow{\partial_j} M_{j-1} \xrightarrow{\partial_{j-1}} \cdots \to \xrightarrow{\partial_1} M_0$$

of objects and morphisms in $\mathfrak{grmod}(A)$, such that $\partial\partial = 0$ everywhere. The sequence of morphisms ∂ above is called the differential for the complex.

The subscripts j seem assigned ambiguously, but here's what we mean: If $(\mathbf{M}, \partial)$ is a complex in $\mathfrak{grmod}(A)$ as above, then the set of elements of M_j of degree i is equal to

$$(M_j)_i \doteq M_{j,i}.$$

In other words, when speaking of a complex in $\mathfrak{grmod}(A)$, a single integer subscript denotes the sequential index of the complex, and a doubly-indexed subscript is read as "first index is the complex index, second is the graded-module index". We will often suppress the internal gradings, so if there is just one subscript, it refers to the "complex index". Hopefully this won't be too confusing.

To further set notation, we will regard any M in $\mathfrak{grmod}(A)$ as a "complex concentrated in degree 0"–this is the complex with all differentials equal to zero, with $M_i = 0$, if the "complex-index" $i \neq 0$, and $M_0 = M$, for "complex-index " 0.

The homology groups of a complex $(\mathbf{M}, \partial)$ are defined as " $\ker \partial / \mathrm{im}\ \partial$" of course, and are also in $\mathfrak{grmod}(A)$:

$$H_j(\mathbf{M})_i = \ker(\partial : M_{j,i} \to M_{j-1,i})/\mathrm{im}(\partial : M_{j+1,i} \to M_{j,i}).$$

Morphisms of graded complexes and short exact sequences of graded complexes are defined in the usual way.

A short exact sequence of graded complexes in $\mathfrak{grmod}(A)$ gives rise to a long exact sequence on homology: if $0 \to \mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C} \to 0$ is a short exact sequence of graded complexes in $\mathfrak{grmod}(A)$, there exists a graded morphism ω of complex degree -1 such that the sequence

$$\cdots \xrightarrow{\omega_{j+1}} H_j(\mathbf{A}) \xrightarrow{\alpha_*} H_j(\mathbf{B}) \xrightarrow{\beta_*} H_j(\mathbf{C}) \xrightarrow{\omega_j} H_{j-1}(\mathbf{A}) \xrightarrow{\alpha_*} \cdots$$

is an exact sequence in $\mathfrak{grmod}(A)$.

Definition 2.14. *The* ****Euler characteristic*** *of a complex $(\mathbf{M}, \partial)$ in $\mathfrak{grmod}(A)$ is defined when $(\mathbf{M}, \partial)$ is such that each A-module M_i has $*\ell_A(M_i) < \infty$ and for all but finitely many i, $*\ell_A(M_i) = 0$. Given these conditions, the following sum is well defined:*

$$*\chi(\mathbf{M}) \doteq \Sigma_i(-1)^i * \ell_A(M_i).$$

Since $*\ell$ sums over short exact sequences, we get the following lemma.

Lemma 2.15. *Let* $0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$ *be a short exact sequence of graded complexes in* $\mathfrak{grmod}(A)$. *Then, the *Euler characteristic of* $\mathbf{B}$ *is defined if and only if the *Euler characteristics of* $\mathbf{A}$ *and* $\mathbf{C}$ *are, and* $*\chi(\mathbf{B}) = *\chi(\mathbf{A}) + *\chi(\mathbf{C})$.

If the two conditions for a well-defined *Euler characteristic of a complex are not met, there may be a way to salvage the situation by passing to homology.

Definition 2.16. *Let* $(\mathbf{M}, \partial)$ *be a complex in* $\mathfrak{grmod}(A)$ *such that for every* i, $H_i(\mathbf{M})$ *has finite* $*length$ *over* A, *and for* $i >> 0$ $*\ell(H_i(\mathbf{M})) = 0$. *We define the *Euler characteristic of the homology to be* $*\chi(\mathbf{H}(\mathbf{M})) \doteq \Sigma_i(-1)^i * \ell_A(H_i(\mathbf{M}))$.

With the proof exactly analogous to that in the ungraded case, we have

Theorem 2.17. *When the *Euler characteristic* $*\chi(\mathbf{M})$ *is defined, then* $*\chi(\mathbf{H}(\mathbf{M}))$ *is also defined, and we have that* $*\chi(\mathbf{M}) = *\chi(\mathbf{H}(\mathbf{M}))$.

Note however that the converse is not necessarily true; i.e. $*\chi$ of the homology may be defined but $*\chi$ of the complex not.

Using the additivity of $*\ell$, and the long exact sequence on homology, if $\mathbf{A} \rightarrowtail \mathbf{B} \twoheadrightarrow \mathbf{C}$ is a short exact sequence of graded complexes in $\mathfrak{grmod}(A)$ such that the *Euler characteristic of the homology of each complex is defined, then,

$$*\chi(\mathbf{H}(\mathbf{B})) = *\chi(\mathbf{H}(\mathbf{A})) + *\chi(\mathbf{H}(\mathbf{C})).$$

If A is a graded ring, we may do homological algebra in $\mathfrak{grmod}(A)$ quite analogously to how it's done in the ungraded case. In particular, graded A-modules $Tor_i^A(M, N) \in \mathfrak{grmod}(A)$ for every $i \geq 0$, and $M, N \in \mathfrak{grmod}(A)$ may be defined mimicking the constructions and definitions in the ungraded category: beginning with the graded tensor product $M \otimes_A N$. (For the definition of the graded tensor product of graded modules over a graded ring, see [13].) The tensor product $M \otimes_A N$ has a natural grading on it: if $m \in M_i$ and $n \in N_j$ are homogeneous elements, then

$\deg(m \otimes n) = i + j$. Then, one proceeds to talk about projective resolutions, and arrives at the definition of $Tor_i^A(M, N) \in \mathfrak{grmod}(A)$. We do not give further details here.

2.2.1 The Koszul complex

Standard properties of the Koszul complex in the ungraded case may be found in [25]. We use Serre's notation: if $\bar{x} \doteq x_1, \ldots, x_u$ is a sequence of elements in A, then the Koszul complex is denoted by $\mathbf{K}(\bar{x}, A)$.

If we pass to the graded category, with A a graded ring, and choose a sequence $\bar{x} \doteq x_1, \ldots, x_u$ of homogeneous elements of A, the definition of the graded Koszul complex is briefly summarized as follows. Recall that the tensor product of graded complexes $\mathbf{C} \otimes_A \mathbf{D}$ is defined exactly analogously to the ungraded case, and is again a graded complex; keep in mind in particular the definition of the differential of a tensor product of complexes: if $c \in C_i$ and $d \in D_j$, then $\partial_{C \otimes_A D}(c \otimes d) = \partial_C(c) \otimes d + (-1)^i c \otimes \partial_D(c)$. Starting with the case $u = 1$, $\mathbf{K}(x_1, A)$ is the two-term complex in $\mathfrak{grmod}(A)$

$$K_1(x_1, A) = A(-d) \stackrel{\cdot x_1}{\to} K_0(x_1, A) = A,$$

where d is the degree of x_1. Then, if $\bar{x} = x_1, \ldots, x_u$ is a sequence of homogeneous elements in A,

$$\mathbf{K}(\bar{x}, A) \doteq \mathbf{K}(x_1, A) \otimes_A \cdots \otimes_A \mathbf{K}(x_u, A).$$

If $M \in \mathfrak{grmod}(A)$, the Koszul complex associated to the graded A-module M and $\bar{x}$ is:

$$\mathbf{K}^A(\bar{x}, M) \doteq \mathbf{K}(\bar{x}, A) \otimes_A M.$$

If we're always regarding a graded abelian group M as an A-module, for a fixed graded ring A, we will often delete the superscript A.

Setting notation, $K_0(x_i, A)$ is identified with A as a free, graded A-module (in other words, the free generator lies in degree zero, and is identified with $1 \in A_0$). For K_1, choose e_{x_i} of $\deg(x_i)$ and identify $K_1(x_i, A)$ with the free graded A-module generated by the homogeneous element e_{x_i}. Then, $K_p(\bar{x}, A)$ is the free graded A-module isomorphic to the free graded A-module generated by the homogeneous elements $e_{x_{i_1}} \otimes \cdots \otimes e_{x_{i_p}}$ of degree $\deg(x_{i_1}) + \cdots + \deg(x_{i_p})$ where $i_1 < \cdots < i_p$, so is isomorphic to the graded exterior product

$$\Lambda^p(A(-\deg(x_1)) \oplus \cdots \oplus A(-\deg(x_u))).$$

In addition, both the pth part of the Koszul complex $\mathbf{K}^A(\bar{x}, M)$, and its differential have exactly the same form as described in [25], IV.A.2 in the ungraded case. A particular consequence is that, as a graded A-module, $K_p^A(\bar{x}, M)$ is a direct sum of $\binom{u}{p}$ copies of M, each shifted: the copy associated to the multi-index $i_1 < \cdots i_p$ looks like $M(-(\deg(x_{i_1}) + \cdots + \deg(x_{i_p})))$; if $\mathcal{I}$ is the graded ideal of A generated by $x_1, \ldots, x_u$, and $k \geq 0$, then $K_p^A(\bar{x}, M)/\mathcal{I}^k K_p^A(\bar{x}, M)$ is, as a graded A-module, isomorphic to $\binom{u}{p}$ copies of $M/\mathcal{I}^k M$, each shifted as described above.

The pth homology group of the graded Koszul complex $\mathbf{K}^A(\bar{x}, M)$ is denoted by $H_p(\bar{x}, M)$, or $H_p^A(\bar{x}, M)$ if we need to emphasize the role of A. These homology groups are also graded A-modules.

Definition 2.18. *Suppose that $x_1, \ldots, x_u$ is a sequence of homogeneous elements in A and M is in $\mathfrak{grmod}(A)$. This sequence is a M-sequence if and only if x_1 is not a zero-divisor on M, and for each $i > 1$, x_i is not a zero-divisor on $M/(x_1, \ldots, x_{i-1})M$.*

The following may all be proved as in the ungraded case (see [25]):

Proposition 2.19. *Let A be a graded ring and $M \in \mathfrak{grmod}(A)$. If $\bar{x}$ is a M-sequence, then the Koszul complex $\mathbf{K}^A(\bar{x}, M)$ is acyclic. As in the ungraded case, $H_0^A(\bar{x}, M) = M/(x_1, \ldots, x_u)M$.*

Conversely, in the *local Noetherian case one has

Proposition 2.20. *If $(A, \mathcal{N})$ is a *local Noetherian ring, and $M \in \mathfrak{grmod}(A)$, then the following are equivalent, for a sequence of homogeneous elements $\bar{x} \doteq x_1, \ldots, x_u$ of $\mathcal{N}$:*

- $H_p^A(\bar{x}, M) = 0$, *for* $p \geq 1$.

- $H_1^A(\bar{x}, M) = 0$.

- $\bar{x}$ *is an* M*-sequence in* A.

The proofs of the above Propositions are exactly analogous as that of IV.A.2, Propositions 2, 3 in [25], replacing any use of Nakayama's lemma with the graded Nakayama's lemma (Lemma 1.28); similarly, IV.A.2, Corollary 2 yields, in the graded case,

Corollary 2.21. *If* $(A, \mathcal{N})$ *is a *local Noetherian ring,* $M \in \mathfrak{grmod}(A)$*, and* $\bar{x} = x_1, \ldots, x_u$ *are homogeneous elements of* $\mathcal{N}$ *that form an* A*-sequence for* A*, then there is a natural isomorphism of graded* A*-modules*

$$\psi : H_i^A(\bar{x}, M) \to Tor_i^A(A/(\bar{x}), M).$$

Example 2.22. *Hilbert's syzygy theorem states that for a finitely generated graded module* M *over a polynomial ring* $R = k[x_1, \ldots, x_n]$*, a free resolution of length at most* n *exists.*

Certainly $\bar{x}$ *forms an* R*-sequence and thus,* $\mathbf{K}^R(\bar{x}, R)$ *is a free and acyclic complex which forms a free resolution of* k*.*

Thus, $Tor_i^R(k, M) \cong Tor_i^R(M, k) = H_i^R(\bar{x}, M)$*. It is a standard fact that* M *has a free resolution of the form* $\cdots \to Tor_i^R(M, k) \otimes_k R \to \cdots \to Tor_0^S(M, k) \otimes_k R \to 0$ *, but each term* $Tor_i^R(M, k) = H_i^R(\bar{x}, M)$ *and since the exterior algebra is* 0 *past the dimension of* n*, so too is this resolution.*

Finally, IV.A.2, Proposition 4, has the analogous

Proposition 2.23. *Suppose that* $(A, \mathcal{N})$ *is a *local graded Noetherian ring and* $M \in \mathfrak{grmod}(A)$*. If* $x_1, \ldots, x_u$ *are homogeneous elements of* $\mathcal{N}$*, then* $(\bar{x}) + Ann_A(M) \subseteq Ann_A(H_i^A(\bar{x}, M))$*.*

As a corollary,

Proposition 2.24. *Suppose that* $(A, \mathcal{N})$ *is a *local graded Noetherian ring , and* $M \in \mathfrak{grmod}(A)$*. Let* $\mathcal{I}$ *be a GIOD for* M*, generated by the homogeneous sequence* $\bar{x} = x_1, \ldots, x_u \in \mathcal{N}$*. Then,* $H_j^A(\bar{x}, M)$ *has finite *length over* A *for every* $j \geq 0$*.*

Proof. Since $\mathcal{I} + Ann_A(M) \subseteq Ann_A(H_j^A(\bar{x}, M))$, and $\{\mathcal{N}\} = *V(\mathcal{I} + Ann_A(M))$, if $H_j^A(\bar{x}, M) \neq 0, \{\mathcal{N}\} = *V(Ann_A(H_j^A(\bar{x}, M)))$.

□

Thus, in the *local case the *Euler characteristic of the homology of the graded Koszul complex is well defined:

Definition 2.25. *Suppose that $(A, \mathcal{N})$ is a *local, Noetherian graded ring and $M \in \mathfrak{grmod}(A)$. Let $\mathcal{I}$ be a GIOD for $M \in \mathfrak{grmod}(A)$ generated by a homogeneous sequence $\bar{x} = x_1, \ldots, x_u$. We define the ****Koszul multiplicity*** $*\chi^A(\bar{x}, M)$ *to be the *Euler characteristic of the homology of the graded Koszul complex:*

$$*\chi^A(\bar{x}, M) = \sum_{i=1}^{u} (-1)^i * \ell_A(H_i^A(\bar{x}, M)).$$

2.2.2 Equality of *Samuel and *Koszul multiplicities

As in the ungraded case [25], IV.A.3, the *Koszul multiplicity is equal to a certain *Samuel multiplicity. This section concludes our account of the theory of multiplicities adapted to the $\mathbb{Z}$-graded category.

Let $(A, \mathcal{N})$ be a *local, Noetherian graded ring, $M \in \mathfrak{grmod}A$ and $\bar{x} = x_1, \ldots, x_u$ a sequence of homogeneous elements contained in $\mathcal{N}$. If $\mathcal{I}$ is the graded ideal of A generated by $\bar{x}$, suppose also that $\mathcal{I}$ is a GIOD for M.

One then filters the graded Koszul complex, yielding graded complexes $\mathcal{F}^i\mathbf{K}$ for every i, with $\mathcal{F}^i K_p \doteq \mathcal{I}^{i-p} K_p$ for every p (we've dropped the arguments $\bar{x}, M$ for expediency). Notice we have three indices now: the filtration index, the complex index and the internal gradings of the various A-modules involved. We are suppressing the internal grading. This filtration defines the associated graded complex $gr(\mathbf{K}) \doteq \oplus_i \mathcal{F}^i\mathbf{K}/\mathcal{F}^{i+1}\mathbf{K}$.

If $gr(A)$ is the associated bigraded ring to the $\mathcal{I}$-adic filtration, then denote the images of $x_1, \ldots, x_u$ in $gr(A)_{1,*}$ by $\xi_1, \ldots, \xi_u$. Let $gr(M)$ be the bigraded $gr(A)$-module associated with the $\mathcal{I}$-adic filtration of M. Then, there is an isomorphism of graded objects $gr(\mathbf{K}) \cong \mathbf{K}(\bar{\xi}, gr(M))$.

Moreover, one argues that the homology modules $H_p(\bar{\xi}, gr(M))$ have finite *length over $gr(A)$, for all p, since $A/(\mathcal{I} + Ann_A(M))$ is *Artinian. This in turn, enables one to argue that there exists an $m \geq u$ such that the graded homology groups of the complex $\mathcal{F}^i\mathbf{K}/\mathcal{F}^{i+1}\mathbf{K}$ all vanish for $i > m$, and so one sees that the graded homology groups the complex $\mathcal{F}^i\mathbf{K}$ all vanish if $i > m$.

Continuing as in [25], IV.A.3, (which is really a spectral sequence argument), this means there is an m such that if $i > m$, then $H_p(\mathbf{K}) \cong H_p(\mathbf{K}/\mathcal{F}^i\mathbf{K})$ for $i > m$ and for all p.

Using the fact that *Euler characteristics don't change when passing to homology, $*\chi(\bar{x}, M) = \sum_p (-1)^p * \ell(H_p(\mathbf{K}/\mathcal{F}^i(\mathbf{K})) = *\chi(\mathbf{K}/\mathcal{F}^i\mathbf{K})$, for $i > m$. As noted in the previous section, $(\mathbf{K}/\mathcal{F}^i\mathbf{K})_p$ is isomorphic as a graded A-module to a direct sum of $\binom{u}{p}$ copies of $M/\mathcal{I}^{i-p}M$, shifted appropriately, and since the *length of a shifted A-module $M(d)$ is the same as the *length of M, the remainder of the proof is argued exactly as in [25], IV.A.3, with length replaced by *length, p (a Samuel polynomial) replaced by $*p$ and e replaced by $*e$.

Thus, we have

Theorem 2.26. *Let $(A, \mathcal{N})$ be a *local Noetherian ring. Let $x_1, \ldots, x_u \in \mathcal{N}$ be homogeneous elements generating a graded ideal of definition $\mathcal{I}$ for $M \in \mathfrak{grmod}(A)$. Then,*

$$*\chi^A(\bar{x}, M) = *e(M, \mathcal{I}, u),$$

so $*\chi^A(\bar{x}, M)$ is a strictly positive integer if ${}^*\dim_A(M) = u$, and $*\chi^A(\bar{x}, M) = 0$ if $u > {}^*\dim_A(M)$.

Chapter 3

Multiplicities and Degree for Positively Graded Rings

In this chapter, we specialize to the case of a positively graded Noetherian ring R with R_0 a field k; all graded modules are in $\mathfrak{grmod}(R)$. Set $\mathfrak{m} = R_+$ and note that $(R, \mathfrak{m})$ is then a *local graded Noetherian ring. We do not want to make the assumption that R is generated by elements in degree 1.

In this chapter we may use the * notation even though we could just as well omit the * (e.g. If $M \in \mathfrak{grmod}(R)$, then $*\dim_R(M) = \dim_R(M)$.) This is done to emphasize the fact that all computations may be done in the graded category using the theory developed in the previous two chapters (which is often simpler than the ungraded theory.)

We introduce the *degree* of a graded module, show how it relates to *multiplicity (Theorem 3.8), and give a sum decomposition of degree by a certain set of minimal primes (Theorem 3.6.) Looking ahead, the main theorem of part 2 of this dissertation (theorem 6.8) is essentially a geometric recasting of the algebraic degree sum formula when applied to equivariant cohomology rings.

Since $R_0 = k$, $\ell_k(M_i) = \mathrm{vdim}_k(M_i)$ for every i, so the Poincaré series for M is equal to

$$P_M(t) = \sum_{i \in \mathbb{Z}} \mathrm{vdim}_k(M_i) t^i.$$

Furthermore, this Laurent series has a pole at $t = 1$ using the Hilbert-Serre theorem, and the order of the pole $d_1(M)$ at $t = 1$ is, by Smoke's dimension theorem, is exactly ${}^*\dim_R(M)$.

This leads to the definition of $\deg_R(M)$:

Definition 3.1. *If R is a positively graded Noetherian ring with $R_0 = k$ a field, $M \in \mathfrak{grmod}(R)$, $M \neq 0$ and $D(M) = {}^*\dim_R(M)$, then*

$$\deg_R(M) \doteq \lim_{t \to 1}(1-t)^{D(M)} P_M(t)$$

is a well-defined, strictly positive, rational number. For convenience, define $\deg_R(0) = 0$.

Often we delete the subscript R and just write $\deg(M)$.

3.1 Multiplicities and Euler-Poincaré series

If $X \in \mathfrak{grmod}(R)$ has finite *length as an R-module, since each R_i is finite-dimensional as a vector space over k, we may use Lemmas 1.15 and 1.16 to conclude that $\ell_R(X) = *\ell_R(X) = \mathrm{vdim}_k(X)$, where $\mathrm{vdim}_k(X)$ is the total dimension $\sum_j \mathrm{vdim}_k(X_j)$ of the graded k-vector space X. We may then prove:

Lemma 3.2. *Suppose R is a positively graded Noetherian ring with $R_0 = k$, a field, and $X \in \mathfrak{grmod}(R)$ is such that $*\ell_R(X) < \infty$. If B is a graded subring of R, Noetherian or not, with $B_0 = k = R_0$, then $X \in \mathfrak{grmod}(B)$, $*\ell_B(X) < \infty$ and $*\ell_B(X) = *\ell_R(X) = \ell_R(X) = \mathrm{vdim}_k(X) < \infty$.*

Proof. For, using Lemma 2.9 applied to R, each X_i is finite-dimensional over k, and there are integers t_0 and J such that $t_0 \le J$ with $X = \oplus_{j=t_0}^{J} X_j$. Also, $*\ell_R(X) = \mathrm{vdim}_k(X)$. However, since $k \subseteq B \subseteq R$, X is a finitely generated B-module. Whether B is Noetherian or not, since $B_j \subseteq R_j$ for every j, and R_j is finite-dimensional over k, so is B_j. Thus, using Lemma 2.9 applied to B, $*\ell_B(X) = \mathrm{vdim}_k(X)$ as well. $\square$

If $M \in \mathfrak{grmod}(R)$, then $\mathfrak{m}$ is a GIOD for M, and we may then calculate a *Samuel polynomial $*p_R(M, \mathfrak{m}, n)$ for M; Remark 2.9 says that this is the ordinary Samuel polynomial $p_R(M, \mathfrak{m}, n)$; Corollary 2.11 says that the degree of this polynomial is

$$D(M) \doteq *d(M) = *s(M) = s_1(M) = d_1(M) = {}^*\dim_R(M) = \dim_R(M).$$

Now, suppose that $\bar{x} = x_1, \ldots, x_{D(M)}$ is a GSOP for the R-module M. If $\mathcal{I}$ is the graded ideal in R generated by $\bar{x}$, $\mathcal{I}$ is a GIOD for M. We can change rings to $B \doteq k\langle x_1, \ldots, x_{D(M)}\rangle$, note that this is a graded polynomial ring over k in the indicated variables. The ideal $\hat{\mathcal{I}}$ generated by $\bar{x}$ in B is also a GIOD in B since clearly $\hat{\mathcal{I}}^n M = \mathcal{I}^n M$ for every n. Therefore Lemma 2.9 and the previous lemma guarantee that, for every n, the polynomials below are all equal, as indicated:

$$*p_R(M, \mathcal{I}, n) = p_R(M, \mathcal{I}, n) = p_B(M, \hat{\mathcal{I}}, n) = *p_B(M, \hat{\mathcal{I}}, n);$$

in particular, they all have the same degree $D(M)$, and the following positive integers are also all equal:

$$*e_R(M, \mathcal{I}, D(M)) = e_R(M, \mathcal{I}, D(M)) = e_B(M, \hat{\mathcal{I}}, D(M)) = *e_B(M, \hat{\mathcal{I}}, D(M)).$$

3.1.1 Euler-Poincaré series

The following lemma is found in Avramov and Buchweitz [1]; [24] contains a similar result.

Lemma 3.3. *(Lemma 7 of [1]) If $M, N \in \mathfrak{grmod}(R)$, then*

a. For each i, the graded R-module $Tor_i^R(M, N)$ has finite dimensional (over $k = R_0$) homogenous components $Tor_i^R(M, N)_j$, for every j; also, for every i, $Tor_i^R(M, N)_j = 0$ for $j << 0$. Thus one may form the Laurent series

$$P_{Tor_i^R(M,N)}(t) \doteq \sum_{j \in \mathbb{Z}} \mathrm{vdim}_k (Tor_i^R(M, N))_j t^j.$$

b. Furthermore,the alternating sum

$$\chi_R(M, N)(t) \doteq \sum_{i \geq 0} (-1)^i P_{Tor_i^R(M,N)}(t),$$

which is by definition the Euler-Poincaré series of M, N, is a well-defined Laurent series with integer coefficients and

$$P_R(t) \chi_R(M, N)(t) = P_M(t) P_N(t).$$

If a GSOP $\bar{x}$ is given for $M \in \mathfrak{grmod}(R)$, $B \doteq k\langle\bar{x}\rangle \subseteq R$ is then a graded polynomial ring over k (Proposition 1.49), and $M \in \mathfrak{grmod}(B)$, using Lemma 1.48. Whether we consider $M \in \mathfrak{grmod}(R)$, or $M \in \mathfrak{grmod}(B)$, the Poincare series of M does not change.

Example 2.22 (the Hilbert Syzygy Theorem) shows that the graded Koszul complex $\mathbf{K}^{k\langle\bar{x}\rangle}(\bar{x}, k)$ is acyclic, thus is a free, finite graded resolution of k as a graded $k\langle\bar{x}\rangle$-module. In particular, we may tensor this resolution with M and use it to compute $Tor_i^{k\langle\bar{x}\rangle}(k, M) = Tor_i^{k\langle\bar{x}\rangle}(M, k)$, showing that

$$Tor_i^{k\langle\bar{x}\rangle}(M, k) = H_i^{k\langle\bar{x}\rangle}(\bar{x}, M).$$

Lemma 3.4. *Let $\bar{x} = x_1, \ldots, x_{D(M)}$ be a GSOP for $M \in \mathfrak{grmod}(R)$, and let $\mathcal{I}$ be the graded ideal in R generated by $\bar{x}$. For every i, $P_{Tor_i^{k\langle\bar{x}\rangle}(M,k)}(t) \in \mathbb{Z}[t, t^{-1}]$, and therefore $\chi_{k\langle\bar{x}\rangle}(M, k)(t) \in \mathbb{Z}[t, t^{-1}]$. Furthermore,*

$$\chi_{k\langle\bar{x}\rangle}(M, k)(t) = \sum_{j=0}^{D(M)} (-1)^j P_{H_j^{k\langle\bar{x}\rangle}(\bar{x},M)}(t),$$

and evaluating this Laurent polynomial at $t = 1$, we compute

$$\chi_{k\langle\bar{x}\rangle}(M, k)(1) = {*}\chi^{k\langle\bar{x}\rangle}(\bar{x}, M) = {*}e_R(\mathcal{I}, M, D(M)) = {*}\chi^R(\bar{x}, M),$$

where $D(M) = {}^*\dim_R(M)$.

Proof. Lemma 3.3 shows the first part of the statement, and since the resolution $\mathbf{K}^{k\langle\bar{x}\rangle}(\bar{x}, k)$ is in any case zero for complex degree larger than $D(M)$, $Tor_i^{k\langle\bar{x}\rangle}(M, k)$ is also zero for $i > D(M)$, so

$$\chi_{k\langle\bar{x}\rangle}(M, k)(t) \doteq \sum_{j \geq 0} (-1)^j P_{Tor_j^{k\langle\bar{x}\rangle}(M,k)}(t) = \sum_{j=0}^{D(M)} (-1)^j P_{H_j^{k\langle\bar{x}\rangle}(\bar{x},M)}(t),$$

being a finite sum of Laurent polynomials, is a Laurent polynomial.

Setting $B \doteq k\langle\bar{x}\rangle$ yields,

$$*\ell_B(H_i^B(\bar{x}, M)) = \ell_B(H_i^B(\bar{x}, M)) = \mathrm{vdim}_k(H_i^B(\bar{x}, M)),$$

so

$$*\chi^B(\bar{x}, M) = \sum_{j=0}^{D(M)} (-1)^j \ell_B(H_j^B(\bar{x}, M)) = \sum_{j=0}^{D(M)} (-1)^j \mathrm{vdim}_k(H_j^B(\bar{x}, M)) \doteq \chi_B(M, k)(1).$$

As noted at the beginning of this section, if $\hat{\mathcal{I}}$ is the ideal generated by $\bar{x}$ in B, then $*e_B(M, \hat{\mathcal{I}}, D(M)) = *e_R(M, \mathcal{I}, D(M))$ and Theorem 2.26 says that

$$*\chi^B(\bar{x}, M) = *e_B(M, \hat{\mathcal{I}}, D(M)) = *e_R(M, \mathcal{I}, D(M)) = *\chi^R(\bar{x}, M).$$

$\square$

3.1.2 Degree of a Graded Module in $\mathfrak{grmod}(R)$

Given $M \in \mathfrak{grmod}(R)$, $\deg(M) > 0$, if $M \neq 0$, we can read off the degree of a module directly from the Poincare series if we expand it as a Laurent series about $t = 1$:

$$P_R(t) = \frac{\deg(M)}{(1-t)^{D(M)}} + \text{"higher order terms"}.$$

Lemma 3.5. *Suppose that* $0 \to N \to M \to P \to 0$ *is an exact sequence in* $\mathfrak{grmod}(R)$. *Then,*

- $D(M) = \max\{D(N), D(P)\}$.
- *If* $D(N) < D(M)$, *then* $\deg(M) = \deg(P)$.
- *If* $D(P) < D(M)$, *then* $\deg(M) = \deg(N)$.
- *If* $D(P) = D(N) = D(M)$, *then* $\deg(M) = \deg(N) + \deg(P)$.
- $\deg(M(d)) = \deg(M)$, *for every integer* d.

This immediately yields:

Theorem 3.6. *Let $M \in \mathfrak{grmod}(R)$, and $\mathcal{D}(M)$ be defined as in Theorem 2.10: this is the set of prime ideals $\mathfrak{p}$ in R, necessarily minimal primes for M and graded, such that ${}^*\dim_R(R/\mathfrak{p}) = {}^*\dim_R(M)$. Then,*

$$\deg(M) = \sum_{\mathfrak{p}\in\mathcal{D}(M)} {*\ell}_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

Proof. Choose a graded filtration $M^\bullet$ of M of the form in Lemma 1.5 We know that if $\mathfrak{p} \in \mathcal{D}(M)$, then the graded R-module $R/\mathfrak{p}$, possibly shifted, occurs exactly ${*\ell}_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \ell_{R_\mathfrak{p}}(M_\mathfrak{p})$ times (using Theorem 1.26) as a subquotient in the filtration. The lemma above then gives the result. □

We want to compare degree to our previously studied multiplicities.

Letting $\bar{x}$ be a GSOP for $M \in \mathfrak{grmod}(R)$, we've seen that $k\langle\bar{x}\rangle$ is a graded polynomial ring, and one directly calculates that

$$P_{k\langle\bar{x}\rangle}(t) = \frac{1}{\prod_{i=1}^{D(M)}(1-t^{d_i})},$$

where d_i is the degree of the homogeneous element x_i.

Now, using $M \in \mathfrak{grmod}(k\langle\bar{x}\rangle)$, and recalling that $P_M(t)$ is the same whether we consider $M \in \mathfrak{grmod}(R)$ or $M \in \mathfrak{grmod}(k\langle\bar{x}\rangle)$, Lemma 3.3 gives that

$$P_k(t)P_M(t) = P_{k\langle\bar{x}\rangle}(t)\chi_{k\langle\bar{x}\rangle}(M,k)(t).$$

Also, $\chi_{k\langle\bar{x}\rangle}(M,k)(t) \in \mathbb{Z}[t,t^{-1}]$. Since $P_k(t) = 1$, we have

Theorem 3.7. *If $M \in \mathfrak{grmod}(R)$ and $\bar{x}$ is a GSOP for M, then*

$$P_M(t) = \frac{\chi_{k\langle\bar{x}\rangle}(M,k)(t)}{\prod_{i=1}^{D(M)}(1-t^{d_i})},$$

with $\chi_{k\langle\bar{x}\rangle}(M,k)(t) \in \mathbb{Z}[t,t^{-1}]$.

Since

$$(1-t)^{D(M)}P_M(t) = \frac{\chi_{k\langle\bar{x}\rangle}(M,k)(t)}{\prod_{i=1}^{D(M)}(1+t+\cdots+t^{d_i-1})},$$

taking the limit as t approaches 1 yields:

Theorem 3.8. *If $M \neq 0$ is in $\mathfrak{grmod}(R)$, and $x_1, \ldots, x_{D(M)}$ of degrees $d_1, \ldots, d_{D(M)}$ form a GSOP for M, generating the graded ideal $\mathcal{I}$ of R, then*

$$\deg(M) = \frac{{}^*e_R(M, \mathcal{I}, D(M))}{d_1 \cdots d_{D(M)}} = \frac{{}^*\chi^R(\bar{x}, M)}{d_1 \cdots d_{D(M)}}.$$

Thus, the ratio

$$\frac{{}^*e_R(M, \mathcal{I}, D(M))}{d_1 \cdots d_{D(M)}}$$

is independent of the choice of system of parameters $x_1, \ldots, x_{D(M)}$ for M.

Note that we can delete the “stars" in the equalities of the above theorem and retain the equalities, using Remark 2.9.

Example 3.9. *Let $R = k[x, y, z]/(zy^2 - x^3)$ have the standard grading. We have that*

$$P_R(t) = \frac{1 + t + t^2}{(1 - t)^2}.$$

If we expand as a Laurent series about $t = 1$ we get

$$P_R(t) = \frac{3}{(1 - t)^2} - \frac{3}{1 - t} + 1.$$

Thus, $deg(R) = 3$.

Now, suppose that $R = k[x, y]/(y^2 - x^3)$, with $\deg(x) = 2$ and $\deg(y) = 3$. Then,

$$\begin{aligned} P_R(t) &= \frac{1}{(1 - t)^1} \left[\frac{1 + t + \cdots + t^5}{(1 + t)(1 + t + t^2)} \right] \\ &= \frac{1}{(1 - t)^1} - t. \end{aligned}$$

Here, the change in grading correspondingly changes the degree: $\deg(R) = 1$. Note that $x \in R$ forms a GSOP for R, since R is finitely generated as a $k\langle x \rangle$-module by $1, y$. Similarly, y is also a

GSOP for R, generated by $1, x, x^2$ as a $k\langle y\rangle$-module. One may compute directly that $*\mathrm{e}(R, x) = 2$ and $*\mathrm{e}(R, y) = 3$. Also, using the above theorem: $\frac{*e(R,x)}{2} = \frac{*e(R,y)}{3} = \deg(R)$.

Part II

Applications to Equivariant Cohomology Rings

In this part of the dissertation we will consider the case where G is a compact Lie group (or finite group) and X is a topological space on which G acts continuously.

Below we will recall some basic definitions. Some references for this material are [4] [7]. A compact Lie group is a group which is also a compact smooth manifold (second countable and Hausdorff), such that the following two maps are smooth with respect to the topology on G: The multiplication map $\mu : G \times G \to G$ defined $\mu(g, h) \doteq gh$, and the inversion map $\nu : G \to G$ defined $\nu(g) \doteq g^{-1}$.

Assume that X has a left G-action. The orbit space X/G is the quotient space on X defined by the equivalence relation $x \sim y$ if and only if there exists a $g \in G$ such that $x = gy$. Given an $x \in X$, the isotropy subgroup is defined $G_x \doteq \{g \in G : gx = x\}$. The reader can verify that G_x is actually a subgroup of G for all $x \in X$. An orbit of $x \in X$ under G is defined $Gx \doteq \{gx \in X : g \in G\}$. It's straightforward to make corresponding definitions for a right G-action on X.

A principal G-bundle is a fiber bundle $p : E \to B$, with a right G-action on E, fiber equivariantly homeomorphic to G, with the standard right G-action, and an added equivariant-type requirement on the usual local triviality condition of fiber bundles. That is, there must exist an open cover $\{U_\alpha\}$ of B, homeomorphisms $\{\phi_\alpha\}$ such that $\phi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$, and $\phi_\alpha(b, g) = \phi_\alpha(b, 1)g$, for all $b \in B, g \in G$. The local triviality condition buys us that G acts freely on E and $E/G \cong B$ (i.e. the orbit space is homeomorphic to the base space.)

Chapter 4

Equivariant Cohomology Introduction

Most of the results of this section can be found in Quillen [22]. For any compact Lie group G (or any finite group), there exists a "universal principal bundle" for G, denoted $EG \to BG$ (See Milnor for a construction [21].) The universal principal bundle for a compact Lie group G is a principal G-bundle such that EG is contractible. As mentioned in the previous section, being a principal G-bundle implies that G acts freely on EG and $EG/G \cong BG$.

The **associated fiber bundle** is formed using the "Borel construction": $EG \times_G X \doteq (EG \times X)/G$, where G acts diagonally on $EG \times X$. The **equivariant cohomology ring** of the G-space X is defined

$$H_G^*(X) \doteq H^*(EG \times_G X).$$

For our purposes, H^* represents singular cohomology, and we will assume that coefficients are taken in a field k. In Quillen's formulation [22] he uses sheaf cohomology, but we don't lose any of the fundamental properties by switching to singular cohomology (this is explained on pg. 1163 in [26].) We now establish the functoriality of $H_G^*(-)$.

If G and G' are compact Lie groups with X a G-space and X' a G' space, consider a pair of maps $(u, f) : (G, X) \to (G', X')$ such that u is a Lie group homomorphism, and f a continuous map that is u-equivariant, i.e. $f(gx) = u(g)f(x)$ for all $g \in G$ and all $x \in X$.

If $P \to B$ and $P' \to B'$ are principal G and G' bundles respectively, then there is a commutative diagram, where G acts on P' using the map u, and on $P \times P'$ diagonally:

$$\begin{array}{ccc} (P \times P') \times_G X & \xrightarrow{p_2} & P' \times_{G'} X \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \bar{f}} \\ P \times_G X & \xrightarrow{\overline{v \times f}} & P' \times_{G'} X' \end{array}$$

In the diagram p_1 and p_2 are induced by the projection maps onto the first and second factors respectively, and $v : P \to P'$. Provided that P' is contractible, we see that $P \times P' \times X \simeq P \times X$

and so $(P \times P') \times_G X \simeq P \times_G X$ (all homotopies need to be G-equivariant for this to be true,) then p_1 is a locally trivial map. As shown in [22], p_1^* is an isomorphism on cohomology. Therefore, $H_G^*(X) \cong H^*((P \times P') \times_G X)$, and so we get a map $(p_1^*)^{-1} p_2^* \bar{f}^* : H^*(P' \times_{G'} X') \to H^*(P \times_G X)$. This map is denoted

$$(u, f)^* : H_{G'}^*(X') \to H_G^*(X).$$

If $G \leq G'$ and $X \subseteq X'$, we call the map **restriction** and denote it by $res_G^{G'} : H_{G'}^*(X') \to H_G^*(X)$. Equivariant cohomology is a contravariant functor from the category of pairs (G, X) to the category of graded k-algebras.

Provided that the map $v : P \to P'$ is u-equivariant, there is a well-defined map $s : P \times_G X \to (P \times P') \times_G X$ defined $[p, x] \mapsto [p, v(p), x]$. In fact, for any $[p, x] \in P \times_G X$, $p_1 \circ s([p, x]) \doteq p_1([p, v(p), x]) \doteq [p, x]$. Therefore, s is a section of p_1. Applying the cohomology functor to $p_1 \circ s = id$ we get $s^* \circ p_1^* = id$, and since p_1^* is an isomorphism we see that $s^* = (p_1^*)^{-1}$.

Now consider the map $(u, f)^* : H_{G'}^*(X') \to H_G^*(X)$. By definition, $(u, f)^* \doteq (p_1^*)^{-1} p_2^* \bar{f}^* = s^* p_2^* \bar{f}^*$. Computing by the definition of each of these maps reveals that $\bar{f} \circ p_2 \circ s([p, x]) = [v(p), f(x)]$. Therefore, if v is a u-equivariant map we have

$$(u, f)^* = \overline{v \times f}^*.$$

As an application we have the following lemma:

Lemma 4.1. *Inner automorphisms act trivially on equivariant cohomology: Pick an element $h \in G$, and consider the pair of maps $(c_h, \mu_h) : (G, X) \to (G, X)$ where c_h is defined for each $g \in G$ by $c_h(g) \doteq hgh^{-1}$, and μ_h is defined for each $x \in X$ by $\mu_h(x) = hx$. Then, $(c_h, \mu_h)^* : H_G^*(X) \to H_G^*(X)$ is the identity map.*

Proof. Define $v : EG \to EG$ by $v(e) \doteq he$ for all $e \in EG$. For all $g \in G$, $e \in EG$, $v(g \cdot e) \doteq h(ge) = (hgh^{-1})he \doteq c_h(g)v(e)$, and thus v is c_h-equivariant. By the statements preceding this lemma we know that $(c_h, \mu_h)^* : H_G^*(X) \to H_G^*(X)$ coincides with the map $\overline{v \times \mu_h}^*$. The map $\overline{v \times \mu_h} : EG \times_G X \to EG \times_G X$ is defined by $[e, x] \mapsto [v(e), \mu_h(x)]$, but by definition

$[v(e), \mu_h(x)] = [he, hx] = [e, x]$, so $\overline{v \times \mu_h}$ is the identity map. Therefore, $\overline{v \times \mu_h}^*$ is the identity on cohomology. $\square$

While we are mainly interested in studying equivariant cohomology rings, it is worthwhile to briefly mention the homotopy theory of universal principal bundles. Say that $\xi \doteq G \to EG \to BG$ is the universal principal G-bundle. Take any paracompact space B, the homotopy classes of maps from B to BG are in bijective correspondence with principal G-bundles over B. Let $[B, BG]$ denote homotopy classes of maps from B to BG. Take any $f \in [B, BG]$, and form the pull-back bundle, $f^*(\xi)$, it turns out that this gives a principal G-bundle over B, and in fact every principal G-bundle over B comes from a pull-back of ξ. For this reason, BG is called a *classifying space* for G (it classifies all principal G-bundles!)

4.1 Group Cohomology

Equivariant cohomology is a generalization of group cohomology. In the case that X is a point and G is discrete, equivariant cohomology and group cohomology coincide. The equivariant cohomology of a point can be seen to be the cohomology of BG since $EG \times_G pt \cong EG/G \cong BG$, and so $H^*_G(pt) \doteq H^*(EG \times_G pt) = H^*(BG)$.

We briefly present the algebraic definition of group cohomology, followed by a theorem which describes the relation to the topological definition of equivariant cohomology. In addition to providing a well-rounded overview of the theory of equivariant cohomology rings, the intent of this section is to present a handful of examples of group cohomology rings. When studying $H^*_G(X)$ from an algebraic viewpoint, there is much to be gained by first studying just the group cohomology ring $H^*_G(pt)$.

Also of note is that the algebraic definitions of group cohomology are amenable to machine computation, see Green [12] for a reference.

Given a ring R and a group G, the collection of maps $\alpha : G \to R$ which are nonzero for only a finite number of $g \in G$ defines an object RG, called the *group ring*. An element of RG is is a formal linear combination $\sum_{g \in G} \alpha(g) \cdot g$. RG is both a ring and a free module (generated by the elements of G over R). The ring structure is defined by $\left(\sum_{g\in G}\alpha(g)g\right)\cdot\left(\sum_{g\in G}\beta(g)g\right) \doteq \sum_{g,h\in G}(\alpha(g)\beta(h))(gh)$ (essentially the distributive property.)

Suppose G is a group and A is an abelian group, if there exists a group homomorphism $\sigma : G \to Aut(A)$, we say that A is a G-module. A G-module A is a module over RG, and conversely if A is an RG-module, then it is also a G-module.

Definition 4.2. *For any group G, the nth cohomology group of G with coefficients in the (left) G-module A is defined*

$$H^n(G,A) = Ext^n_{\mathbb{Z}G}(\mathbb{Z},A).$$

The nth homology group of G with coefficients in the (right) G-module B is defined

$$H_n(G,B) = Tor^{\mathbb{Z}G}_n(B,\mathbb{Z}).$$

In each case, $\mathbb{Z}$ is regarded as the trivial $\mathbb{Z}G$-module.

Recall that Ext is defined as the derived functor of Hom and that Tor is the derived functor of $\otimes$. See [15] for an in-depth description of the theory behind these functors. For example, to compute the group cohomology $H^*(G,A)$, one may start by taking a projective resolution of $\mathbb{Z}$ (thought of as the trivial $\mathbb{Z}G$-module). That is, an exact sequence of the form: $\cdots \to P_i \xrightarrow{\delta_i} \cdots \xrightarrow{\delta_1} P_0 \to 0$, such that each P_j is a projective $\mathbb{Z}G$-module, and $\delta_j \circ \delta_{j-1} = 0$ for each j. Next, apply the functor $Hom_{\mathbb{Z}G}(-,A)$, to the resolution $\mathbf{P}$, and compute homology (kernel mod image). The ith homology group (for $i \geq 1$) defines $H^i(G,A)$, with $H^0(G,A) \doteq Hom_{\mathbb{Z}G}(G,A)$.

Said another way, $H^n(-,A)$ is the nth right derived functor of $Hom_{\mathbb{Z}G}(-,A)$. Recall that if $\mathbf{P_1}$ and $\mathbf{P_2}$ are two projective resolutions of $\mathbb{Z}$, then the chain complexes $Hom_{\mathbb{Z}G}(\mathbf{P_1},A)$ and $Hom_{\mathbb{Z}G}(\mathbf{P_2},A)$ are chain homotopic, meaning that the homology groups of each complex are isomorphic. In practice, a common choice of projective resolution to compute group cohomology is the *bar resolution* ([15] VI.13).

Since $Hom(-,-)$ is a bi-functor, one can alternatively compute $H^n(G,A)$ using injectives: that is, by taking an injective resolution of A, and applying the co-variant functor $Hom_{\mathbb{Z}\mathbb{G}}(\mathbb{Z},-)$.

That this injective formulation of $Ext^*_{\mathbb{Z}G}(\mathbb{Z}, A)$ is naturally equivalent to the projective construction above is a standard result in the theory of derived functors.

Example 4.3.

- *Let A be any G-module. Then, $H^0(G, A) \doteq Ext^0_{\mathbb{Z}G}(\mathbb{Z}, A) \doteq Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$. For any $x \in \mathbb{Z}G$, and any $\phi \in Hom_{\mathbb{Z}G}(\mathbb{Z}, A)$, $\phi(x \cdot 1) = \phi(1)$. This is because G acts trivially on $\mathbb{Z}$, so $\phi(x \cdot 1) = \phi(1)$, and since ϕ is a $\mathbb{Z}G$-module homomorphism, $\phi(x \cdot 1) = x \cdot \phi(1)$. Thus we see that the image of ϕ lies in the fixed point set of A under G. It's straightforward to check that the map $\phi \mapsto \phi(1)$ defines an isomorphism:*

$$H^0(G, A) \cong A^G.$$

- *Let $0 \to A \to B \to C \to 0$ be an exact sequence of G-modules. The theory of derived functors gives us a long exact sequence $0 \to A^G \to B^G \to C^G \to H^1(G, A) \to H^1(G, B) \to H^1(G, C) \to H^2(G, A) \to \cdots$.*

- *If A is an injective G-module, then $H^n(G, A) = 0$ for all $n \geq 1$. If B is a flat G-module, then $H_n(G, B) = 0$ for all $n \geq 1$. This follows from the definitions of injective and flat from homological algebra [15].*

The following two theorems give a topological description of group cohomology.

Theorem 4.4. *([6] pg. 36) Let G be any group, and let K be a connected CW complex such that $\pi_1(K) \cong G$ and $\pi_i(K) = 0$ for all $i \neq 1$ (So by definition, K is an Eilenberg-Maclane space of type $(G, 1)$.) For all $n \geq 0$, and for a ring of coefficients R,*

$$H^n(G, R) \cong H^n(K, R)$$

$$H_n(G, R) \cong H_n(K, R).$$

The left hand side is group cohomology with coefficients in a trivial G-module R, and the right hand side is ordinary cellular cohomology of K with coefficients in R.

Proof. We present a sketch of the proof, leaving the reader to consult [6] for a more detailed explanation. The crux of the proof is that the cellular chain complex of the universal cover of K forms a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Given any free resolution one may compute the group cohomology of G, but in this case since the free resolution comes from the topological chain complex, we can also directly compute the cellular homology groups of K, showing that the two are equal.

Let $E \xrightarrow{p} K$ be the universal cover of K. The theory of covering maps gives that $\pi_i(E) \cong \pi_i(B)$ for all $i \geq 2$, so therefore $\pi_i(E)$ is trivial for all i (since $\pi_i(B) = 0$ for $i \geq 2$, and E is also assumed to be simply connected.) Now, Whitehead's theorem for CW complexes implies that E is contractible, for the inclusion of a 0-cell of E induces an isomorphism on all homotopy groups, and therefore E is homotopy equivalent to a 0-cell (contractible).

We also need that E inherits a CW-structure from K such that the G-action on E freely permutes the cells ([6] pg. 15.) It is said that $E \to K$ is a regular or normal cover, with G equal to the group of deck transformations of E.

The conditions that E is contractible, and G freely permutes the cells of E give us that that augmented cellular chain complex $C_*(E)$: $\cdots \to C_n(E) \to \cdots \to C_0(E) \to \mathbb{Z} \to 0$, which is used to compute the cellular homology of E, is such that each $C_i(E)$ is a free $\mathbb{Z}G$-module. Since E is contractible, we know the homology of this complex is everywhere 0, or in other words, the chain complex above is exact. Therefore, the cellular chain complex of E forms a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules.

Finally, let $C_*(E)_G$ be the complex formed by applying $- \otimes_{\mathbb{Z}G} R$ to $C_*(E)$. By definition, taking homology of $C_*(E)_G$ is the group homology of G. On the other hand, one may show (pg. 34 [6]) that $C_*(E)_G$ is isomorphic to $C_*(K)$, and therefore the group homology is isomorphic to the cellular homology. Applying the hom functor gives the result for cohomology. $\square$

We note that the above theorem holds more generally in the case where R is any G-module if one uses cohomology with *local coefficients*.

Corollary 4.5. *([4] Theorem 2.4.11) For G discrete, Milnor's BG satisfies the hypotheses of K in the previous theorem (with contractible universal cover EG), and therefore $H^*(BG, R) \doteq H^*_G(pt, R) \cong H^*(G, R)$.*

4.2 Examples

Example 4.6. ***(Group Cohomology of Finite Cyclic Groups)*** *Let G be a finite cyclic group of order n. We write G multiplicatively, $G = \langle h \rangle$. We form the following free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules:*

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{1-h} \mathbb{Z}G \xrightarrow{N} \cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{1-h} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where the map N is multiplication by the element $N \doteq 1 + h + \cdots + h^{n-1}$, the map $1 - h$ is multiplication by the element $1 - h$, and the augmentation map is defined by $\epsilon(\sum_{g \in G} n_g g) \doteq \sum_{g \in G} n_g$.

A quick check reveals that this is in fact a resolution: Let $\xi = \sum_{i=0}^{n-1} m_i h^i$ be an element of $\mathbb{Z}G$. Now, $(1-h) \cdot \xi = \sum_{i=0}^{n-1}(m_i - m_{i-1})h^i$, where $m_{-1} \doteq m_{n-1}$. The sum of the coefficients $\sum_i m_i - m_{i-1} = 0$, and it's easy to see that $\operatorname{Im}(1-h)$ equals the set of all $\xi \in \mathbb{Z}G$ whose coefficients sum to zero. By definition, this is also the kernel of ϵ, so $\ker \epsilon = \operatorname{Im}(1-h)$. Further, $(1-n) \cdot \xi = 0$ if and only if $m_i - m_{i-1} = 0$ for each i, and thus $\xi \in \ker(1-h)$ if and only if all coefficients of ξ are equal.

Next, $N \cdot \xi = \sum_{j=0}^{n-1} \left(\sum_{i=0}^{n-1} m_i \right) h^j$, so that $\operatorname{Im}(N)$ equals the set of $\xi \in \mathbb{Z}G$ such that all coefficients of ξ are equal, and $\xi \in \ker(N)$ if and only if $\sum_i m_i = 0$. Thus, $\ker(1-h) = \operatorname{Im}(N)$, and $\ker(N) = \operatorname{Im}(1-h) = \ker(\epsilon)$, and the sequence is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules.

We are now in position to compute the group cohomology: Let A be any $\mathbb{Z}G$-module, and apply the functor $Hom_{\mathbb{Z}\mathbb{G}}(-, A)$ to the free resolution. Note that there is a $\mathbb{Z}G$-module isomorphism $Hom_{\mathbb{Z}G}(\mathbb{Z}G, A) \cong A$. This gives the chain complex

$$\begin{tikzcd}
0 \arrow[r] & Hom_{\mathbb{Z}G}(\mathbb{Z}, A) \arrow[r, "\epsilon^*"] \arrow[d, "\cong"] & Hom_{\mathbb{Z}G}(\mathbb{Z}G, A) \arrow[r, "(1-h)^*"] \arrow[d, "\cong"] & Hom_{\mathbb{Z}G}(\mathbb{Z}G, A) \arrow[r, "N^*"] \arrow[d, "\cong"] & Hom_{\mathbb{Z}G}(\mathbb{Z}G, A) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & A^G \arrow[r, "\epsilon^*"] & A \arrow[r, "(1-h)^*"] & A \arrow[r, "N^*"] & A \arrow[r, "(1-h)^*"] & \cdots
\end{tikzcd}$$

It follows that on the bottom row, $(1-h)^*$ and N^* are defined by multiplication by the indicated elements (this multiplication makes sense because A is a $\mathbb{Z}G$-module.)

Since this complex alternates every other map, it is said to have period 2, and so $H^1(G, A)$ determines every odd dimensional cohomology group, and $H^2(G, A)$ determines every even dimensional cohomology group. Further, since $Hom(-, A)$ is left-exact, we have that $\ker(1-h)^* = \operatorname{Im}(\epsilon^*) = A^G$. *Therefore,*

$H^0(G, A) \cong A^G$,

$H^{2i-1}(G, A) \cong \ker(N^*)/\operatorname{Im}(1-h^*)$ *for* $i \geq 1$,

$H^{2i}(G, A) \cong A^G/\operatorname{Im}(N^*)$ *for* $i \geq 2$.

Suppose that $A = \mathbb{Z}$ is the trivial $G = \mathbb{Z}/n$-module. The computations above show that:

$H^0(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}$,

$H^{2i-1}(\mathbb{Z}/n, \mathbb{Z}) = 0$ *for* $i \geq 1$,

$H^{2i}(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n$ *for* $i \geq 2$.

Example 4.7. *[4]* ***(Principal G-bundles)***

1. *The map $p : \mathbb{R} \to S^1$ defined by $t \mapsto \langle \cos t, \sin t\rangle$ defines a $\mathbb{Z}$-principal bundle over S^1. The fiber $\mathbb{Z}$ acts on $\mathbb{R}$ via addition, and since $\mathbb{R}$ is contractible, it is a universal bundle.*

2. *$p : S^n \to \mathbb{R}P^n$, defined by the map $x \in S^n \subseteq \mathbb{R}^{n+1} \mapsto [x] \in \mathbb{R}P^n$ is a $\mathbb{Z}/2\mathbb{Z}$- principal bundle. By defining S^∞ as the limit of the sequence of inclusions $\cdots \hookrightarrow S^i \hookrightarrow S^{i+1} \hookrightarrow \cdots$, we get the universal principal $\mathbb{Z}/2\mathbb{Z}$-bundle $S^\infty \to \mathbb{R}P^\infty$.*

   Similarly, $p : S^{2n+1} \to \mathbb{C}P^n$ is an S^1-principal bundle, and $S^\infty \to \mathbb{C}P^\infty$ is the universal principal S^1-bundle.

3. *If G is a Lie group and H is a closed subgroup, then $G \to G/H$ is a principal H-bundle.*

4. *There is a map $p : V^n(\mathbb{R}^m) \to G^n(\mathbb{R}^m)$ from the Stiefel manifold to the Grassmanian, defined by sending an orthonormal n-frame to the n-dimensional subspace that it spans. If $Y \in G^n(\mathbb{R}^m)$, then the fiber $p^{-1}(Y)$ is the set of orthonormal n-frames spanning Y, of course we can identify each n-frame bijectively with an element of $O(n)$. The fiber over any point is a copy of $O(n)$, and one may show that $V^n(\mathbb{R}^m) \to G^n(\mathbb{R}^m)$ is a principal $O(n)$-bundle.*

Example 4.8. ***(Equivariant Cohomology with $X = pt$)***

In the examples below where G is a discrete group, Corollary 4.5 implies the cohomology of BG is isomorphic to the group cohomology, thus illustrating the topological viewpoint of group cohomology.

- *If $G = \mathbb{Z}$ then $BG = S^1$. Therefore, $H^*(B\mathbb{Z}, R) \cong R[x]/(x^2)$, where $\deg(x) = 1$.*

- *([7] Theorem 2.5) The following example is key to Quillen's main theorems on equivariant cohomology (5.6, 5.8). Let $p \neq 2$ be a prime, and $G = \mathbb{Z}/p$.*

$$H^*(B(\mathbb{Z}/p), k) \cong k[x] \otimes_k \Lambda[y],$$

  where $x \in H^2$, $y \in H^1$, and Λ is the exterior algebra over k. For $p = 2$, $H^*(B(\mathbb{Z}/2), \mathbb{Z}/2) \cong \mathbb{Z}/2[y]$, where $y \in H^1$.

  Now, if A is the elementary abelian p-group of rank n, i.e. $A \cong \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, applying the Kunneth theorem yields:

$$H^*(BA, k) \cong k[x_1, \ldots, x_n] \otimes_k \Lambda[y_1, \ldots, y_n],$$

  where $\deg(x_i) = 2$ and $\deg(y_i) = 1$. Later, we make use of the fact that, modulo the nilradical, the even degree part of this ring is polynomial.

- *For $G = S^1$, $BS^1 = \mathbb{C}P^\infty$, and $H^*(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}[c]$ where c has degree two. More generally, if $G = U(n)$ then $BG = G_n(\mathbb{C}^\infty)$ (the Grassmananian on $\mathbb{C}^\infty$) and $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$ where $\deg(c_i) = 2i$.*

Example 4.9. ***(Degree of Group Cohomology Rings)***

- *Let $G = U(n)$, then $H^*(BG, k) \cong k[c_1, \ldots, c_n]$ with $\deg(c_i) = 2i$. The Poincare series is:*

$$PS(k[c_1, \ldots, c_n]) = \frac{1}{\prod_{i=1}^{n}(1 - t^{2i})}.$$

Then,

$$\begin{aligned}\deg(H^*(BG)) &= \lim_{t\to 1}(1-t)^n PS(H^*(BG)) \\ &= \lim_{t\to 1}\frac{(1-t)^n}{\prod_{i=1}^{n}(1-t^{2i})} \\ &= \prod_{i=1}^{n}\frac{1}{2i},\end{aligned}$$

which is the reciprocal of the product of the degrees of the generators c_i.

- *Let $H = k[x_1, \ldots, x_r] \otimes_k \Lambda[y_1, \ldots, y_r]$, where $\deg(x_i) = d_i$ and $\deg(y_i) = 1$. The Poincare series of H is:*

$$\begin{aligned}PS(k[x_1, \ldots, x_r] \otimes_k \Lambda[y_1, \ldots, y_r]) &= PS(k[x_1, \ldots, x_r])PS(y_1, \ldots, y_r) \\ &= \frac{1}{\prod_{i=1}^{r}(1-t^{d_i})} \cdot \sum_{i=0}^{r}\binom{r}{i}t^i\end{aligned}$$

Now if A is the elementary abelian p group of rank r, the group cohomology ring of A is $H_A^* = k[x_1, \ldots, x_r] \otimes_k \Lambda[y_1, \ldots, y_r]$ where $\deg(x_i) = 2$ and $\deg(y_i) = 1$. From the formula above we have that,

$$
\begin{aligned}
\deg(H^*) &= \lim_{t\to 1}(1-t)^r \frac{1}{(1-t^2)^r}\sum_{i=0}^{r}\binom{r}{i}t^i \\
&= \lim_{t\to 1}\left(\frac{1}{1+t}\right)^r \sum_{i=1}^{r}\binom{r}{i}t^i \\
&= \left(\frac{1}{2}\right)^r 2^r \\
&= 1
\end{aligned}
$$

Chapter 5

Commutative Algebra of Equivariant Cohomology Rings

Quillen laid the foundation for the study of the commutative algebra of equivariant cohomology rings. The two main theorems of Quillen's papers ([22], [23]) are presented below. These two theorems relate the prime spectrum of $H^*_G(X)$ to the elementary abelian groups which appear as subgroups of G, and have fixed points when acting on X.

Recall that the product structure of a cohomology ring comes from the cup product. By construction, the cup product has the following graded anti-commutative property: If $\alpha \in H^k(X), \beta \in H^j(X)$, then $\alpha\beta = (-1)^{\deg(\alpha)\deg(\beta)}\beta\alpha$.

Thus, in general, $H^*_G(X, R)$ will not be a commutative ring (although when $char(R) = 2$, we do get commutativity.) Further, these rings often have a complicated nilpotent structure. Despite these obstacles, Quillen showed that we can apply the tools of commutative algebra, and work around the nilpotents. If $char(R)$ is odd or 0, then the even degree part of $H^*_G(X)$ (which is a commutative ring) has some nice geometric properties. We denote the even degree part of an equivariant cohomology ring by omitting the star, e.g. $H_G(X)$ is the even degree part of $H^*_G(X)$. If $char(R) = 2$ then $H^*_G(X) = H_G(X)$.

The following results allow for the study of geometric and commutative algebraic properties of equivariant cohomology rings:

Theorem 5.1. *([22] Corollary 2.2) Let G be a compact Lie group, and let X be a topological space on which G acts continuously. Take all cohomology coefficients in a field k. If $H^*(X)$ is a finitely generated k-vector space, then $H^*_G(X)$ is a finitely generated k-algebra.*

Corollary 5.2. *Let k be a finite field of characteristic p. If $H^*(X)$ is a finite dimensional graded k-vector space, then $H_G(X)$ is a finitely generated graded k-algebra.*

A proof may be found in [18], Proposition 4.1.

Lemma 5.3. *$H_G^*(X)$ is a finitely generated, graded module over the Noetherian, graded, and commutative ring $H_G(X)$. Using the notation from part 1, $H_G^*(X) \in \mathfrak{grmod}(H_G(X))$.*

Theorem 5.4. *([22] Corollary 2.3) If $(u, f) : (G, X) \to (G', X')$ is a morphism such that u is injective and $H^*(X)$ is a finitely generated k-module, then $(u, f)^* : H_{G'}^*(X') \to H_G^*(X)$ is finite; i.e. $H_G^*(X)$ is a finitely generated $H_{G'}^*(X')$-module.*

When relating the equivariant cohomology ring of a subgroup G to that of the full group G', the theorem above allows us to take advantage of the theory of **integral extensions** from commutative algebra. A more thorough explanation of the theory of integral extensions can be found in [11]. For a subring R of S, an element $s \in S$ is said to be *integral* over R if it is the root of a monic polynomial with coefficients in R. When $f : R \to S$ is a ring homomorphism, we say that S is integral over R when S is integral over the image $f(R)$.

Given a ring homomorphism $f : R \to S$ where S is a finitely generated R-module, a standard result gives that S must also be integral over R. In the case of equivariant cohomology rings, the theorem above implies that $H_G^*(X)$ is integral over $H_{G'}^*(X')$. A couple of useful theorems about integral extensions are:

- The lying-over theorem. For an integral extension $f : R \to S$, for every $\mathfrak{p} \in Spec(R)$ there exists a $\mathfrak{q} \in Spec(S)$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.

- The going-up theorem. Suppose $j < k$, there is a chain of primes $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_k$ in R, and a chain of primes $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_j$ in S such that $f^{-1}(\mathfrak{q}_i) \cap R = \mathfrak{p}_i$ for $i = 1, \ldots, j$, then there is a way to complete the chain of $\mathfrak{q}$'s; i.e. there exist primes $\mathfrak{q}_{j+1} \subseteq \cdots \subseteq \mathfrak{q}_k$ in S such that $f^{-1}(\mathfrak{q}_i) \cap R = \mathfrak{p}_i$ for all i.

  In the case that S is a finitely generated R-module, one may use the going-up theorem to prove the Krull dimensions are equal: $\dim(S) = \dim(R)$.

5.1 Main Theorems of Quillen

Quillen's main theorems hold in the following setting, which we assume for the remainder of the dissertation: G is a compact Lie group which acts continuously on a space X. We also require that X is Hausdorff, and that X is either compact, or is paracompact with finite mod-p cohomological dimension (see [22] for a definition.) Finally, all cohomology is taken with coefficients in the field $k \doteq \mathbb{Z}/p\mathbb{Z}$ for p a prime.

Given a pair (G, X) which satisfy the conditions above, we denote **Quillen's category of pairs** by $\mathcal{Q}(G, X)$. Objects of this category are pairs (A, c) where A is an elementary abelian p-subgroup of G, $X^A \neq \emptyset$, and c is a connected component of X^A. If (A, c) and (A', c') are objects in $\mathcal{Q}(G, X)$, then there is a morphism between them if there exists an element $g \in G$ such that $gAg^{-1} \leq A'$ and $c' \subseteq gc$.

If there exists a morphism $\theta_g : (A, c) \to (A', c')$, we say that (A, c) is *subconjugate* to (A', c'), also written $(A, c) \lesssim (A', c')$. We define *conjugate* objects, denoted $(A, c) \sim (A', c')$, to be two objects that are isomorphic in $\mathcal{Q}(G, X)$.

The notions of conjugate and subconjugate give a partial ordering on the conjugacy classes of $\mathcal{Q}(G, X)$. If $[A, c]$ and $[A', c']$ are conjugacy classes, then we say $[A, c] \leq [A', c']$ when $(A, c) \lesssim (A', c')$.

Definition 5.5. *For X and G as above,*

$$\mathcal{Q}'(G, X) \doteq \{[A, c] : (A, c) \in \mathcal{Q}(G, X),\ [A, c] \textit{ is maximal w.r.t } \leq\}.$$

Within $\mathcal{Q}'(G, X)$, we are often interested in the elementary abelian subgroups which have maximal rank, so we define

$$\mathcal{Q}'_{max}(G, X) \doteq \{[A, c] \in \mathcal{Q}'(G, X) : rk(A) \geq rk(B),\ \forall\ [B, d] \in \mathcal{Q}'(G, X)\}.$$

We refer the reader to [20] for an example where $\mathcal{Q}'_{max}(G, X)$ is distinct from $\mathcal{Q}'(G, X)$, take $G = GL_n(\mathbb{Z}/p)$ for $n \geq 4$.

Two Main Theorems of Quillen [22] [23]:

Theorem 5.6. *([22] Theorem 7.7) Assume (G, X) as above, and that $H^*(X)$ is finite dimensional as a vector space over k. Then, the Krull dimension of $H^*_G(X)$ equals the maximal rank of an elementary abelian p-subgroup such that $X^A \neq \emptyset$. Recall that all cohomology is taken with coefficients in $k = \mathbb{Z}/p\mathbb{Z}$.*

For example, in the case where X is a point, Quillen's theorem says that Krull dimension of H^*_G equals the maximal rank of an elementary abelian p-group in G.

A useful restatement of Quillen's theorem is in terms of isotropy subgroups. Let A be a max rank elementary abelian p-group of G such that $X^A \neq \emptyset$. For any $x \in X^A$, we have $A \leq G_x$. By Quillen's theorem, $\dim(H^*_{G_x})$ equals a max rank elementary abelian p-group of G_x. Since A was assumed to be of maximal rank, $\dim(H^*_{G_x}) = rk(A)$. Thus, we may restate Quillen's theorem: The Krull dimension of $H^*_G(X)$ equals the maximum Krull dimension from the collection $\{H^*_{G_x} : x \in X\}$.

Definition 5.7. *Let $(A, c) \in \mathcal{Q}(G, X)$, and pick a point $x_0 \in X^A$. We define $\mathfrak{p}_{A,c}$ as the kernel of the following composition*

$$H_G(X) \xrightarrow{res^G_A} H_A(x_0) \to H_A(x_0)/\sqrt{0}.$$

Since $H_A/\sqrt{0}$ is a polynomial ring, the first isomorphism theorem implies that that $\mathfrak{p}_{A,c} \in Spec(H_G(X))$.

Theorem 5.8. *([23] Proposition 11.2)*

- *The correspondence between pairs $(A, c) \in \mathcal{Q}(G, X)$ and primes $\mathfrak{p}_{(A,c)}$ is inclusion reversing. That is, $\mathfrak{p}_{(A,c)} \supseteq \mathfrak{p}_{(A',c')}$ if and only if there exists a morphism in $\mathcal{Q}(G, X)$: $(A, c) \to (A', c')$.*

- *For each $[A, c] \in \mathcal{Q}'(G, X)$ there exists a unique minimal prime $\mathfrak{p} \in Spec(H_G(X))$, and $\mathfrak{p} = \mathfrak{p}_{(A,c)}$. On the other hand, for each minimal prime $\mathfrak{p} \in Spec(H_G(X))$ there exists* *a unique class $[A, c] \in \mathcal{Q}'(G, X)$ such that $\mathfrak{p} = \mathfrak{p}_{(A,c)}$. Therefore, the correspondence of maximal classes $[A, c]$ and minimal primes in $H_G(X)$ is one-to-one.*

5.2 Quillen's Magic Space F

For any compact Lie group G, there exists a (not unique) unitary group U and an embedding of G into U making G closed in U. Given a compact Lie group G, we often begin by fixing such an embedding. Then, define S to be the set of diagonal matrices of order p in U, where p is a fixed prime. Observe that U is equipped with both a left G-action and a right S action.

Quillen defined the space $F \doteq U/S$, and showed how the product space $X \times F$ can be used to encode information about how G acts on X. Some general properties of F are outlined in the following lemma (5.9). Note that if $g \in G$ and $uS \in F$, then the action of G on F is defined by $g \cdot (uS) \doteq guS$. Fix an embedding of the compact Lie group G into a unitary group U, and let $k = \mathbb{Z}/p\mathbb{Z}$ for p an odd prime.

Lemma 5.9. *Let G be a compact Lie group, and fix an embedding into a unitary group U. Let X be a compact G-space, and let $F \doteq U/S$ be defined as above. Let the field of coefficients for cohomology be $k = \mathbb{Z}/p\mathbb{Z}$. Then, with G acting diagonally on $X \times F$,*

1. *For $(A, c) \in \mathcal{Q}(G, X)$, $(X \times F)^A \neq \emptyset$. Also, let $(x, e) \in X \times F$, and say that $e \doteq vS$ for $v \in U$, then $G_{(x,e)}$ is an elementary abelian p-subgroup of G. Specifically, $G_{(x,e)} = G_x \cap vSv^{-1}$.*

2. *Let $[A, c] \in \mathcal{Q}'(G, X)$ (i.e. $[A, c]$ is maximal w.r.t $\leq$,) then for each $(x, e) \in c \times F^A$, $G_{(x,e)} = A$.*

3. *Take $[A, c], [B, d] \in \mathcal{Q}'(G, X)$, and suppose $G \cdot (c \times F^A) \cap G \cdot (d \times F^B) \neq \emptyset$, then $[A, c] = [B, d]$.*

4. *For $[A, c] \in \mathcal{Q}'_{max}(G, X)$, $\dim(H^*_G(X \times F)) = \dim(H^*_G(G \cdot (c \times F^A)))$.*

5. *Let $(x, e) \in X \times F$, $(x, e) \notin \amalg_{[A,c] \in \mathcal{Q}'_{max}(G,X)} G \cdot (c \times F^A)$, then $\dim(H^*_{G_{(x,e)}}) < \dim(H^*_G(G \cdot (c \times F^A)))$, for all $[A, c] \in \mathcal{Q}'_{max}(G, X)$.*

Proof. **(1)** A acts diagonally on $X \times F$, and by definition of Quillen's category of pairs, we know that $X^A \neq \emptyset$. To see that $F^A \neq \emptyset$, consider that S is the diagonal elementary abelian p-group in U, and use that A is abelian to simultaneously diagonalize the matrices in A. That is, there exists a unitary matrix $u \in U$ such that $u^{-1}Au \subseteq S$, and therefore $a(uS) = uS$ for all $a \in A$. In particular, $uS \in F^A$. Thus, $(X \times F)^A \neq \emptyset$.

To prove the second statement, suppose $(x, e) \in X \times F$, with $e \doteq vS$ for $v \in U$. For any $g \in G$ with $g \cdot (x, e) = (x, e)$, it must be true that $gx = x$ and $ge = e$ (G is acting diagonally.) Since $gx = x$ we immediately see that $G_{(x,e)} \subseteq G_x$. Also, $ge = e \Leftrightarrow gvS = vS \Leftrightarrow v^{-1}gv \in S$. Therefore, $G_{(x,e)} = G_x \cap vSv^{-1}$.

Finally note that vSv^{-1} is elementary abelian, and $G_{(x,e)}$ is a subgroup, so $G_{(x,e)}$ is also elementary abelian.

(2) Suppose that $[A, c] \in \mathcal{Q}'(G, X)$. Let $e = uS \in F^A$, and let $x \in c \subseteq X^A$. We have that $A \subseteq G_x$, since $x \in X^A$. Also, for every $a \in A$, $a \cdot (uS) = uS$, so $u^{-1}au \in S \Leftrightarrow a \in uSu^{-1}$. Therefore, $A \subseteq G_x \cap uSu^{-1} = G_{(x,e)}$. Note that we did not need the maximality hypothesis for this direction of the inclusion.

Now we will show that $G_{(x,e)} \subseteq A$. Let $b \in G_x \cap uSu^{-1} = G_{(x,e)}$. Define $A' = \langle A, b\rangle$. Note that b commutes with everything in A since $A \subseteq G_{(x,e)}$, and $G_{(x,e)}$ is elementary abelian. Therefore, A' is commutative and all of its elements have order p, and it too is elementary abelian.

Now, $x \in X^{A'}$ since $A' = \langle A, b\rangle$ and both A and b fix x (Recall that $b \in G_x$ and $x \in X^A$.) Then, there exists a unique connected component c' of $X^{A'}$ which contains x. Note that $(A', c') \in \mathcal{Q}(G, X)$.

By construction we have that $X^{A'} \subseteq X^A$, and so $c' \subseteq c$. Therefore, $[A, c] \leq [A', c']$. By the maximality assumption on $[A, c]$, it must be true that $[A, c] = [A', c']$. This equality means that A and A' are conjugate, but in this case we also have $A \subseteq A'$, so in fact $A = A'$. Therefore, $b \in A$. Since b was chosen arbitrarily, we have shown that $G_{(x,e)} = G_x \cap uSu^{-1} \subseteq A$.

(3) Take $(z, f) \in G \cdot (c \times F^A) \cap G \cdot (d \times F^B)$. Suppose $(z, f) = g_1(x_1, e_1) = g_2(x_2, e_2)$, where $g_1, g_2 \in G$, $x_1 \in c$, $x_2 \in d$, and $e_1 = u_1S \in F^A$, $e_2 = u_2S \in F^B$.

By the previous part of this lemma, $G_{(x_1,e_1)} = A$ and $G_{(x_2,e_2)} = B$. Now, $G_{(z,f)} = g_1 G_{(x_1,e_1)} g_1^{-1}$ $= g_1 A g_1^{-1}$, and $G_{(z,f)} = g_2 G_{(x_2,e_2)} g_2^{-1} = g_2 B g_2^{-1}$. Let $h = g_1^{-1} g_2$, then $A = hBh^{-1}$.

Then, $X^A = hX^B$. Since multiplication by h is a homeomorphism, in particular, it establishes a bijection between the connected components of X^B and X^A. Since $x_1 \in c$ and $x_2 \in d$ and $x_1 = hx_2$, we conclude that $c = hd$. Therefore, $(A, c) \sim (B, d)$.

(4) Let $(A, c) \in \mathcal{Q}'_{max}(G, X)$, Quillen's theorem implies that $\dim(H_G^*(X)) = rk(A)$. From part 1 of this lemma, $X^A \neq \emptyset \Leftrightarrow (X \times F)^A \neq \emptyset$, so using Quillen's theorem for the space $X \times F$, we have $\dim(H_G^*(X \times F)) = rk(A)$. Another application of Quillen's theorem implies that $\dim(H_G^*(G \cdot (c \times F^A)) = max\{\text{p-rk}(G_{(z,f)}) : (z, f) \in G \cdot (c \times F^A)\}$.

Pick a $(z, f) \in G \cdot (c \times F^A)$, we may write $(z, f) = g \cdot (x, e)$ for some $g \in G, x \in c, e \in F^A$. Then, $G_{(z,f)} = gG_{(x,e)}g^{-1}$. By part 2 of this lemma, $G_{(x,e)} = A$, so $G_{(z,f)} \sim A$. Therefore, all the isotropy for points in $G \cdot (c \times F^A)$ are maximal rank elementary abelian subgroups of G. Thus, $\dim(H_G^*(G \cdot (c \times F^A)) = rk(A)$.

(5) Let $[A, c] \in \mathcal{Q}'_{max}(G, X)$, so by Quillen's theorem $\mathrm{d}(H_G^*(X)) = rk(A)$. Take $(x, e) \in X \times F$ with $(x, e) \notin \amalg_{[A,c] \in \mathcal{Q}'_{max}(G,X)} G \cdot (c \times F^A)$. Define $B \doteq G_{(x,e)}$, and note that B is an elementary abelian p-subgroup of G.

By definition of B, $(x, e) \in (X \times F)^B$ which implies that $X^B \neq \emptyset$. Therefore, $B \in \mathcal{Q}(G, X)$. Since A is of max rank, $rk(B) \leq rk(A)$.

Suppose that $rk(B) = rk(A)$. Let d be the unique component of X^B containing x. Then by definition, $[B, d] \in \mathcal{Q}'_{max}(G, X)$. This implies that $(x, e) \in d \times F^B$, with $[B, d] \in \mathcal{Q}'_{max}(G, X)$, but this contradicts the assumption on (x, e). Therefore, $rk(B) < rk(A)$, and by Quillen's theorem this is the same as $\mathrm{d}(H_{G_{(x,e)}}^*) < \mathrm{d}(H_G^*(X))$.

□

Quillen used the following result to prove the two main theorems of the last section(5.6, 5.8), and we make use of this result in the next section. Recall that a sequence of modules $A \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} C$ is called an *equalizer sequence* when $0 \to A \xrightarrow{h} B \xrightarrow{f-g} C$ is an exact sequence.

Lemma 5.10. *([22] Lemma 6.5) The following is an equalizer sequence of $H_G(X)$-modules:*

$$H^*_G(X) \to H^*_G(X \times F) \rightrightarrows H^*_G(X \times F^2);$$

defined by applying the equivariant cohomology functor to the sequence

$$X \times F \times F \overset{\pi_{12}}{\underset{\pi_{13}}{\rightrightarrows}} X \times F \xrightarrow{\pi_1} X,$$

where π is the projection map onto the indicated components.

Quillen then shows there is an isomorphism of $H_G(X)$-modules

$$H^*_G(X \times F) \cong H^*_G(X) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F).$$

If $r \in H_G(X)$, $x \otimes y \in H^*_G(X) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F)$, then the $H_G(X)$-module structure is defined by $r \cdot (x \otimes y) \doteq (x \cdot r) \otimes y$, i.e. by only acting on the first factor.

The following description of the space F comes in handy when proving lemma 5.14 of the next section. Describe a point in F as a pair $\{(l_1, \ldots, l_n); (v_1^*, \ldots, v_n^*)\}$. Each l_i is a line in $\mathbb{C}^n$, and all lines are mutually orthogonal. Since we are working over $\mathbb{C}$, where a line has two real dimensions, we can pick a vector v_i which lives on the unit circle of the plane defined by l_i. Then, v_i^* is defined to be the orbit of v_i under the group of pth roots of unity living in the corresponding copy of $S^1 \subseteq \mathbb{C}^n$.

For any $g \in G$, and $\{(l_1, \ldots, l_n); (v_1^*, \ldots, v_n^*)\} \in F$, the group action is defined by

$$g \cdot \{(l_1, \ldots, l_n); (v_1^*, \ldots, v_n^*)\} = \{(gl_1, \ldots, gl_n); (gv_1^*, \ldots, gv_n^*)\}.$$

For example, let $u \in U(n)$ be a matrix which is a representative for a point $uS \in F$. Using the notation above, the ith column of u spans the line l_i. Since the columns of a unitary matrix form an orthonormal basis of $\mathbb{C}^n$, the l_i's are mutually orthogonal. Express the matrix u by its columns as: $u = [u_1|\cdots|u_n]$. We can choose the vectors v_i (as in the notation above) to equal the u_i. We see this as follows: Suppose that $w \in U(n)$ and $wS = uS$, so that $w^{-1}u \in S$. This implies that there exists a diagonal matrix $D \in S$, such that $u = wD$. Say the ith diagonal element of D is ξ_i, a pth root of unity. We see that $u = [\xi_1 w_1|\cdots|\xi_n w_n]$, or put another way, each column of w is found by rotating a column of u by a pth root of unity in the corresponding plane. In fact, we see that every unitary matrix which can serve as a representative for the point uS can be found in this way, which validates the description of points in F as pairs $\{(l_1,\ldots,l_n);(v_1^*,\ldots,v_n^*)\}$.

5.3 A Localization Theorem of Duflot

The material developed in this section is leveraged to prove the main theorem of this paper (Theorem 6.8). We do not present all of the details of Duflot's localization proof, however we will sketch a proof by explaining the most important ideas. Much like Quillen, Duflot passes a problem about $H_G^*(X)$ to the space $H_G^*(X \times F)$, and uses the nice properties of F established in the last section to prove her result.

Definition 5.11. *Let G be a group, X a G-space, and let $(A, c) \in \mathcal{Q}(G, X)$.*

- $N_G(A, c) \doteq \{g \in G : gAg^{-1} = A\} \cap \{g \in G : gc = c\}$
- $C_G(A, c) \doteq \{g \in G : ga = ag,\ \forall a \in A\} \cap \{g \in G : gc = c\}$
- $W_G(A, c) \doteq N_G(A, c)/C_G(A, c)$

In the case that X is just a point, we omit the c's from the notation. e.g. $N_G(A)$ instead of $N_G(A, c)$.

Duflot Localization Theorem:

Theorem 5.12. *([9] Theorem 3.2) Suppose that (A, c) is a maximal pair of $\mathcal{Q}(G, X)$. Then there is an isomorphism*

$$H_G^*(X)_{[\mathfrak{p}_{(A,c)}]} \xrightarrow{(res)_{\mathfrak{p}_{(A,c)}}} H_{C_G(A,c)}^*(c)_{[\mathfrak{p}_{(A,c)}]}^{W_G(A,c)}.$$

Remark: In the original paper [9], it was not clear whether the localization was ordinary localization or graded localization. In [10], Duflot makes this distinction clear, and we've added the brackets here to indicate graded localization.

First, let's make explicit how $W_G(A,c)$ acts on $H^*_{C_G(A,c)}(c)$ (forget any localization for the time being.) For any $n \in N_G(A,c)$, we have the conjugation map on the pair $(C_G(A,c),c) \to (C_G(A,c),c)$ which sends (g,x) to (ngn^{-1},nx). Refer to this map as $\cdot n$, and the induced map on equivariant cohomology as $n^* : H^*_{C_G(A,c)}(c) \to H^*_{C_G(A,c)}(c)$. Recall that inner-automorphisms act trivially on equivariant cohomology, therefore if $n \in C_G(A,c)$ then $n^* = id$ on $H^*_{C_G(A,c)}(c)$. Therefore, $W_G(A,c)$ has a well-defined action on $H^*_{C_G(A,c)}(c)$.

Another preliminary observation is that the restriction map takes $H^*_G(X)$ into the ring of invariants $H^*_{C_G(A,c)}(c)^{W_G(A,c)}$. This follows from the commutative diagram of pairs:

$$\begin{tikzcd}
(G,X) \arrow[r, "\cdot n"] & (G,X) \\
(C_G(a,c),c) \arrow[u, "i"] \arrow[r, "\cdot n"] & (C_G(A,c),c) \arrow[u, "i"']
\end{tikzcd}$$

In the diagram $\cdot n$ is the conjugation map defined above, and i is inclusion. By applying equivariant cohomology we get a corresponding commutative diagram:

$$\begin{tikzcd}
H^*_G(X) \arrow[d, "res^G_C"'] & H^*_G(X) \arrow[l, "n^*"'] \arrow[d, "res^G_C"'] \\
H^*_{C_G(A,c)}(c) & H^*_{C_G(A,c)}(c) \arrow[l, "n^*"']
\end{tikzcd}$$

Now, lemma 4.1 implies that the conjugation map n^* on $H^*_G(X)$ is the identity map, so commutativity implies that $n^* \circ res^G_C = res^G_C$, thus demonstrating that restriction lands in the ring of invariants.

The remainder of the section comprises a sketch of a proof of Duflot's localization theorem. The central method of her proof is to take advantage of the following sequence of maps for a pair

$[A, c] \in \mathcal{Q}'_{max}(G, X)$:

$$H^*_G(X) \xrightarrow{①} H^*_G(G \cdot (c \times F^A)) \xrightarrow{②} H^*_{N_G(A,c)}(c \times F^A) \xrightarrow{③} H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)} \qquad (\star)$$

Let's investigate each of these maps in turn. Map (1) is a restriction map coming from the inclusion of pairs $\big(G, G \cdot (c \times F^A)\big) \hookrightarrow (G, X)$. Map (2) is an isomorphism and is explained in the following lemma and proof:

Lemma 5.13. *([9] Lemma 3.4) For a maximal pair* $[A, c] \in \mathcal{Q}'_{max}(G, X)$*, there is an isomorphism*

$$H^*_G(G \cdot (c \times F^A)) \cong H^*_{N_G(A,c)}(c \times F^A)$$

Proof. Let $\theta : G \times_{N_G(A,c)} \big(c \times F^A\big) \to G \cdot \big(c \times F^A\big)$ be the map defined by $\theta([g, (x, f)]) \doteq (gx, gf)$. On the left side, $N_G(A, c)$ acts diagonally on $G \times (c \times F^A)$. For $n \in N_G(A, c)$ and $g \in G$ the action of $N_G(A, c)$ on G is defined by $n \cdot g = gn^{-1}$. Similarly, on the right side, G acts diagonally on $c \times F^A$. It's then straight-forward to check that θ is well-defined, continuous, surjective, and open. By construction, θ is G-equivariant.

We now show that θ is injective, and is thus a G-equivariant homeomorphism. Suppose that $(gx, gf) = (hy, he)$ are images of $[g, (x, f)]$ and $[h, (y, e)]$ respectively. Take the element $n = g^{-1}h$. If we can show that $n \in N_G(A, c)$, then $[x, (x, f)] = [h, (y, e)]$.

By lemma 5.9, G_f is an elementary abelian p-group. Then, $B \doteq G_x \cap G_f$ is also an elementary abelian p-group. Further, since $x \in X^A$ and $f \in F^A$ we have $A \subseteq B$, which implies $X^B \subseteq X^A$. Since $x \in X^B$, there exists a component $d \subseteq X^A$ which contains x, and therefore $(A, c) \subseteq (B, d) \in Q(G, X)$. However, (A, c) was assumed to be a maximal pair, and therefore $(A, c) = (B, d)$. The same argument on $B' = G_y \cap G_e$ implies that $(A, c) = (B, d) = (B', d')$.

By hypothesis, $(h^{-1}g)x = y$ and $(h^{-1}g)f = e$. Since $n \doteq g^{-1}h$, we must have that $n \cdot G_y \cdot n^{-1} = G_x$, and $n \cdot G_e \cdot n^{-1} = G_f$ (note: the action on the right by n^{-1} means multiply by n on the left.) Therefore, $n \cdot B' \cdot n^{-1} = B$, and also $nd = d'$. Putting it all together, since $A = B = B'$ and

$c = d = d'$, we have $n \cdot A \cdot n^{-1} = A$ and $nc = c$ proving that $n \in N_G(A, c)$, and θ is a G-equivariant homeomorphism.

Now, θ induces the homeomorphism
$\theta_G : EG \times_G \big(G \times_{N_G(A,c)} \left(c \times F^A\right)\big) \to EG \times_G \big(G \cdot \left(c \times F^A\right)\big)$ where $[e, [g, (x, f)]] \mapsto [e, (gx, gf)]$.
By this definition we get:

$$\begin{aligned} EG \times_G \left(G \times_{N_G(A,c)} \left(c \times F^A\right)\right) &\cong [EG \times_G G] \times_{N_G(A,c)} [c \times F^A] \\ &\cong EG \times_{N_G(A,c)} (c \times F^A) \end{aligned}$$

Observe that EG serves as a model for $EN_{G(A,c)}$, so that $EG \times_{N_G(A,c)} (c \times F^A)$ is the total space of the associated fiber bundle for the pair $(N_G(A, c), c \times F^A)$. Thus, θ_G gives a homeomorphism on the total spaces, and taking cohomology gives the result.

□

Let's return to the third map in Duflot's sequence ($\star$) of equivariant cohomology maps,

$$H^*_{N_G(A,c)}(c \times F^A) \xrightarrow{③} H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)}.$$

This map is a restriction map coming from the inclusion of pairs
$(C_G(A, c), c \times F^A) \hookrightarrow (N_G(A, c), c \times F^A)$. Duflot shows that the map is an isomorphism. We outline the relevant ideas in what follows.

Since we have fixed an embedding of $G \hookrightarrow U(n)$, we get a unitary representation of A by restricting the embedding map, let's call this representation ϕ. Since A is abelian, the set of matrices in $\phi(A)$ commute, so we can simultaneously diagonalize these matrices by some matrix Q. i.e. for all $a \in \phi(A)$, $Q^{-1}aQ$ is a diagonal matrix. Further, since the matrices of $\phi(A)$ are unitary, Q is unitary too (it is a change of basis matrix between orthonormal bases.) Then, for any $a \in \phi(A)$, the columns of Q form a set of eigen-vectors for a with corresponding eigenvalues coming from the diagonal entries of $Q^{-1}aQ$. Thinking geometrically, each column of Q spans a complex line in $\mathbb{C}^n$ which is fixed by all the transformations $a \in \phi(A)$.

In the language of representation theory, we can be said to have found a set of 1-dimensional irreducible representations of ϕ, whose corresponding characters χ_i assign to each transformation $a \in \phi(A)$ its ith eigenvalue (the ith diagonal entry of $Q^{-1}aQ$). Say there are k distinct 1-dimensional characters $\{\chi_1, \ldots, \chi_k\}$, the corresponding columns of Q give an orthogonal decomposition: $\mathbb{C}^n = V_1 \perp \cdots \perp V_k$. If $\mathrm{vdim}(V_i) = n_i$, then $n = n_1 + \cdots n_k$.

Recall from the last section the description of the space $F = U/S$ where a point in F is given by a pair $\{(l_1, \ldots, l_n); (v_1^*, \ldots, v_n^*)\}$. Using this description of F and the decomposition of $\mathbb{C}^n$ by the irreducible characters of ϕ, we get the following description of the fixed point set F^A:

$$F^A = \dot{\bigcup}_{\sigma \in \Sigma(n_1, \ldots, n_k)} \sigma \cdot (F(V_1) \times \cdots \times F(V_k)) \,. \tag{$\star\star$}$$

Here $F(V_j)$ denotes the flag space of the vector space V_j, elements of the flag space are a choice of ordering on the basis lines of V_j, and the corresponding orbits in the unit circle for each line. The action by σ serves to permute the ordering of the basis given by the flag on each V_j. Then, if $\sigma = (i_{11}, \ldots, i_{1n_1}; i_{12}, \ldots, i_{1n_2}; \ldots; i_{k1}, \ldots i_{kn_k})$, we have a permutation which indicates how to swap the ordered basis lines on each $F(V_j)$. i.e.

$$\begin{aligned}\sigma \cdot \left\{(l_1, \ldots, l_{n_1}; l_{n_1+1}, \ldots, l_{n_1+n_2}; \ldots) \,; (v_1^*, \ldots, v_{n_1}^*; v_{n_1+1}^*, \ldots v_{n_1+n_2}^*; \ldots)\right\} \mapsto \\ \left\{\left(l_{i_{11}}, \ldots, l_{i_{1n_1}}; l_{i_{21}}, \ldots, l_{i_{2n_2}}; \ldots\right); (v_{i_{11}}^*, \ldots, v_{i_{1n_1}}^*; \ldots)\right\}\end{aligned}$$

Also, it's clear that $\sigma \cdot F(V_j)$ gives a different flag than $F(V_j)$, and so the disjoint union in equation ($\star\star$) is justified. To see that equation ($\star\star$) really is an equality, we only need to recall that the basis lines for each V_j come from simultaneously diagonalizing the matrices in the representation of A. On the right hand side of ($\star\star$) we have different orderings of these basis lines, but clearly they are still fixed by A. On the other hand, any ordered collection of lines (and orbits) in F which are preserved by A have to come from this same collection of lines.

As topological spaces, $\sigma \cdot F(V_j)$ is homeomorphic to $F(V_j)$ for every σ (since the only difference is an ordering of bases). And since $F(V_1) \times \cdots \times F(V_k)$ is connected and closed, we see that these really are the connected components of F^A.

Lemma 5.14. *([9] pg. 99) For any pair* $(A, c) \in \mathcal{Q}(G, X)$*,* $W_G(A, c)$ *acts freely on the set of components of* $(c \times F^A)^i, i = 1, 2$.

Proof. Let's show that $C_G(A)$ fixes each component of F^A. A component of F^A has the form $\sigma \cdot (F(V_1) \times \cdots F(V_k))$. Since σ only permutes the order of lines in each flag space, we begin by showing that for a given line $l \in V_j$, and for any choice $g \in C_G(A)$, $g \cdot l \in V_j$. Say that l is the $\mathbb{C}$-span of a unit vector v. For any $a \in A$, $ga = ag$, so that $ga \cdot v = ag \cdot v$. On the left hand side, we know that a acts on v by multiplication by the jth irreducible character of the unitary representation of A, so $ga \cdot x = g\chi_j(a)v = \chi_j(a)g \cdot v$. Therefore, $a \cdot (g \cdot v) = \chi_j(a)(g \cdot v)$. In other words, $g \cdot v$ is an eigenvector for a with eigenvalue $\chi_j(a)$, and so g preserves the line spanned by v. Also note that the orbit of v under the pth roots of unity in the line l is carried along by the action of g. Since we've shown that $C_G(A)$ preserves each line in V_j, it must preserve the flag $F(V_j)$ (which just orders these lines). Since $C_G(A)$ preserves each flag, it preserves the product $F(V_1) \times \cdots \times F(V_k)$, and hence any permutation of the ordering of bases, i.e. any $\sigma \cdot (F(V_1) \times \cdots \times F(V_k))$. Now, since $C_G(A, c) \leq C_G(A)$, it's true that for all $g \in C_G(A, c)$ and for all components d of F^A, $gd = d$.

Finally, we need to verify that any element of $N_G(A)$ which fixes a component of F^A is in the centralizer. Let $g \in N_G(A)$ be such that $g \cdot (F(V_1) \times \cdots \times F(V_k)) = (F(V_1) \times \cdots F(V_k))$. Then, g takes lines in V_j to lines in V_j for each j. Then, if $a \in A$, and $x \in V_j$, we have $a \cdot (gx) = \chi_j(a)gx = g \cdot (\chi_j(a)x) = g \cdot ax$, so $a^{-1}g^{-1}agx = x$ for every x and every a. Therefore, $a^{-1}g^{-1}ag$ is the identity transformation in $U(n)$, and we have that g centralizes A.

Since $N_G(A, c) \leq N_G(A)$, and the components of $c \times F^A$ are of the form $c \times d$, where d is a component of F^A, if $n \in N_G(A, c)$, we know that $n(c \times d) \doteq nc \times nd \doteq c \times nd$. Then, $n(c \times d) = c \times d$ if and only if $nd = d$, which by the statements above happens only when $n \in C_G(A)$, and since n was assumed to be in $N_G(A, c)$, it follows by definition that $n \in C_G(A, c)$. $\square$

Corollary 5.15. *([9] Lemma 3.5) If $(A,c) \in \mathcal{Q}(G,X)$, then $W_G(A,c)$ acts freely on $\pi_0(c \times (F^A)^i)$, and $H^q_{C_G(A,c)}(c \times (F^A)^i)$ is a free $\mathbb{Z}/p\mathbb{Z}[W_G(A,c)]$-module for all $i \geq 1$ and all $q \geq 0$.*

Lemma 5.16. *([9] Lemma 3.6) For $(A,c) \in \mathcal{Q}(G,X)$, there is an isomorphism*

$$H^*_{N_G(A,c)}(c \times F^A) \cong H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)}.$$

Proof. Let EN be a choice of model for the total space of the universal principal $N_G(A,c)$-bundle. Since $C_G(A,c)$ is a subgroup of $N_G(A,c)$, we can take EN as a model for the total space of the universal principal $C_G(A,c)$-bundle. Let $Z = c \times F^A$, and let $W = W_G(A,c)$. Consider the map $EN \times_{C_G(A,c)} Z \to EN \times_{N_G(A,c)} Z$, which sends a class represented by (e,z) to a class represented by (e,z) (only the quotient has changed.) We see that the fiber over any point $[e,z] \in EN \times_{N_G(A,c)} Z$ is in bijective correspondence with elements of W. In fact, we get a principal W-bundle: $W \to EN \times_{C_G(A,c)} Z \to EN \times_{N_G(A,c)} Z$.

The homotopy theory of universal principal bundles tells us that every W-bundle is "classified" by the universal W-bundle, so we get a commutative diagram of G-bundles:

$$\begin{tikzcd}
EN \times_{C_G(A,c)} Z \arrow[r] \arrow[d] & EW \arrow[d] \\
EN \times_{N_G(A,c)} Z \arrow[r] & BW
\end{tikzcd}$$

Now, the map $EN \times_{C_G(A,c)} Z \to EN \times_{N_G(A,c)} Z \to BW$ is a Serre fibration, and we may apply the Serre spectral sequence (see [22] for a definition). The $E2$-term of the Serre spectral sequence converges: $H^p(BW; \{H^q_{C_G(A,c)}(Z)\}) \Rightarrow H^{p+q}_{N_G(A,c)}(Z)$. By the previous corollary, the coefficients $H^q_{C_G(A,c)}(Z)$ are free modules for all $q \geq 0$. Since W is a finite group, we can use the homological algebraic tools of group cohomology to conclude that $H^p(BW; \{H^q_{C_G(A,c)}(Z)\}) = 0$ for all $p > 0$ and all $q \geq 0$ (essentially since all higher Ext terms vanish on free resolutions).

Thus, the spectral sequence degenerates so that $H^0\left(BW, \{H^*_{C_G(A,c)}(Z)\}\right) \cong H^*_{N_G(A,c)}(Z)$. Finally, group cohomology gives us that $H^0\left(BW, \{H^*_{C_G(A,c)}(Z)\}\right) \cong H^*_{C_G(A,c)}(Z)^W$, and the result is proved. $\square$

We are now ready to present a sketch of a proof of Duflot's localization theorem (5.12).

Theorem 5.17. *([9] pg. 100) Fix an $[A,c] \in \mathcal{Q}'_{max}(G,X)$ with corresponding minimal prime $\mathfrak{p}_{(A,c)} \doteq \mathfrak{p}$. The following diagram is commutative, each of the vertical arrows are graded isomorphisms of $R_{[\mathfrak{p}]} = H_G(X)_{[\mathfrak{p}]}$-modules, and the first and last rows are equalizer sequences.*

$$\begin{tikzcd}
H^*_G(X)_{[\mathfrak{p}]} \arrow[r] & H^*_G(X\times F)_{[\mathfrak{p}]} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "1"] & H^*_G(X\times F^2)_{[\mathfrak{p}]} \arrow[d] \\
 & H^*_G(G\cdot(c\times F^A))_{[\mathfrak{p}]} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "2"] & H^*_G(G\cdot(c\times (F^A)^2))_{[\mathfrak{p}]} \arrow[d] \\
 & H^*_{N_G(A,c)}(c\times F^A)_{[\mathfrak{p}]} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "3"] & H^*_{N_G(A,c)}(G\cdot(c\times (F^A)^2))_{[\mathfrak{p}]} \arrow[d] \\
 & H^*_{C_G(A,c)}(c\times F^A)^{W_G(A,c)}_{[\mathfrak{p}]} \arrow[r, shift left] \arrow[r, shift right] & H^*_{C_G(A,c)}(G\cdot(c\times (F^A)^2))^{W_G(A,c)}_{[\mathfrak{p}]} \\
H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{[\mathfrak{p}]} \arrow[r] & H^*_{C_G(A,c)}(c\times F)^{W_G(A,c)}_{[\mathfrak{p}]} \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "4"] & H^*_{C_G(A,c)}(G\cdot(c\times F^2))^{W_G(A,c)}_{[\mathfrak{p}]} \arrow[u]
\end{tikzcd}$$

Taking the claims of this theorem for granted, Duflot's localization result is obtained from this diagram since each square is commutative, all of the vertical arrows are isomorphisms, and then relating the exact sequence of the first row to the exact sequence of the last row. Let's investigate this diagram a bit more thoroughly.

We've seen already that vertical arrows ② and ③ are isomorphisms without the localization. Vertical arrow ① comes from a restriction map, but is only an isomorphism after localization. We have not explained how this map is an isomorphism, the details are explained in lemma 3.3 of [9]. Similarly, arrow ④ comes from a restriction map, but is only an isomorphism after localizing and taking invariants under $W_G(A,c)$. That this map is an isomorphism is also an application of lemma 3.3 of [9].

The exactness of the first and final rows of Duflot's diagram 5.17 come from lemma 5.10. Specifically, apply the result to the pairs (G, X) and $(C_G(A, c), c)$ respectively, to get the equalizer sequences: $H^*_G(X) \to H^*_G(X \times F) \rightrightarrows H^*_G(X \times F^2)$, and $H^*_{C_G(A,c)}(c) \to H^*_{C_G(A,c)}(c \times F) \rightrightarrows H^*_{C_G(A,c)}(c \times F^2)$. Since localization is exact, and taking invariants is left exact, we get exactness of the top and bottom rows of Duflot's diagram 5.17, and conclude that $H^*_G(X)_{[\mathfrak{p}]} \cong H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{[\mathfrak{p}]}$.

Chapter 6

The Degree of an Equivariant Cohomology Ring

Recall from section 3 of part 1 the definition of the degree of a graded module $M \in \mathfrak{grmod}(R)$: $\deg(M) \doteq \lim_{t\to 1}(1-t)^{*\dim_R(M)} PS_M(t)$. We proved (Theorem 3.8) a relationship between this numerical measure and the different notions of graded multiplicity from both commutative algebra and homological algebra. We also presented (Theorem 3.6) a decomposition of the degree by considering the irreducible components of $Spec(R)$. In this section we prove the main result of this part of the dissertation - For equivariant cohomology rings, the algebraic decomposition of degree by irreducible components has an interpretation using Quillen's identification of maximal rank elementary abelian subgroups with minimal primes, and Duflot's localization result.

6.1 Degree Decomposition of Lynn

This short section is intended to be a snapshot of Lynn's results [18], and serves simply to give a flavor of Lynn's methods, rather than rigorously go through her approach. Her main result is a summation decomposition of the degree of an equivariant cohomology ring in the case where X is taken to be a point. This work was the starting point for our more general degree decomposition of the full equivariant cohomology ring (Theorem 6.8.)

The reason Lynn is unable to produce a degree summation formula for the entire equivariant cohomology ring seems to stem from a heavy reliance on topological methods (e.g. the Gysin sequence) to derive results which we have produced using commutative algebra. We also point out that the results of Lynn require stronger topological hypotheses than our results, namely, her results require X to be a compact smooth manifold.

Theorem 6.1. *([18] Theorem 4.21) Let G be a compact Lie group, and let X be a smooth, compact G-manifold. Let $Z = \cup_{i=1}^n Z_i$, where the Z_i's are closed, G-invariant, disjoint submanifolds of X such that ν_{Z_i} (the normal bundle) is orientable for all i. Assume that* $\dim(H_G^*(X)) = \dim(H_G^*(Z_i)$

for all i, and if $z \notin Z$, then $\dim(H^*_{G_z}) < \dim(H^*_G(Z_i))$ *for all i. Then,*

$$\deg(H^*_G(X)) = \sum_{i=1}^{n} deg(H^*_G(Z_i)).$$

By lemma 5.9, for any $[A, c] \in \mathcal{Q}'_{max}(G, X)$ the sub-space $G\cdot(c\times F^A)$ satisfies the hypotheses of Z_i in the above theorem, so we have:

Corollary 6.2. *For X a compact, smooth manifold with G a compact Lie group acting smoothly on X,* $\deg(H^*_G(X)) = \sum_{[A,c]\in\mathcal{Q}'_{max}(G,X)} \deg(H^*_G(G\cdot(c\times F^A))$

Lynn argues by descent (taking X to be a point), to derive the following summation decomposition of degree.

Main theorem of R.Lynn:

Theorem 6.3. *[18] Let G be a compact Lie group, and let $\mathcal{Q}'_{max}(G)$ be the set of conjugacy classes of maximal rank elementary abelian p-groups of G. Then,*

$$\deg(H^*_G) = \sum_{[A]\in\mathcal{Q}'_{max}(G)} \frac{1}{|W_G(A)|}\deg(H^*_{C_G(A)}).$$

6.2 Main Theorem on Degree

As usual, we assume that G is compact Lie, and it acts continuously on the Hausdorff space X, which is compact or paracompact with finite mod-p cohomological dimension. Fix an odd prime p, and take $k = \mathbb{Z}/p\mathbb{Z}$ to be the field of coefficients for cohomology.

We will make use of Duflot's localization result 5.12, so let's look a little more closely at the ring $H^*_{C_G(A,c)}(c)$ for a given pair $(A, c) \in \mathcal{Q}(G, X)$. A first observation is that $H^*_{C_G(A,c)}(c)$ may be thought of as a module in the category $\mathfrak{grmod}(H_{C_G(A,c)}(c))$, or as a module in $\mathfrak{grmod}(H_G(X))$. Its structure as a finitely generated, graded module over $H_G(X)$ comes from the restriction map $res^G_C : H^*_G(X) \to H^*_{C_G(A,c)}(c)$ - an application of theorem 5.4. The following lemma relates the minimal primes of $H^*_{C_G(A,c)}(c)$ given the two different module structures.

Lemma 6.4. *Let* $[A,c] \in \mathcal{Q}'_{max}(G,X)$. *Let* $R = H_G(X)$, $S = H_{C_G(A,c)}(c)$, *and denote the restriction map by* $res^G_C : R \to S$. *Observe that* $H^*_{C_G(A,c)}(c)$ *is naturally an* S*-module, and it is an* R*-module via the restriction map. Then,*

i. $\mathcal{Q}'_{max}(C_G(A,c),c) = \{[A,c]\}$

ii. $\mathfrak{p}^C \doteq \ker\left(H_{C_G(A,c)}(c) \to H_A/\sqrt{0}\right)$ *is the unique minimal prime for* $H^*_{C_G(A,c)}(c)$ *as an* S*-module.*

iii. $(res^G_C)^{-1}\left(\mathfrak{p}^C\right) = \mathfrak{p}$, *where* $\mathfrak{p} \doteq \ker\left(H_G(X) \to H_A/\sqrt{0}\right)$.

iv. $\mathfrak{p}$ *is the unique minimal prime for* $H^*_{C_G(A,c)}(c)$ *as an* R*-module.*

v. $*\ell_{S_{[\mathfrak{p}^C]}}\left(H^*_{C_G(A,c)}(c)_{[\mathfrak{p}^C]}\right) < \infty$ *and* $*\ell_{R_{[\mathfrak{p}]}}\left(H^*_{C_G(A,c)}(c)_{[\mathfrak{p}]}\right) < \infty$*; also,*

$$\begin{aligned}\deg\left(H^*_{C_G(A,c)}(c)\right) &= *\ell_{S_{[\mathfrak{p}^C]}}\left(H^*_{C_G(A,c)}(c)_{[\mathfrak{p}^C]}\right)\deg(S/\mathfrak{p}^C)\\ &= *\ell_{R_{[\mathfrak{p}]}}\left(H^*_{C_G(A,c)}(c)_{[\mathfrak{p}]}\right)\deg(R/\mathfrak{p}).\end{aligned}$$

Proof. **(i)** We use Quillen's identification 5.8 of minimal primes with maximal rank classes of elementary abelian subgroups to show that $H^*_{C_G(A,c)}(c)$ has a unique minimal prime. We have that $A \leq C_G(A,c)$ since it commutes with its own elements (it's abelian by definition) and c is a component of X^A. Also, since we supposed (A,c) was a maximal rank pair in $\mathcal{Q}(G,X)$, it is necessarily maximal in $\mathcal{Q}(C_G(A,c),c)$. Suppose that $(B,d) \in \mathcal{Q}(C_G(A,c),c)$ was some other maximal rank pair.

Consider the group $D \doteq \langle A,b\rangle$ where b is any non-trivial element in B. D is a subgroup of $C_G(A,c)$, and in fact must be an elementary abelian subgroup, for every element $a \in A$ has order p, so the element b has order p, and b commutes with every $a \in A$ since it is an element of the centralizer of A. Thus, D is an abelian group for which every element has order p, i.e. it is elementary abelian. By the maximal rank assumption on A in G, D and A are elementary abelian subgroups of the same rank; moreover, since $A \subseteq D$, they are on the nose equal. Since b

was chosen arbitrarily in B, we have that $A = B$. Observe that $d = c$ since $d \subseteq c^B$, but c was assumed to be a connected component, and is maximal by definition. d is non-empty by definition of Quillen's category of pairs, so d and c must be equal. Thus, there is a unique maximal class $[A, c] \in \mathcal{Q}(C_G(A, c), c)$. Then, Quillen's theorem implies that $\mathfrak{p}^C$ is a unique minimal prime in $H^*_{C_G(A,c)}(c)$ proving **(ii)**.

(iii) The following diagram is commutative, the arrows are inclusions, and pt is any point in c:

$$\begin{tikzcd}
(G, X) & (C_G(A, c), c) \arrow[l, hook'] \\
(A, pt) \arrow[u, hook] \arrow[ur, hook]
\end{tikzcd}$$

By functoriality, there is a commutative diagram:

$$\begin{tikzcd}
H_G(X) \arrow[r, "res^G_C"] \arrow[d, "\pi \circ res^G_A"'] & H_{C_G(A,c)}(c) \arrow[dl, "\pi \circ res^C_A"] \\
H_A(pt)/\sqrt{0}
\end{tikzcd}$$

By commutativity, $\mathfrak{p} = res^{G-1}_C(\mathfrak{p}^C)$.

(iv) By Quillen's theorem, $\mathfrak{p}$ is a minimal prime for $H^*_G(X)$ as an R-module. Quillen's finiteness theorem gives that $res^G_C : R \to S$ is an integral extension. Let's show that $\mathfrak{p}$ is a minimal prime for $H^*_{C_G(A,c)}(c)$ as an R-module. We need to show that $\mathfrak{p}$ is minimal over $Ann_R(H^*_{C_G(A,c)}(c)) \doteq \left\{ r \in R : res^G_C(r) \cdot x = 0, \text{ for all } x \in H^*_{C_G(A,c)} \right\}$. But $H^*_{C_G(A,c)}(c)$ is a unital ring, so it is clear that $Ann_R(H^*_{C_G(A,c)}(c)) = \ker(res^G_C)$, and by commutativity of the diagram $\ker(res^G_C) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is minimal in R, it must be minimal over $Ann_R(H^*_{C_G(A,c)}(c))$ and thus is a minimal prime for $H^*_{C_G(A,c)}(c)$ as an R-module.

Finally, we need to show that $\mathfrak{p}$ is the unique minimal prime for $H^*_{C_G(A,c)}(c)$ as an R-module. We use the "lying over theorem" (chapter 5 section 0.) Let $\mathfrak{q} \subseteq R$ be another minimal prime for $H^*_{C_G(A,c)}(c)$ as an R-module, "Lying Over" implies there exists a $\tilde{\mathfrak{q}} \in Spec(S)$ such that

$(res^G_C)^{-1}(\tilde{\mathfrak{q}}) = \mathfrak{q}$. But, $\mathfrak{p}^C$ is the only minimal prime in $Spec(S)$, so $\mathfrak{p}^C \subseteq \tilde{\mathfrak{q}}$. Thus, $(res^G_C)^{-1}(\mathfrak{p}^C) \subseteq (res^G_C)^{-1}(\tilde{q})$ which implies that $\mathfrak{p} \subseteq \mathfrak{q}$, but by assumption $\mathfrak{q}$ is minimal, and therefore $\mathfrak{p} = \mathfrak{q}$.

(v) For ease of notation, let $N \doteq H^*_{C_G(A,c)}(c)$. Parts ii and iv of this lemma show that $\mathfrak{p}$ is a minimal prime for N as an R-module and $\mathfrak{p}^C$ is a minimal prime for N as an S-module. By Theorem 1.26 from part 1, $N_{[\mathfrak{p}]}$ is a *Artinian $R_{[\mathfrak{p}]}$-module, and $N_{[\mathfrak{p}^C]}$ is a *Artinian $S_{[\mathfrak{p}^C]}$-module, which proves the claim on finite *length.

To prove the claim on degree, we use the algebraic summation decomposition of degree (Theorem 3.6 part 1.) We restate the result here: For R any positively graded Noetherian ring with $R_0 = k$ a field, if $M \in \mathfrak{grmod}(R)$, then $\deg(M) = \sum_{\mathfrak{p}\in\mathcal{D}(M)} *\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p})$, where $\mathcal{D}(M)$ is the set of minimal primes of M such that $*\dim(R/\mathfrak{p}) = *dim_R(M)$.

We showed above, in parts ii and iv, that $\mathfrak{p}$ is the unique minimal prime for N as an R-module, and $\mathfrak{p}^C$ is the unique minimal prime for N as an S-module. Therefore, by applying the algebraic summation decomposition to N as an R-module, there is only one summand: $\deg(N) = *\ell_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]})\deg(R/\mathfrak{p})$, and similarly the summation decomposition applied to N as an S-module: $\deg(N) = *\ell_{S_{[\mathfrak{p}^C]}}(N_{[\mathfrak{p}^C]})\deg(S/\mathfrak{p}^C)$. Since either formula computes the same number, they are equal. $\square$

We state the following theorem in a general algebraic setting, and will demonstrate an application (Theorem 6.6) to the cohomology ring $H^*_{C_G(A,c)}(c)^{W_G(A,c)}$. As it relates to our main theorem on the degree of equivariant cohomology rings, we aim to localize the invariant ring at the minimal prime $\mathfrak{p}_{[A,c]} \in Spec(H_G(X))$ to produce a decomposition which allows for the computation of *length. Specifically, we show that $*\ell\left(H^*_{C_G(A,c)}(c)^{W_G(A,c)}_{\mathfrak{p}_{[A,c]}}\right)$ may be computed from the order of the group $W_G(A,c)$ and $*\ell\left(H^*_{C_G(A,c)}(c)_{\mathfrak{p}_{[A,c]}}\right)$. In general settings, computing the length of an invariant ring can be complicated, and may require some knowledge of invariant theory. For this particular case, the free $W_G(A,c)$-action simplifies things considerably.

Theorem 6.5. *Let L be a $\mathbb{Z}$-graded k-algebra for a field k (concentrated in degree 0,) and let P be a graded L-module. Suppose the following:*

- *π_0 is a finite set indexing a direct sum decomposition of P by graded sub-modules. Specifically, $P = \oplus_{c\in\pi_0} P_c$, where each P_c is a finite dimensional vector space over k in each degree.*

- *W is a finite group which acts as a group of graded L-module automorphisms on P.*

- *W acts freely on the set π_0, and in such a way that respects the decomposition of P; that is, for any $x \in P_c$, $w \cdot x \in P_{w\cdot c}$.*

Given these suppositions, we prove that:

i. *Let $c_1, \ldots, c_t \in \pi_0$ be a set of representatives for the orbits of W on π_0. Note that $t = |\pi_0|/|W|$. For each j from 1 to t, define $P[j] = \oplus_{w\in W} P_{wc_j}$. Then, $P[j]$ is a graded submodule of P with respect to both L and $k[W]$. Also, $P = \oplus_{j=1}^{t} P[j]$.*

ii. *P is a free $k[W]$-module. Here, $k[W]$ is regarded as a graded ring concentrated in degree 0.*

iii. *P^W is isomorphic as a graded L-module to $\oplus_{j=1}^{t} P_{c_j}$.*

iv. *If P is a *Artinian L-module, then so is P^W and,*

$$*\ell_L(P) = |W| * \ell_L(P^W).$$

Proof. **(i)** Note that for each j, $P[j]$ really is an "internal" direct sum since for any $w, \tilde{w} \in W$, $P_{wc_j} = P_{\tilde{w}c_j}$ if and only if $w = \tilde{w}$ (this is due to the free action of W on the components.) Therefore, $P_{wc_j} \cap P_{\tilde{w}c_j} = \{0\}$.

The fact that $P[j]$ is a graded L-submodule follows because the given decomposition $P = \oplus_c P_c$ is by graded L-submodules, and $P[j]$ is a just a particular sum of these sub-modules, so certainly the L-module structure is preserved. The $k[W]$-multiplication on $P[j]$ is closed since the W-action on P is free on components, and $P[j]$ contains the full orbit of P_{c_j} under W. Note that all W-actions preserve grading by hypothesis.

Using the given decomposition $P = \oplus_c P_c$ along with the fact that we've picked $c_1, \ldots, c_t$ as representatives for the orbits under W (and hence $P[j] \cap P[i] = 0$ for $i \neq j$) gives us the decomposition by orbits: $P = \oplus_{j=1}^t P[j]$.

(ii) To show that P is a free $k[W]$-module, we appeal to the graded structure on P and use the decomposition $P = \oplus_{j=1}^t P[j]$ to produce a $k[W]$-basis. Specifically, let $\{e_1, \ldots, e_n\}$ be a k-vector space basis for the degree d homogeneous component of P_{c_j}. We claim that $\{e_1, \ldots, e_n\}$ is a $k[W]$-basis for the degree d component $P[j]_d \subseteq P[j]$. By picking a k-basis for every homogeneous component, and every j, we get a $k[W]$-basis for $P = \oplus_{j=1}^t P[j]$ (note that the basis will not be finite in general.)

Let $x \in P[j]_d$, and use the decomposition by orbits to write x as the unique sum: $x = \sum_{w \in W} x_w$, $x_w \in P_{w \cdot c_j}$. By properties of the W-action, $w^{-1}x_w \in P_{w^{-1}wC_j} = P_{cj}$ for each w. Use the vector space basis of P_{c_j} to write $w^{-1}x_w = \sum \alpha(w)e_i$ where $\alpha(w) \in k$. Thus, $x = \sum_w x_w = \sum_w \sum_i w\alpha(w)e_i$, which is in the $k[W]$-span of the $\{e_1, \ldots, e_n\}$.

It remains to be seen that the basis is linearly independent. Suppose $\sum_i \xi_i e_i = 0$ for $\xi_i \in k[W]$. For each i, write ξ_i as a linear combination $\sum_{w \in W} \alpha(w)w$, $\alpha(w) \in k$. Then, $\sum_i \xi_i e_i = \sum_i \sum_w \alpha(w)we_i = \sum_w \sum_i (\alpha(w)we_i)$. Now for each $w \in W$, $w \cdot \{e_1, \ldots, e_n\}$ is contained in P_{wc_j}, and $w \cdot \{e_1, \ldots, e_n\}$ is a vector space basis for the degree d homogeneous component of P_{wc_j} because w is a graded automorphism of P.

Using the fact that $P[j]$ is an internal direct sum (since W acts freely on components), we get that $\sum_w \sum_i (\alpha(w)we_i) = 0$ if and only if $\sum_i (\alpha(w)we_i) = 0$ for each $w \in W$. Put another way, we are in the situation where for each $w \in W$ there is a k-subspace P_{wc_j}, and taken all together, these subspaces are mutually disjoint. Then, for every individual subspace we picked a set of basis vectors, and took a linear combination of all the basis vectors across all of the subspaces. This linear combination can only sum to 0 if the linear combination taken on each individual subspace is 0, because of the fact that all subspaces are mutually disjoint.

Now, $\sum_i (\alpha(w)we_i) = 0$ for each w, so it must be true that $w^{-1}\sum_i (\alpha(w)we_i) = 0$. Distributing the w^{-1} gives $\sum_i (\alpha(w)e_i) = 0$. Since $\{e_1, \ldots, e_n\}$ is a vector space basis, we get that $\alpha(w) = 0$ for every i and for every w. Therefore $\xi_i = 0$ for all i.

(iii) Define $\theta_j : P_{c_j} \to P[j]$ by $x \mapsto \sum_{w\in W} w \cdot x$. Since each w acts as a graded L-module homomorphism, and θ_j is a linear combination of these homomorphisms, it's clear that θ_j is a graded L-module homomorphism. Further, θ_j is injective since $P[j]$ is an internal direct sum, so the only way $\sum_{w\in W} w \cdot x = 0$ is if $w \cdot x = 0$ for every w. Assuming W is a non-trivial group, this occurs only if $x = 0$.

We claim that the image of θ_j is $P[j]^W$. If we can show the claim to be true, the result is obtained since invariants distribute over direct sums of $k[W]$-modules. i.e. $P^W = \oplus_{j=1}^t \left(P[j]^W\right) \xrightarrow{\oplus_j \theta_j} \oplus_{j=1}^t P_{c_j}$ is an isomorphism.

Let $x \in P_{c_j}$. We may write x uniquely as $x = \sum_{w\in W} x_w$. Then, for all $g \in W$,

$$\begin{aligned}
g \cdot \theta_j(x) &= g \cdot \sum_{w\in W} w \cdot x \\
&= \sum_{w\in W} g \cdot w \cdot x \\
&= \sum_{w\in W} (g \cdot w) \cdot x \\
&= \sum_{\tilde{w}\in W} \tilde{w} \cdot x \\
&= \theta_j(x).
\end{aligned}$$

The re-indexing by $\tilde{w}$ is due to the fact that W is a finite group, so any element $g \in W$ acts as an automorphism of W. In other words, multiplication by g just permutes the order of the summands, but of course that doesn't change the sum itself. Therefore, $Im(\theta_j) \subseteq P[j]^W$.

Let $x \in P[j]^W$, and write x uniquely as $\sum_{w\in W} x_w$. For every $g \in W$, $g \cdot \sum_{w\in W} x_w = \sum_{w\in W} gx_w$. Since $gx_w \in P_{gwc_j}$, and $\sum_{w\in W} g \cdot x = x$ for every g, we see that the action of every $g \in G$ is simply to permute the summands x_w of x. Let x_{w*} be the summand of x that is in P_{c_j}. For each $g \in W$, there is one summand of x corresponding to $x_{g^{-1}w}$, and $g \cdot x_{g^{-1}} \in P_{c_j}$, so $g \cdot x_{g^{-1}} = x_{w^*}$. Therefore, each $g \in W$ sends the summand x_{w^*} to a unique summand x_{gw^*} of x, and taking the orbit of x_{w^*} under W recovers all of the summands of x. In other words, $\theta_j(x_{w*}) = \sum_{w \in W} w x_{w*} = x$, and therefore, $P[j]^W \subseteq Im(\theta_j)$.

(iv) Note that for each w and each c_j, $*\ell(P_{cj}) = *\ell(P_{wc_j})$ since W acts as a group of automorphisms, and must preserve length of submodules. Now we use that *length adds over direct sums along with the decompositions established in this theorem: $*\ell_L(P) = \sum_{j=1}^{t} \sum_{w \in W} *\ell_L(P_{wc_j}) = |W| \sum_{j=1}^{t} *\ell_L(P_{c_j}) = |W| * \ell_L(P^W)$.

$\square$

Theorem 6.6. *Let $[A, c] \in \mathcal{Q}'_{max}(G, X)$. Consider $H^*_{C_G(A,c)}(c \times F^A)$ as an $H_G(X)$-module. Let $\mathfrak{p} \in Spec(H_G(X))$ be the minimal graded prime corresponding to $[A, c]$ under Quillen's identification. Then,*

i. Consider $H^*_{C_G(A,c)}(c \times F^A)$ as a graded $H_G(X)$-module. Let π_0 be the connected components of $c \times F^A$, and let $W = W_G(A, c)$. Then,

(a) $H^*_{C_G(A,c)}(c \times F^A)$ is a free $k[W_G(A, c)]$-module.

(b) If c_i, $1 \leq i \leq t$, are the representatives for the orbits of $W_G(A, c)$ acting on the set of components of $c \times F^A$, then

$$H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)} \cong \oplus_{i=1}^{t} H^*_{C_G(A,c)}(c_i).$$

ii. $*\ell_{H_G(X)_{[\mathfrak{p}]}} \left(H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)}_{[\mathfrak{p}]} \right) = \frac{1}{|W_G(A,c)|} * \ell_{H_G(X)_{[\mathfrak{p}]}} \left(H^*_{C_G(A,c)}(c \times F^A)_{[\mathfrak{p}]} \right)$

Proof. We begin by showing that the hypotheses of the previous theorem (6.5) are met: Recall that the W-action on $H^*_{C_G(A,c)}(c \times F^A)$ is induced by $N_G(A, c)$ acting by conjugation on $C_G(A, c)$, and by multiplication on c and F. In corollary 5.15 we demonstrated that W acts freely on the set of connected components of $c \times F^A$ for any $[A, c]$ in Quillen's category. In section 5.3, we gave a description of the connected components of F^A by permutations of a flag space, of which there are only a finite number, and therefore there are only a finite number of connected components of F^A. Correspondingly, $\pi_0(c \times F^A)$, the set of connected components of $c \times F^A$, is a finite set, say of size t.

For each $w \in W$, since w^* has inverse $(w^{-1})^*$, each w^* is an automorphism (graded) of $H^*_{C_G(A,c)}(c \times F^A)$. Further, Quillen shows that W is a finite group ([23]).

An elementary property of cohomology is that it adds over connected components. In this case, for any $q \geq 0$, $H^q_{C_G(A,c)}(c \times F^A) = \oplus_{c_i \in \pi_0} H^q_{C_G(A,c)}(c_i)$. Since W acts freely on $\pi_0(c \times F^A)$, the action on cohomology respects this direct sum decomposition.

Therefore, we are in position to apply Theorem 6.5: $H^*_{C_G(A,c)}(c \times F^A)$ is a free $k[W_G(A,c)]$-module frollows from part ii of 6.5, and $H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)} \cong \oplus_{i=1}^t H^*_{C_G(A,c)}(c_i)$ follows from part iii.

Finally, localizing $H^*_{C_G(A,c)}(c \times F^A)$ at the minimal prime $\mathfrak{p}$ gives us a *Artinian $H_G(X)_{\mathfrak{p}}$-module (Corollary 1.30), and so we apply part iv of 6.5 which completes the proof. $\square$

Theorem 6.7. *Let $R = H_G(X)$, and suppose that $\mathfrak{p} \in Spec(R)$ is a graded minimal prime such that $*\dim(R/\mathfrak{p}) = *\dim(R)$, and $(A,c) \in \mathcal{Q}(G,X)$ is a maximal pair which corresponds to $\mathfrak{p}$ under Quillen's identification (5.8). Then,*

$$*\ell_{R_{[\mathfrak{p}]}}(H^*_G(X)_{[\mathfrak{p}]}) = \frac{1}{|W_G(A,c)|} * \ell_{R_{[\mathfrak{p}]}}(H^*_{C_G(A,c)}(c)_{[\mathfrak{p}]}).$$

Proof. In this computation abbreviate $C_G(A,c)$ to C_G, $W_G(A,c)$ to W, let $R = H_G(X)$, and $k = \mathbb{Z}/p$. We prove this theorem in five steps.

1) We use lemma 5.10, which is Quillen's result that $H^*_G(X \times F) \cong H^*_G(X) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F)$. Now, F is a finite dimensional compact manifold, so $H^*(F)$ is a finite dimensional graded vector space over k, let's say it has dimension m. We use that *length adds over the tensor (lemma 1.17.) Specifically,

$$*\ell_{R_{[\mathfrak{p}]}}(H^*_G(X \times F)_{[\mathfrak{p}]}) = *\ell_{R_{[\mathfrak{p}]}}(H^*_G(X)_{[\mathfrak{p}]} \otimes *\ell_{\mathbb{Z}/p\mathbb{Z}}(H^*(F)) = *\ell_{R_{[\mathfrak{p}]}}(H^*_G(X)_{[\mathfrak{p}]} \cdot m.$$

2) Refer back to the diagram of theorem 5.17. The first vertical arrow of this diagram is an isomorphisms: $H^*_G(X \times F)_{[\mathfrak{p}]} \cong H^*_{C_G}(c \times F^A)^W_{[\mathfrak{p}]}$.

3) Part ii of Theorem 6.6, proved that we can rewrite the *length of $H^*_{C_G(A,c)}(c \times F^A)_{[\mathfrak{p}]}$:

$$*\ell_{H_G(X)_{[\mathfrak{p}]}}\left(H^*_{C_G(A,c)}(c \times F^A)^{W_G(A,c)}_{[\mathfrak{p}]}\right) = \frac{1}{|W_G(A,c)|} * \ell_{H_G(X)_{[\mathfrak{p}]}}\left(H^*_{C_G(A,c)}(c \times F^A)_{[\mathfrak{p}]}\right).$$

4) By referring again to the diagram of theorem 5.17, the fourth vertical arrow gives an isomorphism:

$$H^*_{C_G(A,c)}(c \times F^A)_{[\mathfrak{p}]} \cong H^*_{C_G(A,c)}(c \times F)_{[\mathfrak{p}]}.$$

5) As in step 1, we apply Quillen's result: $H^*_G(X \times F) \cong H^*_G(X) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F)$, except this time we replace G by C_G, and X by c. Thus, $H^*_G(c \times F) \cong H^*_{C_G}(c) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F)$.

Putting these five results together gives us the following computation:

$$\begin{aligned} *\ell_{R_{[\mathfrak{p}]}}(H^*_G(X)_{[\mathfrak{p}]}) &\overset{(1)}{=} \frac{1}{m} * \ell_{R_{[\mathfrak{p}]}}(H^*_G(X \times F)_{[\mathfrak{p}]}) \\ &\overset{(2)}{=} \frac{1}{m} * \ell_{R_{[\mathfrak{p}]}}(H^*_{C_G}(c \times F^A)^W_{[\mathfrak{p}]})) \\ &\overset{(3)}{=} \frac{1}{m}\frac{1}{|W|} * \ell_{R_{[\mathfrak{p}]}}(H^*_{C_G}(c \times F^A)_{[\mathfrak{p}]}) \\ &\overset{(4)}{=} \frac{1}{m}\frac{1}{|W|} * \ell_{R_{[\mathfrak{p}]}}(H^*_{C_G}(c \times F)_{[\mathfrak{p}]}) \\ &\overset{(5)}{=} \frac{1}{m}\frac{1}{|W|} \cdot m \cdot *\ell_{R_{[\mathfrak{p}]}}(H^*_{C_G}(c)_{[\mathfrak{p}]}) \\ &= \frac{1}{|W|} * \ell_{R_{[\mathfrak{p}]}}(H^*_{C_G}(c)_{[\mathfrak{p}]}) \end{aligned}$$

$\square$

We are now in position to prove our main result of the dissertation.

A Geometric Decomposition of Degree for Equivariant Cohomology:

Theorem 6.8. *Let G be a compact Lie group which acts continuously on a topological space X. All cohomology will be taken with coefficients in $\mathbb{Z}/p$ for p prime. Let X be either compact or paracomapact with finite mod-p cohomological dimension. Then,*

$$\deg(H_G^*(X)) = \sum_{[A,c]\in \mathcal{Q'}_{max}(G,X)} \frac{1}{|W_G(A,c)|} \deg(H^*_{C_G(A,c)}(c)).$$

Also, recall the algebraic decomposition of degree (theorem 3.6) for a graded module $M \in \mathfrak{grmod}(R)$:

$$\deg(M) = \sum_{\mathfrak{p}\in \mathcal{D}(M)} *\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

Our proof establishes that for $M = H_G^*(X)$, and $R = H_G(X)$, each decomposition is indexed over equivalent sets and the summands are equal term-by-term. Thus, the result can be understood as a "geometric" interpretation of the algebraic sum formula.

Proof. For ease of notation let $M = H_G^*(X)$, and $R = H_G(X)$. Let $N = H^*_{C_G(A,c)}(c)$, and $S = H_{C_G(A,c)}(c)$.

Let's apply the algebraic decomposition of degree to M as an R-module (Theorem 3.6, which is applicable since $H_G^*(X) \in \mathfrak{grmod}(H_G(x))$.) Then,

$$\deg(M)= \sum_{\mathfrak{p}\in \mathcal{D}(M)} *\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

Recall that $\mathcal{D}(M) \doteq \{\mathfrak{q} \in Spec(R) : \dim(R/\mathfrak{q}) = \dim_R(M)\}$. By Quillen's main theorem 5.6, there is a bijective correspondence between $\mathcal{D}(M)$ and $\mathcal{Q}'_{max}(G,X)$, the set of maximal classes of maximal rank in Quillen's category.

Now, the *length factor in each summand of the degree formula may be re-written using the previous theorem (6.7):

$$*\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \frac{1}{|W|} * \ell_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]}).$$

Let $[A, c]$ be any element of $\mathcal{Q}'_{max}(G, X)$, and consider the corresponding primes $\mathfrak{p} \doteq \ker\left(res^G_A : H^*_G(X) \to H_A/\sqrt{0}\right)$, and $\mathfrak{p}^C \doteq \ker\left(res^G_A : H^*_{C_G(A,c)}(c) \to H_A/\sqrt{0}\right)$. In Lemma 6.4, we compared the sum formula for the degree of N given its R and S-module structures. In particular, we showed that the degree of N has only one term in the sum decomposition, which may be computed using either module structure as:

$$\begin{aligned} \deg(N) &= *\ell_{S_{[\mathfrak{p}^C]}}\left(N_{[\mathfrak{p}^C]}\right)\deg(S/\mathfrak{p}^C) \\ &= *\ell_{R_{[\mathfrak{p}]}}\left(N_{[\mathfrak{p}]}\right)\deg(R/\mathfrak{p}). \end{aligned}$$

We therefore have the computation:

$$\begin{aligned} \deg(M) &= \sum_{\mathfrak{p}\in\mathcal{D}(M)} *\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}) \\ &= \sum_{[A,c]\in\mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} * \ell_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]})\deg(R/\mathfrak{p}) \\ &= \sum_{[A,c]\in\mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} * \ell_{S_{[\mathfrak{p}^C]}}(N_{[\mathfrak{p}^C]})\deg(S/\mathfrak{p}^C) \\ &= \sum_{[A,c]\in\mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|}\deg(N). \end{aligned}$$

$\square$

Chapter 7

Future Work

Chapters 1 and 2 of part 1 of this dissertation provided an account of the transition of Serre's theory of multiplicities to the $\mathbb{Z}$-graded category. Chapter 3 narrowed the line of inquiry to the positively graded case, and we reviewed the degree invariant. Our account of graded multiplicities could be made richer by re-working chapter 3 for the $\mathbb{Z}$-graded category. If A is a $\mathbb{Z}$-graded ring with A_0 Artinian, and M a finitely generated graded A-module, one could define a modified Poincare series for $M \in \mathfrak{grmod}(A)$ by $P_M(t) \doteq \sum_{-\infty}^{\infty} \ell_{A_0}(M_i)t^i$. It would then be desirable to relate this to a modified degree invariant.

In the positively graded case, we saw two important features of the degree. (1) Its relation to multiplicity: $\deg(M) = \frac{*e_R(M,\mathcal{I},D(M))}{d_1 \cdots d_{D(M)}}$. (2) The algebraic summation decomposition: $\deg(M) = \sum_{\mathfrak{p}\in\mathcal{D}(M)} *\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p})$. We hope that similar results could be produced in the $\mathbb{Z}$-graded category, for *local rings A. In fact, since we've already defined graded multiplicity and graded systems of parameters in $\mathfrak{grmod}(A)$, one might simply take the definition of degree in the $\mathbb{Z}$-graded case to be (1).

For $M \in \mathfrak{grmod}(R)$, R a positively graded Noetherian ring, the algebraic summation decomposition of degree is indexed by the minimal primes of M which have maximal dimension. In [27], Vasconcelos considers various ways of extending this measure. He defines a geometric multiplicity by summing over all minimal primes, not just the ones of maximal dimension, and he defines the arithmetic multiplicity by summing over all associated primes. Now, if $\mathfrak{p}$ is an associated prime, and not a minimal prime, $M_{[\mathfrak{p}]}$ is no longer *Artinian and so the term $*\ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]})$ in the sum decomposition needs to be replaced. A suitable replacement is found in the zeroth local cohomology group $H^0_{\mathfrak{p}}(M)$, whose graded localization turns out to be a *Artinian $R_{[\mathfrak{p}]}$-module (see [11].) Thus, the arithmetic degree is defined

$$Adeg(M) = \sum_{\mathfrak{p} \in Ass_R(M)} *\ell_{R_{[\mathfrak{p}]}}(H^0_{\mathfrak{p}}(M)_{\mathfrak{p}}) \cdot \deg(R/\mathfrak{p}).$$

The zeroth local cohomology group has a natural interpretation in the context of irredundant primary decompositions of M ([11]). In [10], Duflot proves a result on the primary decomposition of equivariant cohomology rings with embedded components. Perhaps this result can be used to ascertain a better understanding of these extended degree formulas. A good starting point would be to work out the extended degree in the case where G is the extra-special p group of order p^3 and exponent p. Duflot [10] explicitly describes all associated primes of the cohomology ring of G, and in fact, there is only one embedded prime. Given her efforts, the extended degree for this example may be computed without too much difficulty.

A more long-term research goal will be to understand how these algebraic results on localization, multiplicity, degree, etc. fit into the homotopy theoretic picture of equivariant cohomology using Lannes' T-Functor [17]. The basic idea is that mod p cohomology is a functor from the category of topological spaces into the category of graded, unstable modules over the Steenrod algebra, $\mathcal{U}_{\mathfrak{p}}$. In this category, for every elementary abelian p-group A, $H^*(BA, \mathbb{Z}/p)$ is an injective object. We can define N as a nilpotent module if and only if $Hom_{\mathcal{U}_p}(N, H^*(BA, \mathbb{Z}/p)) = 0$ for all elementary abelian p-groups A.

For each elementary abelian p-subgroup of G, there is the restriction map $res^G_A : H^*_G(X) \to H^*_A$. Using restriction, we get the map $H^*_G(X) \to lim.inv_{(A,c) \in \mathcal{Q}(G,X)} H^*_A$. Quillen's F-isomorphism theorem (pg. 575 of [23]) states that this map is a homomorphism with certain nilpotent phenomenon. This theorem may be re-interpreted in terms of nilpotent modules. Compare the following theorem [14] to Quillen's F-isomorphism theorem: A group G is a Quillen group if and only if the following is an isomorphism,

$$Hom_{\mathcal{U}_p}\left(H^*(BG, k), H^*(BA, k)\right) \leftarrow Hom_{\mathcal{U}_p}\left(inv.lim_{V \in \mathcal{Q}(G,X)} H^*(BV, k), H^*(BA, k)\right).$$

Using Lanne's T-Functor, one can obtain higher order information about $H^*_G(X)$ using nilpotent modules. We would like to understand if algebraic results on $H^*_G(X)$, like those that were the focus of this dissertation, are compatible with the homotopy theoretic perspective of Lannes'.

Bibliography

[1] L. Avramov and R. Buchweitz, *Lower Bounds for Betti Numbers*, Compositio Mathematica, **86** (1993).

[2] A. Adem and R.J Milgram, *Cohomology of Finite Groups. Second Edition*, Springer-Verlag Berlin Heidelberg, 2004.

[3] M.F Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, MA, 1969.

[4] D.J. Benson, *Representations and Cohomology II: Cohomology of Groups and Modules*, Cambridge University Press, 1991.

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings. Revised Edition,* Cambridge University Press, Cambridge, U.K., 1998.

[6] K. Brown, *Cohomology of Groups,* Springer-Verlag, New York, 1982.

[7] T. Dieck, *Transformation Groups,* De Gruyter studies in mathematics, New York, 1987.

[8] J. Duflot, *Depth and Equivariant Cohomology,* Comentarii Mathematici Helvetici, **56** (1981).

[9] J. Duflot, *Localization of Equivariant Cohomology Rings,* Transactions of the American Mathematical Society, **288** (1984).

[10] J. Duflot, *Numerical Invariants for Equivariant Cohomology,* The Quarterly Journal of Mathematics, **64** (2013), 665-720.

[11] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1995.

[12] D. Green, *Grobner Bases and the Computation of Group Cohomology,* Springer-Verlag, Berlin Heidelberg, 2003.

[13] A. Grothendieck, *Elements de geometrie algebrique (rediges avec la collaboration de Jean Dieudonne) : II,* Etude globale elementaire, de quelques classes de morphismes, Publications mathematiques de lÂŠI.H.E.S. 8 (1961).

[14] H.W Henn, *Unstable modules over the Steenrod algebra and cohomology of groups,* Proc. Sympos. Pure Math., **63** 1996, 277-300.

[15] P.J. Hilton and U. Stammbach, *A Course in Homological Algebra,* Springer-Verlag, Berlin Heidelberg, 1970.

[16] W. Hsiang, *Cohomology Theory of Topological Transformation Groups,* Springer-Verlag, Berline, Heidelberg, 1975.

[17] J. Lannes, *Sur la cohomologie modulo p des p-groupes abéliens élémentaires*, Proceedings of the Durham Symposium on Homotopy theory 1985, Cambridge University Press 1987.

[18] B. Lynn, *A Degree Formula for Equivariant Cohomology,* Transactions of the American Mathematical Society, **366** 2014.

[19] J.A. Maiorana, *Smith Theory for p-Groups,* Transactions of the American Mathematical Society, **223** 1976, $253-266$.

[20] R.J. Milgram and B. Priddy, *Invariant Theory and $H^*(GL_n(\mathbb{F}_p;\mathbb{F}_p)$*, Journal of Pure and Applied Algebra 44, 1987, 291-302.

[21] J. Milnor, *Construction of Universal Bundles I and II,* Annals of Mathematics, **63** 1956, $272-284$, $430-436$.

[22] D. Quillen, *The spectrum of an equivariant cohomology ring: I.* Annals of Mathematics, **94** 1971, 459-572.

[23] D. Quillen, *The spectrum of an equivariant cohomology ring: II.* Annals of Mathematics, **94** 1971, 573-602.

[24] W. Smoke, *Dimension and Multiplicity for Graded Algebras,* Journal of Algebra, **21** 1972.

[25] J.P. Serre, *Local Algebra,* Springer-Verlag, Berlin (2000).

[26] P. Symonds, *On the Castelnuovo-Mumford Regularity of the Cohomology Ring of a Group,* Journal of the American Mathematical Society, **23** No. 4, 2010, 1159−1173.

[27] W. Vasconcelos, *The Homological Degree of a Module,* Transactions of the American Mathematical Society, **350** No. 3, 1998, 1167-1179.