TABLE II

ORDER N OF THE DIGITAL DIFFERENTIATORS FOR HIGH FREQUENCIES. N_1: FOR MINIMAX RELATIVE ERROR DIFFERENTIATORS [1]; N_2: FOR MAXIMALLY LINEAR DIFFERENTIATORS (PROPOSED)

Relative Error →	$\leqslant$ -40 dB (1%)		$\leqslant$ -60 dB (0.1%)		$\leqslant$ -80dB (0.01%)		$\leqslant$ -100 dB (0.001%)		
Frequency/π (w_p to π) ↓	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2	Remarks
0.95 to 1.0	22	2	> 128	4	$\gg$ 128	4	$\gg$128	6	1. The minimax relative error differentiators give uniform performance from $w = 0$ to π 2. Multiplications required per input sample for the minimax DD is $N_1/2$ and that for the proposed DD is N_2.
0.90 to 1.0	"	4	"	4	"	6	"	6	
0.80 to 1.0	"	4	"	6	"	8	"	10	
0.70 to 1.0	"	6	"	8	"	10	"	14	
0.60 to 1.0	"	8	"	12	"	16	"	20	
0.50 to 1.0	"	10	"	16	"	22	"	> 22	

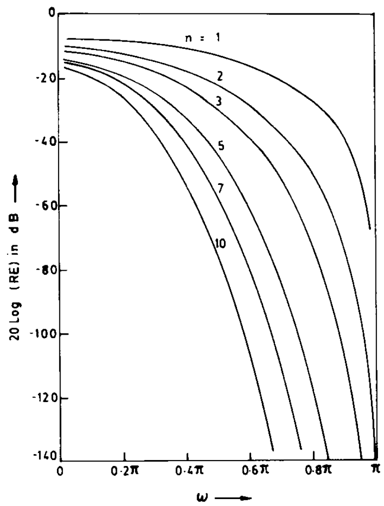

Fig. 3. Relative error (RE), in decibels, of frequency response $D(w)$ for the proposed digital differentiators for $n = 1$, 2, 3, 5, 7, and 10.

frequency bandwidths w_p to π and relative errors less than or equal to -40, -60, -80, and -100 dB. To be fair, we compare order N_1 of the minimax DD's with twice the order N_2 of the proposed differentiators since the proposed design contains twice as many coefficients as the minimax design of equal order. It may be seen that the maximally linear DD's have an edge over their minimax counterparts. If high accuracies are desired the maximally linear DD's are much superior to the minimax ones; for example, for the frequency coverage of $0.5\pi \leqslant w \leqslant \pi$ and a $|\text{RE}| \leqslant 0.1$ percent, the proposed DD requires only 16 multiplications per input sample as compared to 64 in the case of minimax DD. The designed digital differentiators are specially suitable for high frequencies (upto $w = \pi$) and for achieving extremely low RE's.

The values of the coefficients c_i and d_i, required in the proposed design, are computed by using the mathematical formulas (8), as against the lengthy optimization algorithms needed to realize the minimax differentiators. As in the minimax design, [1], [5] a half sample delay ($z^{-1/2}$) is imperative for the maximally linear differentiators; otherwise the proposed design is canonic. Since in multirate systems half-sample delay can be easily realized, the suggested design would be particularly suitable for such systems.

V. CONCLUSION

An efficient FIR digital differentiator structure, especially suitable for high frequency ranges, has been proposed. Mathematical relations for calculating the exact values of the weighting coefficients, required in the design, have been derived.

ACKNOWLEDGMENT

The authors thank the reviewers for their constructive suggestions and comments.

REFERENCES

[1] L. R. Rabiner and R. W. Schafer, "On the behaviour of minimax relative error FIR digital differentiators," *Bell. Syst. Tech. J.*, vol. 53, pp. 333–362, Feb. 1974.

[2] P. J. Davis, *Interpolation and Approximation*. New York: Dover, 1975.

[3] C. Froberg, *Introduction to Numerical Analysis*. Reading, MA: Addison Wesley, 1969.

[4] R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1983.

[5] L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1975.

[6] C. C. Ko and Y. C. Lim, "Approximation of a variable length delay line by using tapped delay line processing," *Signal Processing*, vol. 14, no. 4, pp. 363–369, June 1988.

New Results in Strip Kalman Filtering

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***Abstract*—The Strip Kalman filtering proposed in [1] for image restoration is reconsidered. The procedure given in this reference for parameter estimation of the image model does not take into account the vector nature of the image process, and as a result can lead to incorrect identification. It**

Manuscript received March 10, 1988; revised August 23, 1988 and January 24, 1989. This paper was recommended by Associate Editor H. Gharavi.

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IEEE Log Number 8827413.

is also shown that for the composite dynamic model derived in this reference the standard Kalman filtering equations cannot be applied, as the blur states in this model should be estimated one step ahead. These issues are addressed in this paper.

I. Introduction

In [1] Suresh and Shenoi proposed a strip Kalman filtering process which makes use of a vector scanning scheme. The image process is modeled by a finite-order vector autoregressive (AR) model which relates a column of pixels to the past columns in a certain region within a strip. It is assumed that the image process is wide sense stationarity within each strip. Based upon this assumption and ergodicity property the parameters of the vector AR model are evaluated using a Yule–Walker system of equations. This model is then arranged into a state-space form. The effect of a linear shift invariant (LSI) blur is modeled by a 2-D state-space structure implemented by a 1-D structure with intrastrip and interstrip recursion characteristics. These two 1-D models are then combined to yield a composite dynamic structure in which part of the state variables which correspond to the blur need to be estimated one step ahead of those associated with the image.

In this paper we have shown that the assumption of wide sense stationarity within each strip is not valid since the image is modeled by a vector or multichannel AR process. The assumption of column wide sense stationarity within each strip is more appropriate for these processes [2], [3]. In this connection, new procedures for estimating the parameters of the vector AR model are suggested. In addition, new Kalman filtering equations are derived which account for the combination of filtering for the image state and one-step prediction for the blur state.

II. A Vector AR Model for the Image Process

Consider an $N \times N$ image which is vector scanned horizontally in strips of width W. The direction of scanning is assumed to be from left-to-right to top-to-bottom. Each strip is processed independently. The image is assumed to be represented by a vector (or multichannel) Markovian process and modeled within each strip by a Pth-order vector AR process with a causal quarter-plane region of support, $\boldsymbol{R}$ (see Fig. 1). This AR model is given by

$$Z(k)=\phi_1^t Z(k-1)+\phi_2^t Z(k-2)+\cdots+\phi_P^t Z(k-P)+U(k) \tag{1}$$

and $Z(k)$ represents a $W \times 1$ vector with elements that are the pixel intensity values in the kth column of a given (say ith) strip in the image, i.e.,

$$Z(k)=\left[z_{(i-1)W,k}\, z_{(i-1)W+1,k} \cdots z_{iW-1,k}\right]^t \tag{2}$$

where $z_{m,n}$ denotes the intensity of the pixel at location (m,n). Vector $U(k)$ which is defined similar to $Z(k)$, represents a white noise vector process which drives the autoregression. The statistics of this process are

$$E[U(k)]=0$$
$$E\left[U(k)U^t(k-l)\right]=Q_U\delta(l) \tag{3}$$

where Q_U is the covariance matrix of the error vector $U(k)$; $\delta(l)$ represents the Kronecker delta function and E denotes the expectation operator. Matrices $\phi_1,\phi_2,\cdots,\phi_P$ are $W \times W$ matrices that have to be identified in each strip. It is interesting to note

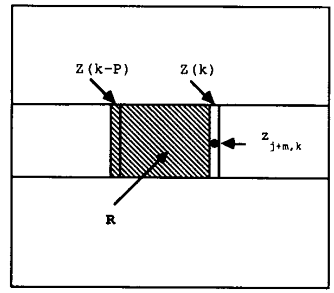

Fig. 1. Region of support of the vector AR model.

that although $U(k)$ is vectorially an uncorrelated process, this does not imply that the elements within each vector are also mutually uncorrelated. This fact can be evident when for each pixel in $Z(k)$ a scalar relationship is derived from (1). This relationship which represents a particular pixel in $Z(k)$ in terms of all the pixels in the support region $\boldsymbol{R}$ and the corresponding scalar error term, is a semicausal representation. The noncausality in this scalar model occurs along the vertical direction. It has been shown in [4]–[5] that semicausal finite order models are driven by a noise process which is white along the causal direction and color along the noncausal direction. Thus the elements of $U(k)$ are mutually correlated as they are along the noncausal (vertical) direction, whereas the elements of $U(k)$ and $U(l)$, $k \neq l$, are uncorrelated since these are along the causal (horizontal) direction. To elaborate on this issue, let us consider the minimum variance vector representation

$$Z(k)=\hat{Z}(k)+U(k) \tag{4}$$

where

$$\hat{Z}(k)=\sum_{i=1}^{P}\phi_i^t Z(k-i) \tag{5}$$

is the minimum variance estimate of $Z(k)$. The orthogonal properties of this estimator give

$$E\left[Z(k-l)U^t(k)\right]=Q_U\delta(l)$$

and hence

$$E\left[\hat{Z}(k)U^t(k)\right]=\mathbf{0}. \tag{6}$$

The scalar representation for each pixel $z_{j+m,k}$ in $Z(k)$ is given by

$$z_{j+m,k}=\sum\sum_{p,q\in S} a_{p,q}^m z_{j+p,k-q}+u_{j+m,k}, \qquad m\in[0,W-1];$$
$$j=(i-1)W,\ i=1,2,\cdots,N/W \tag{7a}$$

where

$$S=\{(p,q),0\leqslant p\leqslant W-1,1\leqslant q\leqslant P\} \tag{7b}$$

and

$$a_{p,q}^m=\varphi_q(m,p) \tag{7c}$$

where $\varphi_q(i,j)$ is the (i,j)th entry of matrix ϕ_q and $u_{j+m,k}$ is the mth element of $U(k)$. To show that $u_{j+m,k}$'s for $m\in[0,W-1]$ are correlated, let us form

$$r_u(m-n,0)=E[u_{j+m,k}u_{j+n,k}]$$
$$=E\left[\left[z_{j+m,k}-\sum\sum_{p,q\in S}a_{p,q}^m z_{j+p,k-q}\right]u_{j+n,k}\right]. \tag{8}$$

Now using the orthogonality property of (6) the covariance function $r_u(m-n,0)$ becomes

$$r_u(m-n,0) = E[z_{j+m,k}u_{j+n,k}] \neq 0 \tag{9}$$

since a vector model is used. Thus, in this case $u_{i,j}$ cannot be modeled as a white noise process. In other words, for any multichannel process described by a vector AR model, it is not generally true to assume that each channel is also individually an AR process. Furthermore, considering the properties of the vector scan, the assumption of wide sense stationarity cannot be valid and thus must be changed to column wide sense stationarity within each strip which is more suitable for multichannel processes [2], [3]. Consequently, the procedure given in [1] for obtaining the AR model parameters is not valid since the correlation matrices are obtained as if the process is wide sense stationary and described by a series of decoupled single channel processes. This inconsistency in modeling and the parameter identification may result in a vector AR model with instability problems [2], [3].

III. Parameter Identification for the Vector AR Model

To estimate the parameter matrices $\phi_1, \phi_2, \cdots, \phi_P$ of the model, transpose (1), premultiply both sides by $Z(k-r)$ and then take the expectation. This yields

$$\begin{aligned}\rho_r &= E[Z(k-r)Z'(k)]\\ &= \rho_{r-1}\phi_1 + \rho_{r-2}\phi_2 + \cdots + \rho_{r-P}\phi_P + E[Z(k-r)U'(k)]\end{aligned} \tag{10}$$

where ρ_r is the covariance matrix of $Z(k)$. Putting $r = 0,1,\cdots,P$ in this normal equations gives the following vector Yule–Walker system of equations which must be solved for $\phi_1, \phi_2, \cdots, \phi_P$ and Q_U.

$$\begin{bmatrix} \rho_0 & \rho_1' & \cdots & \rho_P' \\ \rho_1 & \rho_0 & \cdots & \rho_{P-1}' \\ \vdots & \vdots & & \\ \rho_P & \rho_{P-1} & \cdots & \rho_0 \end{bmatrix} \begin{bmatrix} I \\ -\phi_1 \\ \vdots \\ -\phi_P \end{bmatrix} = \begin{bmatrix} Q_U \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11a}$$

where $\rho_{-r} = \rho_r'$ and I is an identity matrix of appropriate order. This equation in compact form is

$$T\Phi = d \tag{11b}$$

From (11a) one gets

$$\begin{bmatrix} I \\ -\phi_1 \\ \vdots \\ -\phi_P \end{bmatrix} = T^{-1} \begin{bmatrix} Q_U \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{12a}$$

Thus we can write

$$I = [T^{-1}]_{1,1} Q_U \tag{12b}$$

or

$$Q_U = [T^{-1}]_{1,1}^{-1}$$

and

$$\phi_i = -[T^{-1}]_{i+1,1} Q_U, \qquad i = 1,2,\cdots,P. \tag{12c}$$

Note that the definition of ρ_r given in [1, eq. (25)] does not apply here since it relates to the wide sense stationary case. In order to solve (11) estimates of $\hat{\rho}_r$'s are required. Invoking column wide sense stationarity assumption, reasonable estimates of these matrices can be obtained by ergodicity property and using

$$\hat{\rho}_r = \frac{1}{N-r} \sum_{k=r}^{N-1} Z(k-r)Z'(k). \tag{13}$$

The parameter identification algorithm can be implemented online to update the model parameters at each stage of the process. Let us denote the estimate of ρ_r based upon n vectors in a given strip by $\hat{\rho}_r(n)$. Thus we can write

$$\hat{\rho}_r(n) = \frac{1}{n-r} \sum_{k=r}^{n-1} Z(k-r)Z'(k), \qquad r \in [0,P]. \tag{14}$$

Then the estimate based upon $n+1$ vectors is

$$\begin{aligned}\hat{\rho}_r(n+1) &= \frac{1}{(n+1-r)} \sum_{k=r}^{n} Z(k-r)Z'(k)\\ &= \hat{\rho}_r(n) + \frac{1}{(n+1-r)}[Z(n-r)Z'(n) - \hat{\rho}_r(n)]\\ &= \hat{\rho}_r(n) + \delta\hat{\rho}_r(n).\end{aligned} \tag{15}$$

Writing (15) for each constituent block in T yields

$$\hat{T}(n+1) = \hat{T}(n) + \delta\hat{T}(n) \tag{16a}$$

where

$$\delta\hat{T}(n) = \begin{bmatrix} \delta\hat{\rho}_0(n) & \delta\hat{\rho}_1'(n) & \cdots & \delta\hat{\rho}_P'(n) \\ \vdots & & & \\ \delta\hat{\rho}_P(n) & \delta\hat{\rho}_{P-1}(n) & \cdots & \delta\hat{\rho}_0(n) \end{bmatrix}. \tag{16b}$$

Similarly $\hat{d}(n+1) = \hat{d}(n) + \delta\hat{d}(n)$. Now using the matrix inversion lemma [1]

$$\hat{T}^{-1}(n+1) \approx \hat{T}^{-1}(n)[I - \delta\hat{T}(n)\hat{T}^{-1}(n)]. \tag{17}$$

The solution of the vector Yule–Walker equation (11) at iteration $(n+1)$ can then be written [1] as

$$\begin{aligned}\hat{\Phi}(n+1) &= \hat{T}^{-1}(n+1)\, d(n+1)\\ &= \hat{T}^{-1}(n)[I - \delta\hat{T}(n)\hat{T}^{-1}(n)]\, d(n+1)\\ &= \hat{\Phi}(n) + \hat{T}^{-1}(n+1)\delta\hat{d}(n).\end{aligned} \tag{18}$$

This equation provides a simple recursive scheme for estimating the vector AR model parameters when the original image is assumed to be given. The identification of the parameters from the corrupted image for multivariable stochastic systems is complicated due to lack of unique minimal realization for these systems [6].

IV. Kalman Filtering Equations

In [1] the vector AR model for the image is arranged into a 1-D state-space form, namely

$$x_1(k) = A_1 x_1(k-1) + B_1 U(k) \tag{19a}$$

where

$$x_1(k) = [Z'(k) \quad Z'(k-1) \cdots Z'(k-P+1)]' \tag{19b}$$

and

$$A_1 = \begin{bmatrix} \phi_1' & \phi_2' & \cdots & \phi_P' \\ I & 0 & & 0 \\ 0 & I & 0 & \\ & & \ddots & \\ 0 & & & I0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{19c}$$

The vector of interest, $Z(k)$ can be extracted from $x_1(k)$ by

$$Z(k) = C_1 x_1(k) \tag{20}$$

where $C_1 = [I \quad 0 \cdots 0]$. The degradation process for a separable LSI blur and additive WG noise is modeled by a 1-D state-space model with interstrip and intrastrip propagation characteristics [1]. The result of this modeling can be reduced to the following state-space formulation [1].

$$\begin{aligned} x_2(k+1) &= A_2 x_2(k) + B_2 Z(k) \\ Y(k) &= C_2 x_2(k) + D_2 Z(k) + V(k) \end{aligned} \tag{21}$$

where $x_2(k)$ consists of the horizontal and the vertical states associated with the blur; $Y(k)$ is a $W \times 1$ vector of the degraded image; $V(k)$ is a $W \times 1$ vector of the additive WG noise with zero mean and variance σ_v^2 and matrices A_2, B_2, C_2 and D_2 are defined explicitly in [1] in terms of the coefficients of the blur transfer function. Combining (19a), (20), and (21) gives the following composite dynamic model:

$$\begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 A_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(k-1) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 C_1 B_1 \end{bmatrix} U(k)$$

$$Y(k) = [D_2 C_1 \quad C_2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + V(k) \tag{22}$$

in which the state vector x_2 for the blur is estimated one-step ahead. Thus the conditional mean estimate of $x_1(k)$ given observation vectors up to $Y(k)$ gives the "filtered estimate" of the image while the conditional mean estimate of $x_2(k+1)$ based upon these observation vectors leads to the "one-step prediction estimates". Any rearrangement to synchronize (20) and (21) would lead to estimates for the image that are not the "true filtered estimates". In what follows the relevant Kalman filtering equations for the composite dynamic model in (22) are derived.

Let $\xi(k) = \{Y(0) \quad Y(1) \cdots Y(k)\}$ be the observation set containing all the output vectors up to the kth vector. The estimate of the state vector given the observations up to $Y(k)$ can be written as

$$E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \middle| \xi(k) \right\} = E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \middle| \xi(k-1) \right\} + E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \middle| \eta(k) \right\}$$

or

$$\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \hat{\hat{x}}_1(k) \\ \hat{\hat{x}}_2(k+1) \end{bmatrix} + E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \middle| \eta(k) \right\} \tag{23}$$

where "$\hat{\hat{}}$" and "$\hat{}$" denote the *a priori* (before the updating) and the *a posteriori* (after the updating) estimates, respectively, and $\eta(k)$ is the "innovation sequence" given by

$$\eta(k) = Y(k) - \overline{C} \begin{bmatrix} \hat{\hat{x}}_1(k) \\ \hat{\hat{x}}_2(k) \end{bmatrix} \tag{24}$$

where

$$\overline{C} = [D_2 C_1 \quad C_2].$$

The second term in (23) can be expressed using the orthogonal projection lemma [7], i.e.,

$$E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \middle| \eta(k) \right\} = E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \eta'(k) \right\} \cdot \left[E[\eta(k)\eta'(k)] \right]^{-1} \eta(k). \tag{25}$$

Defining the Kalman gain matrix by

$$K(k) \triangleq E\left\{ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \eta'(k) \right\} \left[E[\eta(k)\eta'(k)] \right]^{-1} \tag{26}$$

equation (23) becomes

$$\begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \hat{\hat{x}}_1(k) \\ \hat{\hat{x}}_2(k+1) \end{bmatrix} + K(k)\eta(k). \tag{27}$$

Since $V(k)$ is uncorrelated with the image and the blur states, the term $E[\eta(k)\eta'(k)]$ in (26) can be written as

$$\begin{aligned} E[\eta(k)\eta'(k)] &= E\left\{ \left[\overline{C} \begin{bmatrix} \tilde{\tilde{x}}_1(k) \\ \tilde{\tilde{x}}_2(k) \end{bmatrix} + V(k) \right] \cdot \left[\begin{bmatrix} \tilde{\tilde{x}}_1(k) \\ \tilde{\tilde{x}}_2(k) \end{bmatrix}' \overline{C}' + V'(k) \right] \right\} \\ &= \overline{C} \hat{P}(k) \overline{C}' + \sigma_v^2 I \end{aligned} \tag{28}$$

where

$$\tilde{x} \triangleq x - \hat{x} \tag{29a}$$

$$\tilde{\tilde{x}} \triangleq x - \hat{\hat{x}} \tag{29b}$$

and $\hat{P}(k)$ is the *a priori* error covariance matrix before the updating, i.e.,

$$\hat{P}(k) = E\left[\begin{bmatrix} \tilde{\tilde{x}}_1(k) \\ \tilde{\tilde{x}}_2(k) \end{bmatrix} \left[\tilde{\tilde{x}}_1'(k) \quad \tilde{\tilde{x}}_2'(k) \right] \right]. \tag{29c}$$

The other term in (26) can be expressed as

$$\begin{aligned} E\left[\begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} \left[\left[\tilde{\tilde{x}}_1'(k) \quad \tilde{\tilde{x}}_2'(k) \right] \overline{C}' + V'(k) \right] \right] \\ = A' \hat{P}(k) \overline{C}' \end{aligned} \tag{30}$$

where

$$A' \triangleq \begin{bmatrix} I & 0 \\ B_2 C_1 & A_2 \end{bmatrix}.$$

This is obtained using the following orthogonal properties

$$E\left[\begin{bmatrix} \hat{\hat{x}}_1(k) \\ \hat{\hat{x}}_2(k) \end{bmatrix} \left[\tilde{\tilde{x}}_1'(k) \quad \tilde{\tilde{x}}_2'(k) \right] \right] = 0 \tag{31a}$$

and

$$E\left[\begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} V'(k) \right] = 0. \tag{31b}$$

Thus the expression for the Kalman gain becomes

$$K(k) = A' \hat{P}(k) \overline{C}' \left[\overline{C} \hat{P}(k) \overline{C}' + \sigma_v^2 I \right]^{-1}. \tag{32}$$

Now, in order to compute the Kalman gain matrix, $\hat{P}(k)$ must be evaluated recursively at every stage. Considering (19) we can write

$$\hat{P}(k)=(A_1\oplus I)\hat{\hat{P}}(k-1)(A_1\oplus I)'+\begin{bmatrix}B_1\\0\end{bmatrix}Q_U\begin{bmatrix}B_1' & 0\end{bmatrix} \quad (33)$$

where

$$\hat{\hat{P}}(k-1)\triangleq E\left[\begin{bmatrix}\tilde{\hat{x}}_1(k-1)\\ \tilde{\hat{x}}_2(k)\end{bmatrix}\begin{bmatrix}\tilde{\hat{x}}_1'(k-1) & \tilde{\hat{x}}_2'(k)\end{bmatrix}\right]$$

and $\oplus$ denotes the direct sum operation. $\hat{\hat{P}}(k-1)$ is the *a posteriori* error covariance matrix at stage $k-1$. To find the expression for $\hat{\hat{P}}(k)$, let us subtract (27) from the state vector at present stage k, i.e.,

$$\begin{bmatrix}x_1(k)\\x_2(k+1)\end{bmatrix}-\begin{bmatrix}\hat{\hat{x}}_1(k)\\ \hat{\hat{x}}_2(k+1)\end{bmatrix}=\begin{bmatrix}x_1(k)\\x_2(k+1)\end{bmatrix}-\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k+1)\end{bmatrix}-K(k)\eta(k)$$

or

$$\begin{bmatrix}\tilde{\hat{x}}_1(k)\\ \tilde{\hat{x}}_2(k+1)\end{bmatrix}=A'\begin{bmatrix}\tilde{x}_1(k)\\ \tilde{x}_2(k)\end{bmatrix}-K(k)\eta(k). \quad (34)$$

Now, transposing both sides, postmultiplying by (34) and taking expectation yields

$$\hat{\hat{P}}(k)=[A'-K(k)\overline{C}]\hat{P}(k)A''. \quad (35)$$

As a result, the Kalman filtering equations for computing the filtered estimate of the image and the one-step prediction estimate of the blur, are given in order by

$$\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k+1)\end{bmatrix}=\begin{bmatrix}A_1 & 0\\ B_2C_1A_1 & A_2\end{bmatrix}\begin{bmatrix}\hat{\hat{x}}_1(k-1)\\ \hat{\hat{x}}_2(k)\end{bmatrix} \quad (36a)$$

$$\hat{P}(k)=(A_1\oplus I)\hat{\hat{P}}(k-1)(A_1\oplus I)'+\begin{bmatrix}B_1\\0\end{bmatrix}Q_U\begin{bmatrix}B_1' & 0\end{bmatrix} \quad (36b)$$

$$K(k)=A'\hat{P}(k)\overline{C}'\left[\overline{C}\hat{P}(k)\overline{C}'+\sigma_v^2 I\right]^{-1} \quad (36c)$$

$$\begin{bmatrix}\hat{\hat{x}}_1(k)\\ \hat{\hat{x}}_2(k+1)\end{bmatrix}=\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k+1)\end{bmatrix}+K(k)\left[Y(k)-\overline{C}\begin{bmatrix}\hat{x}_1(k)\\ \hat{x}_2(k)\end{bmatrix}\right] \quad (36d)$$

$$\hat{\hat{P}}(k)=[A'-K(k)\overline{C}]\hat{P}(k)A'' \quad (36e)$$

$$\hat{\hat{Z}}(k)=\begin{bmatrix}C_1 & 0\end{bmatrix}\begin{bmatrix}\hat{\hat{x}}_1(k)\\ \hat{\hat{x}}_2(k+1)\end{bmatrix}. \quad (36f)$$

The form of the above equations [7] and the presence of matrix A' with its special structure clearly reveal the fact that these equations represent the combination of filtering and one-step prediction.

V. Conclusion

New procedures for parameter identification of the image model in strip Kalman filtering is suggested. This method which takes into account the vector nature of the image model, leads to a vector Yule–Walker system of equations. An algorithm for on-line adaptation of the image model parameters is also given. In the composite dynamic model for the image and degradation processes, the blur states are computed one-step ahead of those of the image. Thus a new set of Kalman filtering equations which accounts for both one-step prediction for the blur states and filtering for the image states was required. The derivations of these equations are presented.

References

[1] B. R. Suresh and B. A. Shenoi, "New results in two-dimensional Kalman filtering with applications to image restoration", *IEEE Trans. Circuits Syst.*, vol. CAS-28, pp. 307–319, Apr. 1981.

[2] S. L. Marple, *Digital Spectral Analysis with Applications.* Englewood Cliffs, NJ: Prentice-Hall 1987.

[3] E. A. Robinson, *Multichannel Time Series Analysis With Digital Computer Programs*, Houston. TX: Goose Pond Press, 1983, 2nd edition.

[4] A. K. Jain and J. R. Jain, "Partial differential equations and finite difference methods in image processing—Part II: Image restoration," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 817–834, October 1978.

[5] A. K. Jain, "Advances in mathematical models for image processing," *Proc. IEEE*, vol. 69, pp. 502–528, May 1981.

[6] M. Gevers and V. Wertz, "Uniquely identifiable state-space and ARMA parameterization for multivariable linear systems," *Automatica*, vol. 20, no. 3, pp. 333–347, 1984.

[7] A. P. Sage and J. L. Melsa, *Estimation Theory With Applications to Communication and Control.* New York, McGraw-Hill, 1971.

Quenching Phenomena in Synchronized Van der Pol Systems

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Abstract—**Quenching of sinusoidally driven Van der Pol systems is considered and conditions developed under which harmonic synchronous quenching occurs.**

For this, a multiple time expansion is used to solve the Van der Pol equation with the sinusoidal forcing term. By controlling the expansion in terms of a small parameter and equating terms of like order in the parameter, the solutions of asynchronous and synchronous state are obtained. From these solutions the conditions for harmonic synchronous quenching are developed and a plot given to show the quenching region.

I. Introduction

In sinusoidally forced systems the existence of asynchronous quenching was detected many years ago. In these systems if the ratio of the external forcing frequency to the free-oscillation frequency is m/n, where m and n are not commensurable integers ($m>1$ and $n>1$), then the excitation is nonresonant and the responses consist of a combination of two different frequency components; one from the free-oscillations and the other from the forced oscillation. However, the free oscillations can tend to zero with increasing t under certain conditions; we call these phenomena "asynchronous quenching."

Minorsky [3] in 1962 and Nayfah [4] in 1979 did some theoretical analyses about asynchronous quenching, but they could not find out the relationship between asynchronous quenching and the harmonic synchronization (that is, when free oscillation of the system is synchronized to a harmonic of the external force). In engineering practice sometimes, the phenomenon that the oscillation in a synchronous system stops suddenly is not able to

Manuscript received February 4, 1988; revised September 20, 1988 and December 6, 1988. This work was supported by Professor R. Newcomb. This paper was recommended by Associate Editor T. Matsumoto.

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IEEE Log Number 8826409.

0098-4094/89/0600-0897$01.00 ©1989 IEEE