Technical Report No. 153

SOME GRAPHS AND THEIR FUNCTIONAL FORMS

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TABLE OF CONTENTS

ABSTRACT

The modelling of biological systems often involves the insertion of a functional form to represent the response of a state variable to a stimulus. This technical report depicts the curves, describes a functional form for that curve, and illustrates the effects of parameter variation. Other functions have similar graphs and the FORTRAN code implementing the function is not unique.

INTRODUCTION

In developing models of biological systems a number of functional forms (actually the curves) are encountered repeatedly. We have attempted to present some of the most important such curves and to display a number of their characteristics.

For each of these functions we have displayed in the upper right corner of the first page of the description a curve giving the general shape of the graph of that function. To get this curve some choice was made of the parameters indicated for that function. Next one finds a functional form for that function and for its derivative. Following these expressions is a definition of each of the variables and parameters appearing in the expression.

For each function we then display a series of graphs in which the parameters are varied over a few values to illustrate the effects of these parameters on the graph of the function. The reader is warned that, when varying several of these parameters simultaneously, quite different shapes may arise. To avoid erroneous use, it is wise to compute a few values of the function in the neighborhood of the value(s) of interest. This will provide the user with a "feel" for the function.

The final section of each description contains one or more FORTRAN implementations of the function. These have all been used on the CDC 6400 at Colorado State University and, to our knowledge, work correctly.

ARCTANGENT FUNCTION

Functional Form

$$f(x,a,b,c,d) = b + \frac{c}{\pi} \arctan [\pi d(x - a)]$$

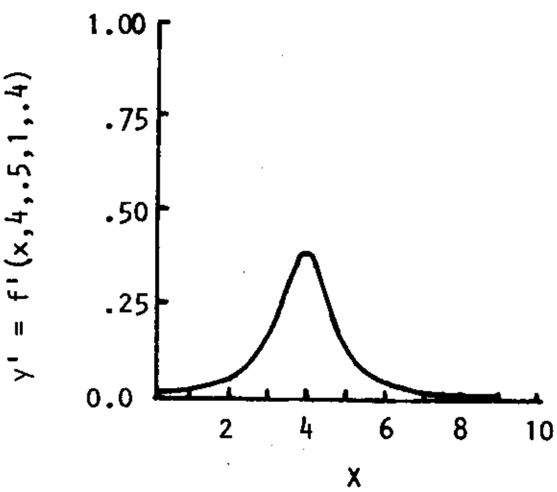
Derivative

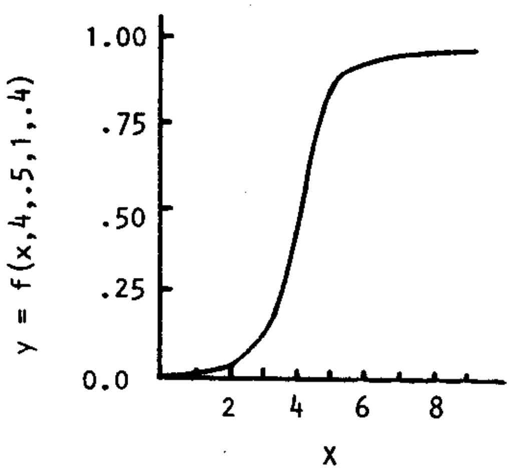
$$f'(x,a,b,c,d) = cd \left(\frac{1}{1 + [\pi d(x - a)]^2}\right)$$

Parameter Definitions

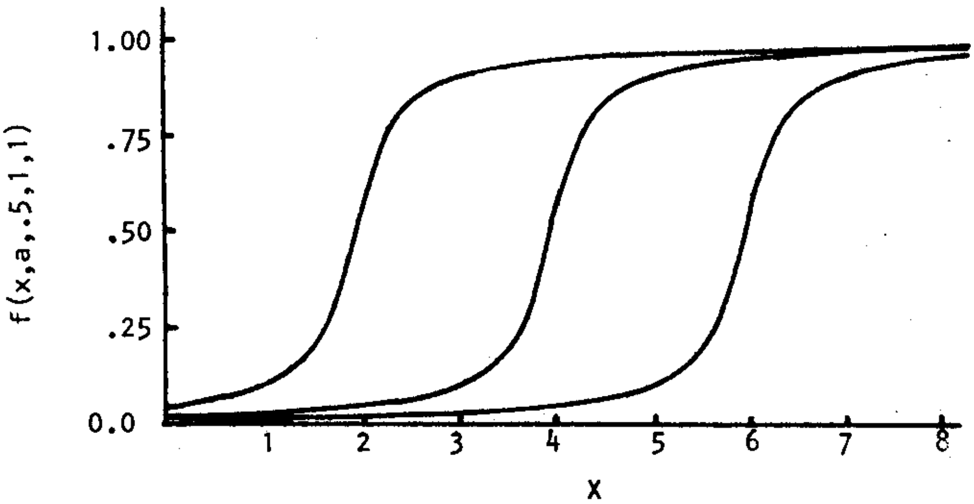
a = "x" location of inflection point
b = "y" location of inflection point
c = step size (distance from the maximum point to the minimum point)
d = slope of line at inflection point

Graphs

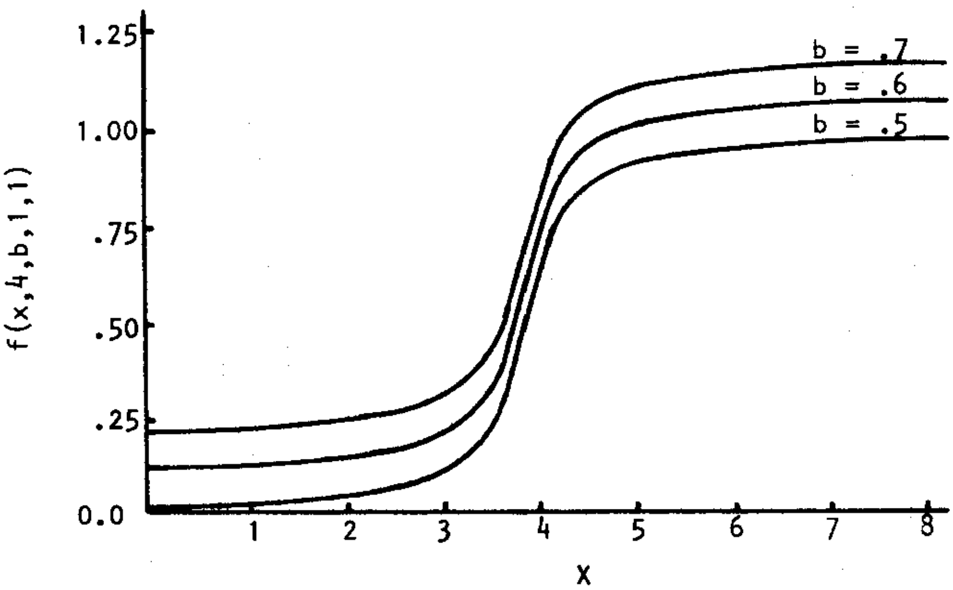




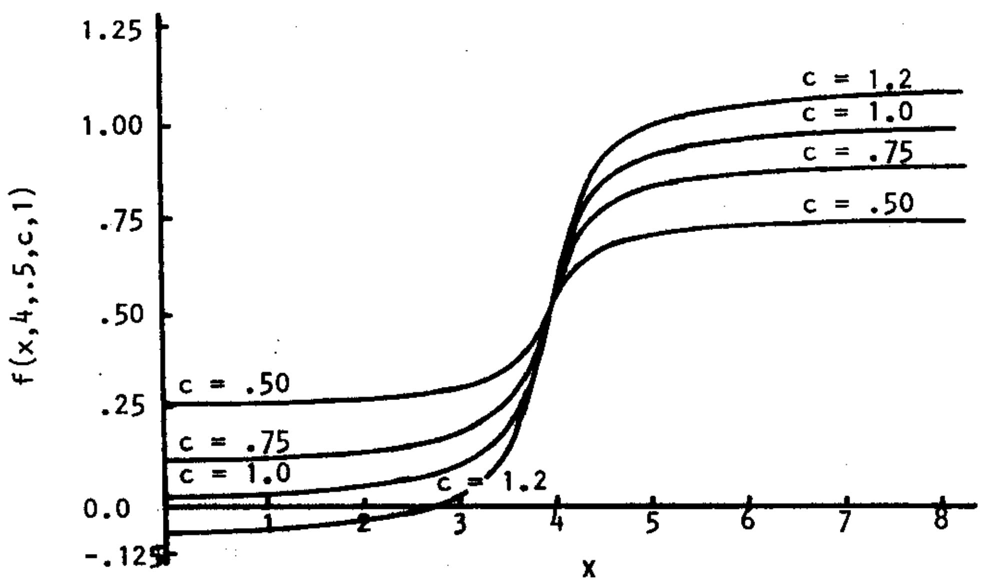
vary "a"

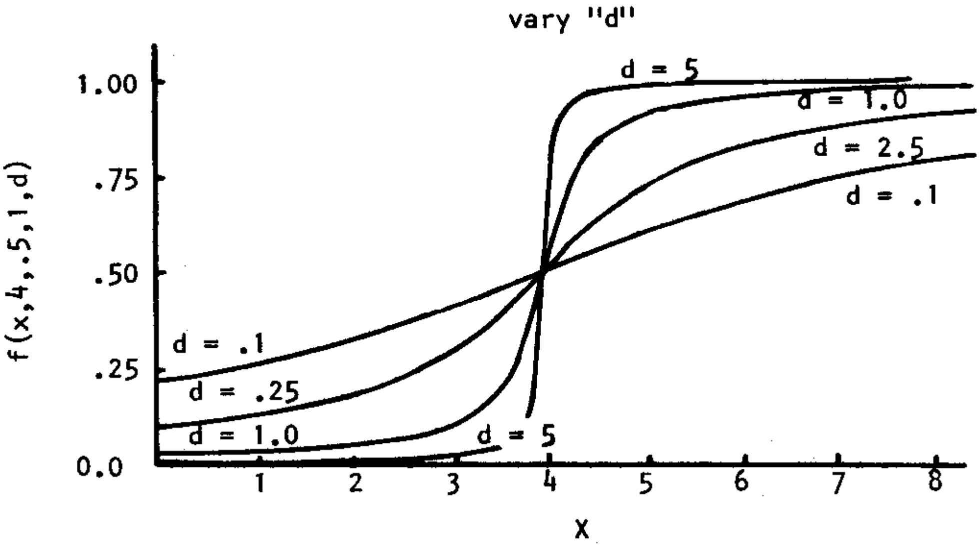


vary "b"



vary "c"





FORTRAN CODE

```
      FUNCTION ATANF(X,A,B,C,D)
      ATANF=B+(C/3.14159)*ATAN(3.14159*D*(X-A))
      RETURN
      END
C...  NOTE THAT 3.14159 IS AN APPROXIMATION TO PI
C...  AND THAT ATAN IS A SYSTEM SUPPLIED ROUTINE
C...  THAT COMPUTES THE PRINCIPAL BRANCH (PI/2>
C...  ATAN(X)>-PI/2)
```

GENERALIZED POISSON DENSITY FUNCTION

Functional Form

$$f(x,a,b,c,d) = \left(\frac{b-x}{b-a}\right)^{c} \cdot e^{\left(\frac{c}{d}\right)\cdot\left[1-\left(\frac{b-x}{b-a}\right)^{d}\right]}$$

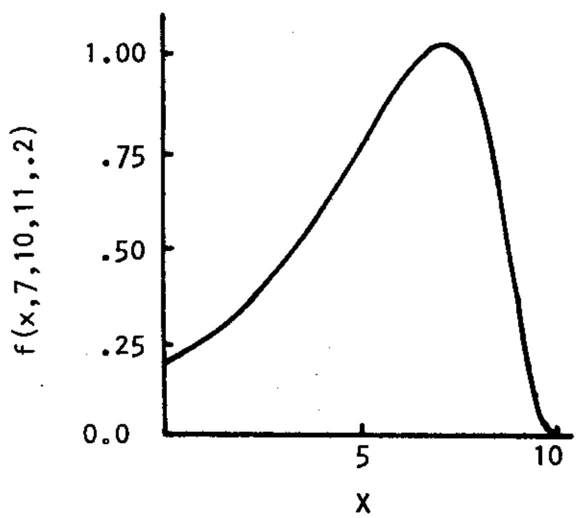
Derivative

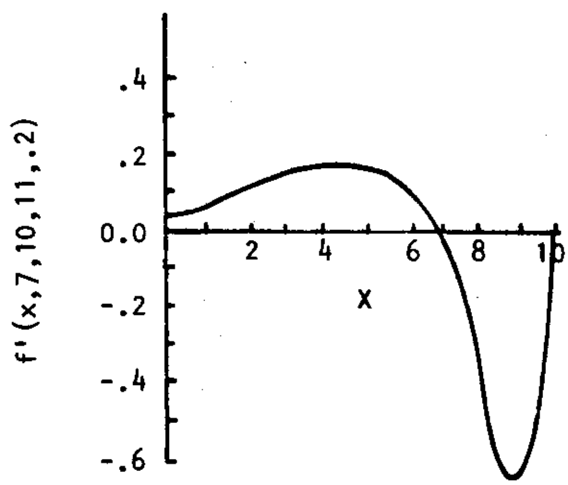
$$f'(x,a,b,c,d) = e^{\left(\frac{c}{d}\left[1-\left(\frac{b-x}{b-a}\right)^{d}\right]\right)} \cdot \left(\frac{b-x}{b-a}\right)^{c-1} \cdot \frac{c}{b-a} \cdot \left(\left(\frac{b-x}{b-a}\right)^{d} - 1\right)$$

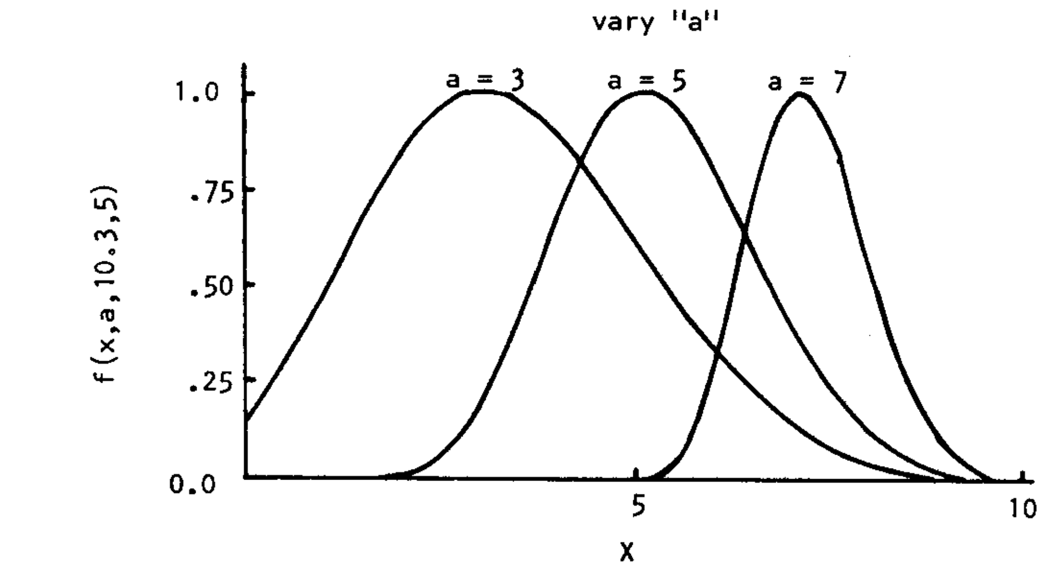
Parameter Definitions

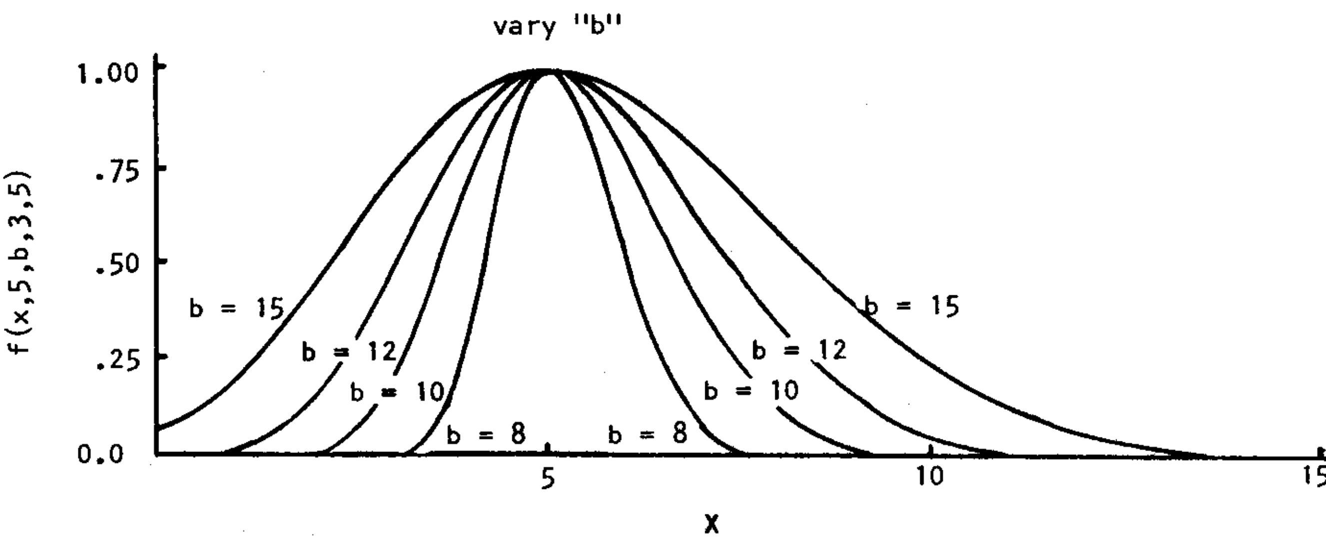
- a = value of "x" where $f(x) = 1.0$
- b = value of "x" where $f(x) = 0.0$ $(x < b)$
- c = shape parameter for part of the curve to the right of $x = a$
- d = shape parameter for part of the curve to the left of $x = a$
- e = base of the natural logarithm ≅ 2.71828

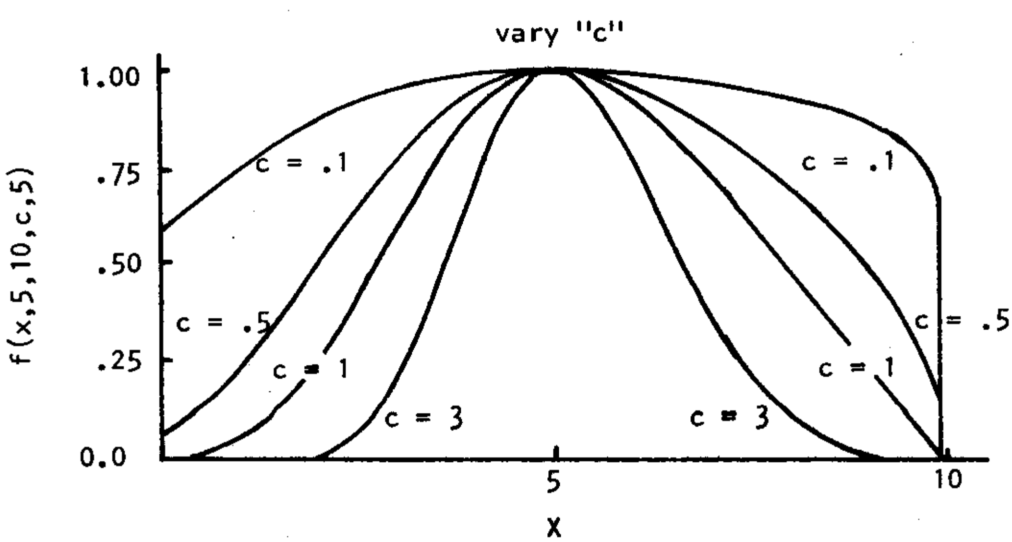
Graphs

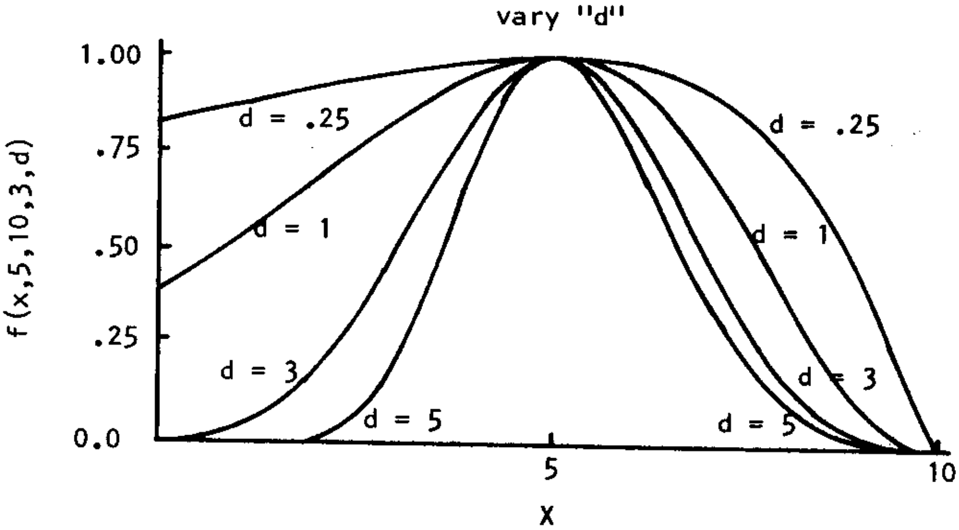





vary "a"
a = 3
a = 5
a = 7
1.0
.75
.50
.25
0.0
f(x,a,10.3,5)
5
10
X

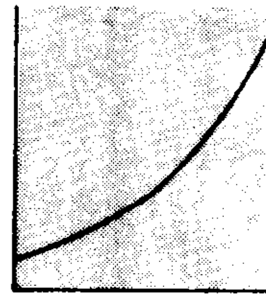

vary "b"
1.00
.75
.50
.25
0.0
f(x,5,b,3,5)
b = 15
b = 12
b = 10
b = 8
b = 8
b = 10
b = 12
b = 15
5
10
15
X


vary "c"
1.00
.75
.50
.25
0.0
f(x,5,10,c,5)
c = .1
c = .1
c = .5
c = .5
c = 1
c = 1
c = 3
c = 3
5
10
X



FORTRAN CODE

```
      FUNCTION GPDF(X,A,B,C,D)
      FRAC=(B-X)/(B-A)
      IF (FRAC.LE.0.) GO TO 1
      GPDF=EXP(C/D*(1.-FRAC**D))
      GPDF=(FRAC**C)*GPDF
      RETURN
    1 GPDF=0
      RETURN
      END
C...  EXP IS A SYSTEM SUPPLIED FUNCTION WHICH
C...  EXPONENTIATES (BASE E) THE ARGUMENT.
C...  THE IF STATEMENT IS INCLUDED TO ASSURE
C...  THAT ONE DOES NOT ATTEMPT TO EXPONENTIATE
C...  A NEGATIVE NUMBER. N.B. IF X>B, GPDF=0.
C...  ALTHOUGH THE FUNCTION IS REALLY NOT DEFINED
C...  THERE.
```



EXPONENTIAL FUNCTION

Functional Form

$$f(x,a,b) = ae^{bx}$$

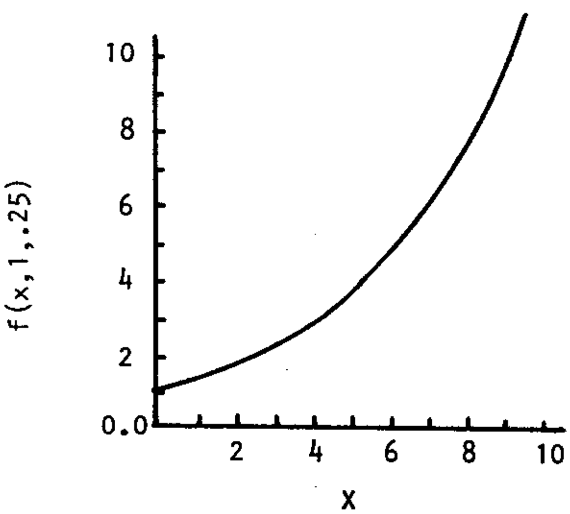
Derivative

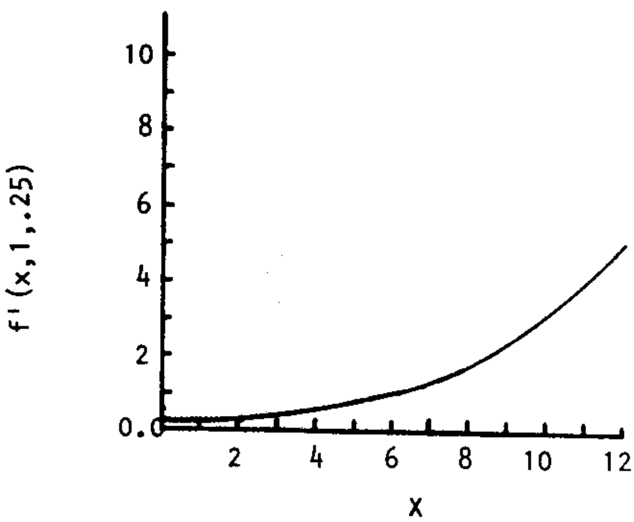
$$f'(x,a,b) = abe^{bx}$$

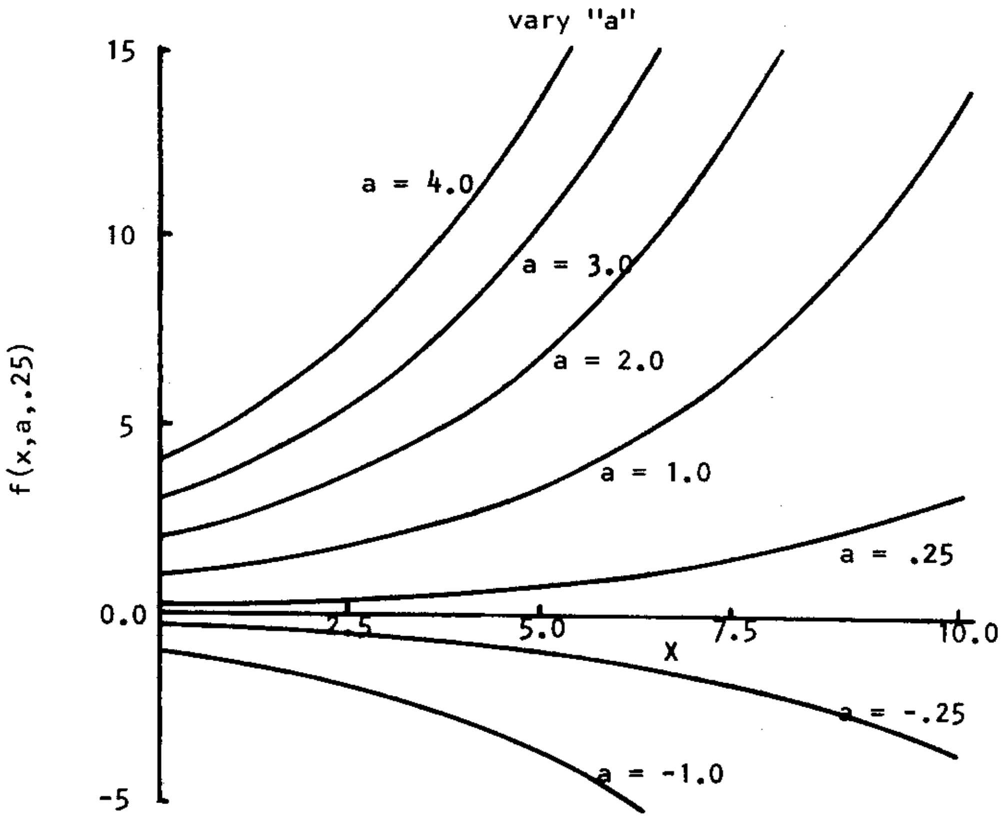
Parameter Definitions

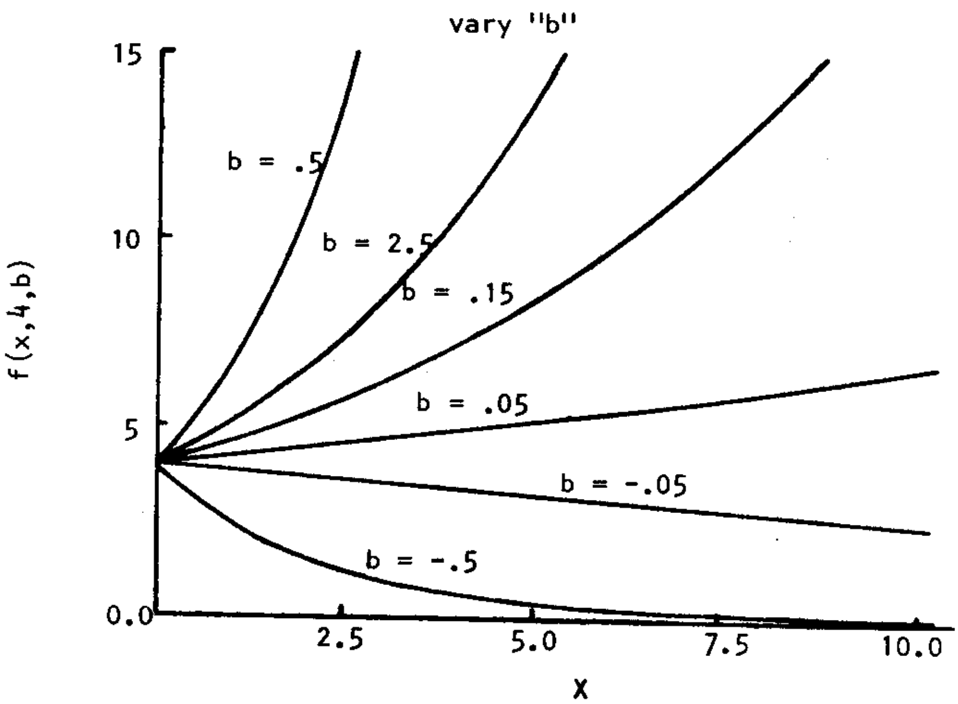
a = the value of $f(x)$ when $x = 0.0$
b = shape parameter for the curve

Graphs









FORTRAN CODE

```
      FUNCTION EF(X,A,B)
      EF=A*EXP(B*X)
      RETURN
      END
C...  EXP IS A SYSTEM SUPPLIED FUNCTION WHICH
C...  EXPONENTIATES (BASE E) THE ARGUMENT.
```

MODIFIED EXPONENTIAL FUNCTION

Functional Form

$$f(x,a,b) = ab^x$$

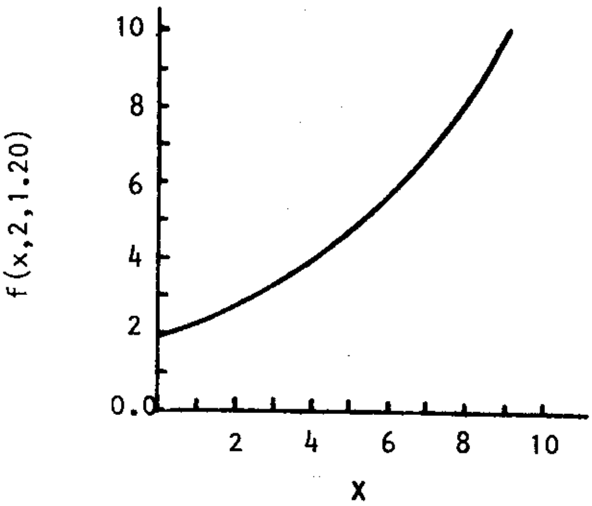
Derivative

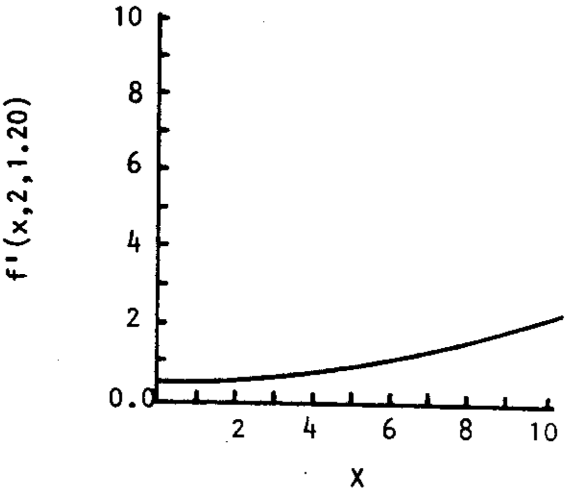
$$f'(x,a,b) = ab^x \ln_e b$$

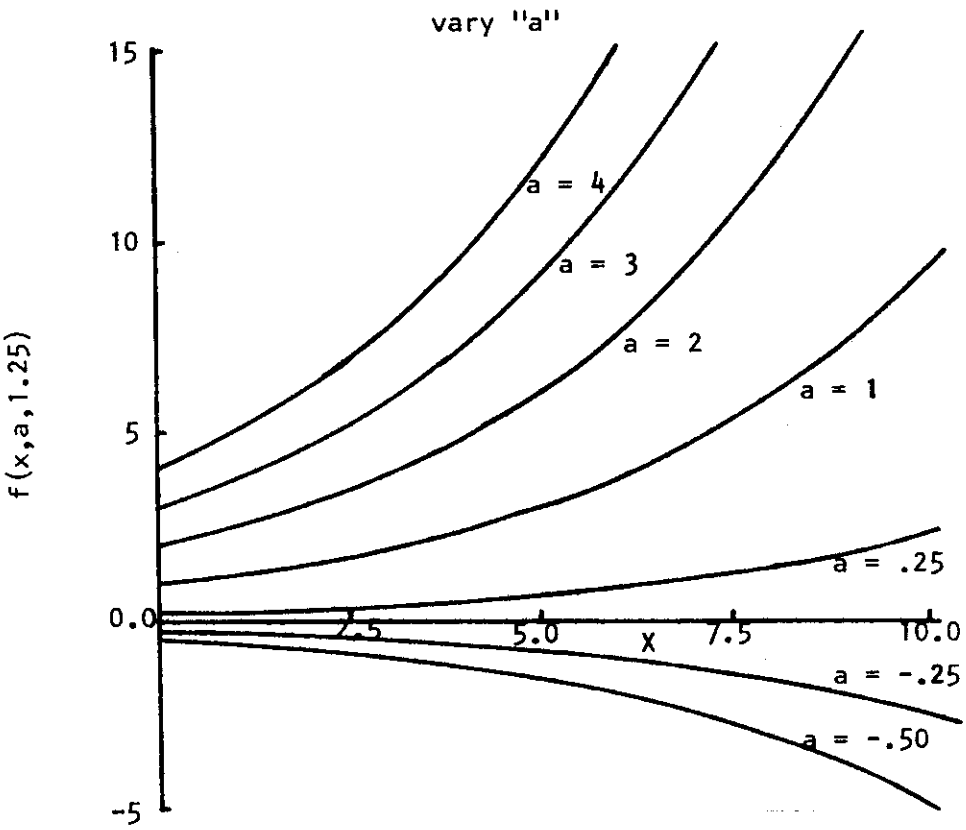
Parameter Definitions

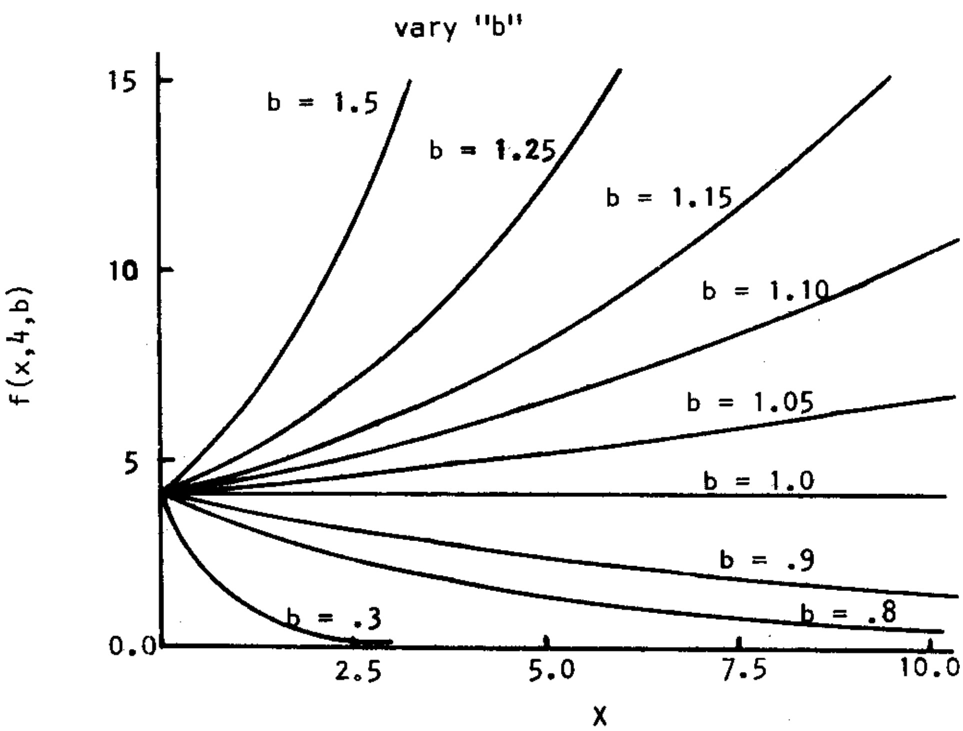
a = the value of f(x) when x = 0.0
b = shape parameter for the curve, b > 0

Graphs



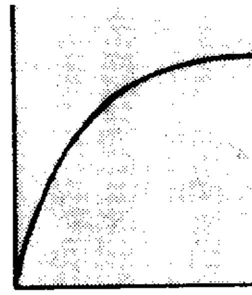



vary "a"
15
10
5
0.0
-5
f(x,a,1.25)
a = 4
a = 3
a = 2
a = 1
a = .25
a = -.25
a = -.50
2.5
5.0
7.5
10.0
X


vary "b"
15
10
5
0.0
f(x,4,b)
b = 1.5
b = 1.25
b = 1.15
b = 1.10
b = 1.05
b = 1.0
b = .9
b = .8
b = .3
2.5
5.0
7.5
10.0
X

FORTRAN CODE

```
      FUNCTION EFM(X,A,B)
      IF (B.LE.0.) GO TO 1
      EFM = A*B**X
      RETURN
    1 EFM = 0.
      RETURN
      END
C...  IF B≤0. THIS FUNCTION IS UNDEFINED
C...  (MULTIPLE VALUED) ALTHOUGH THE ROUTINE
C...  RETURNS THE VALUE 0.
```



NATURAL GROWTH FUNCTION

Functional Form

$$f(x,a,b) = a(1 - e^{-bx})$$

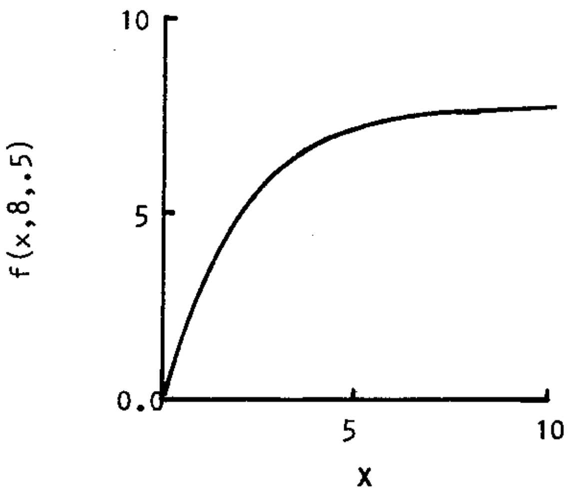
Derivative

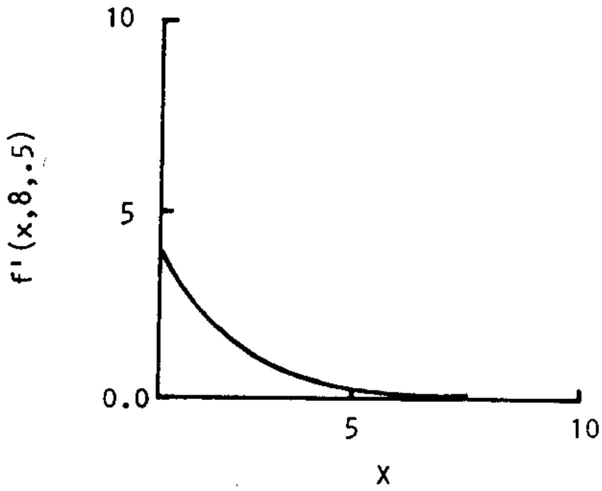
$$f'(x,a,b) = abe^{-bx}$$

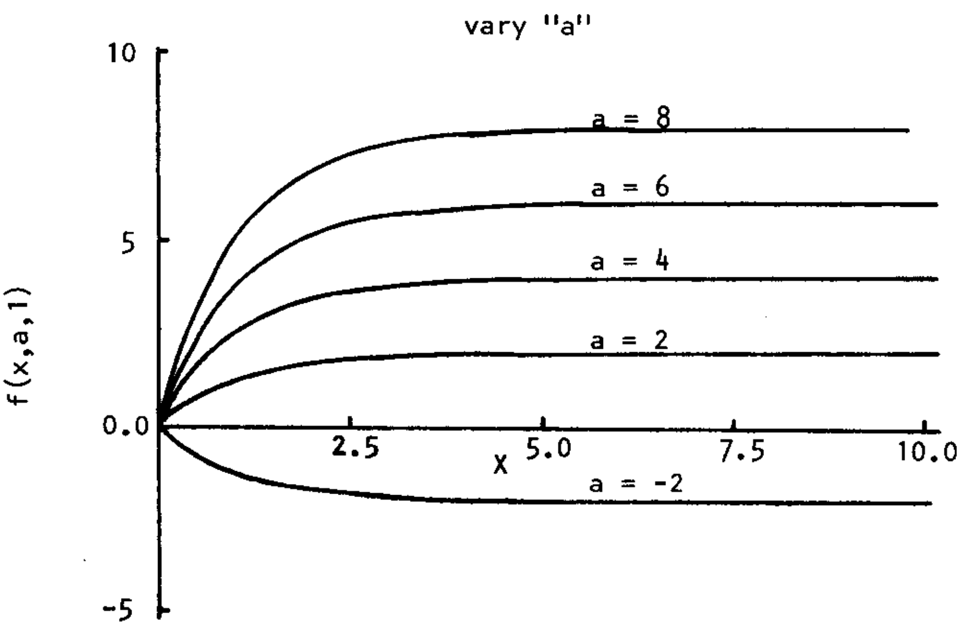
Parameter Definitions

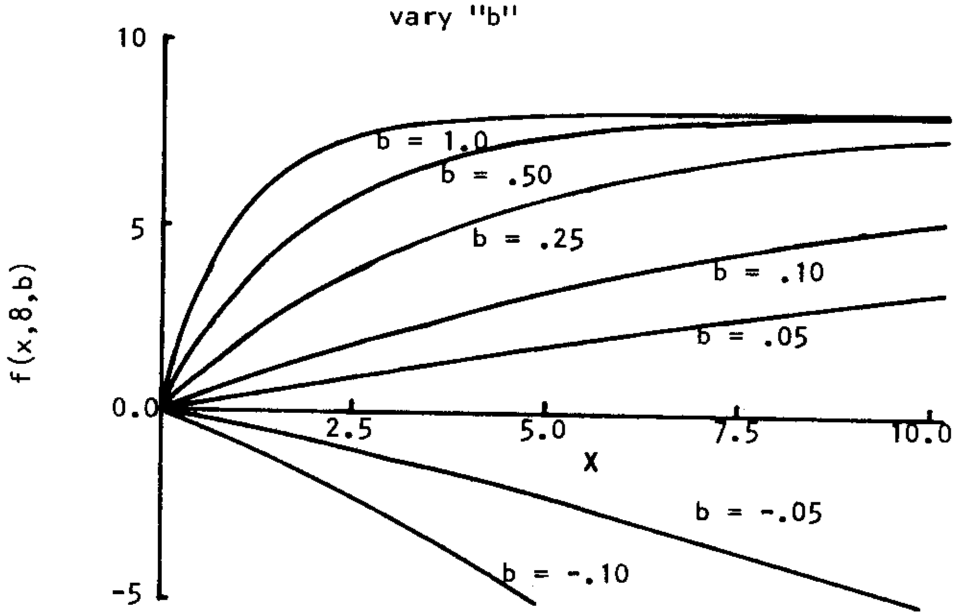
a = the maximum or minimum value of f(x)
b = parameter that controls the rate which f(x) approaches "a"

Graphs



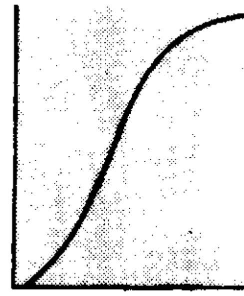






FORTRAN CODE

```
      FUNCTION GFN(X.A.B)
      GFN = A*(1.-EXP(B*X))
      RETURN
      END
```



LOGISTIC FUNCTION

Functional Form

$$f(x,a,b,c) = \frac{a}{1 + be^{-cx}}$$

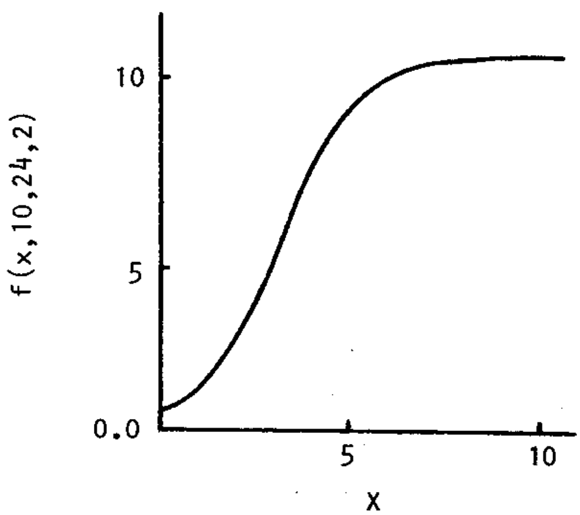
Derivative

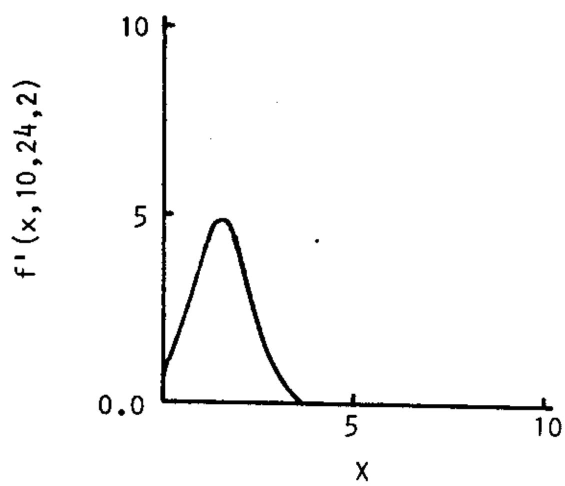
$$f'(x,a,b,c) = \frac{+abce^{-cx}}{(1 + be^{-cx})^2}$$

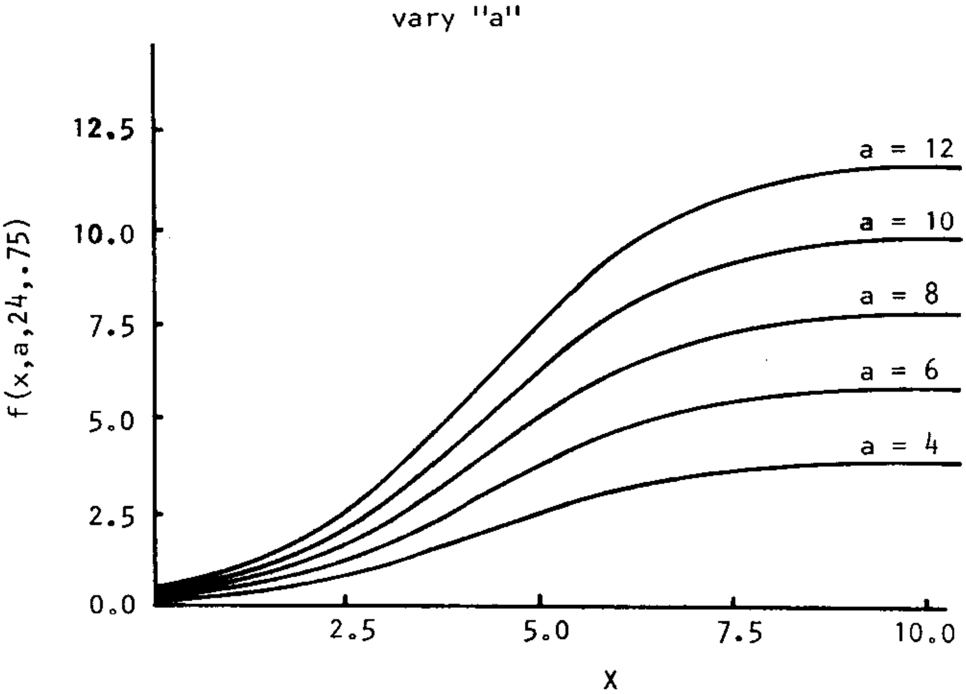
Parameter Definitions

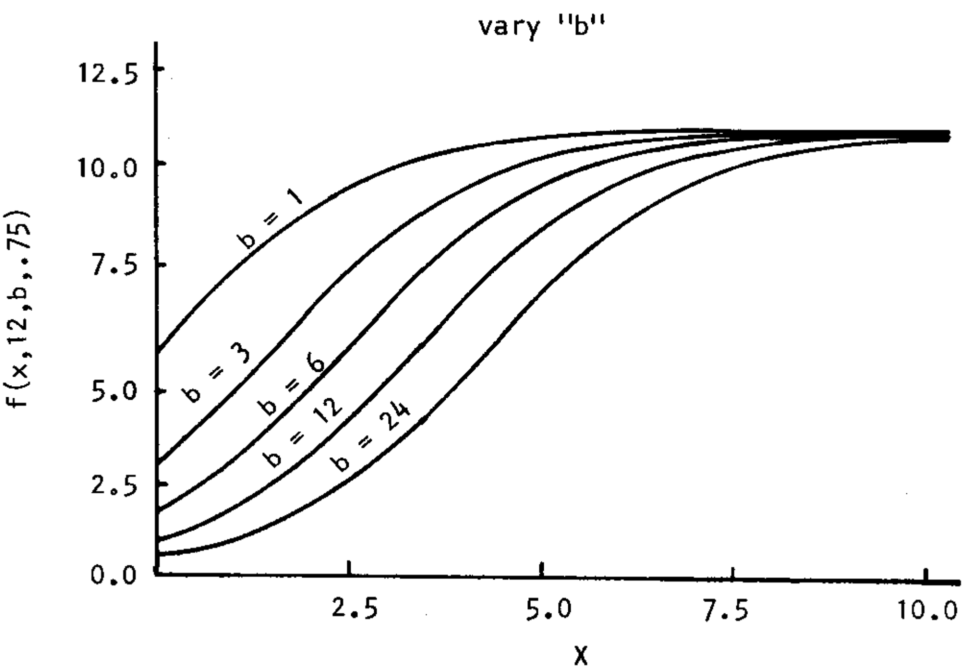
a = the maximum value of f(x) [f(x) equals $\frac{a}{2}$ at the inflection point of the curve]
b = control parameter for value of f(x) when x = 0.0
c = control parameter for the value of "x" at the inflection point of the curve

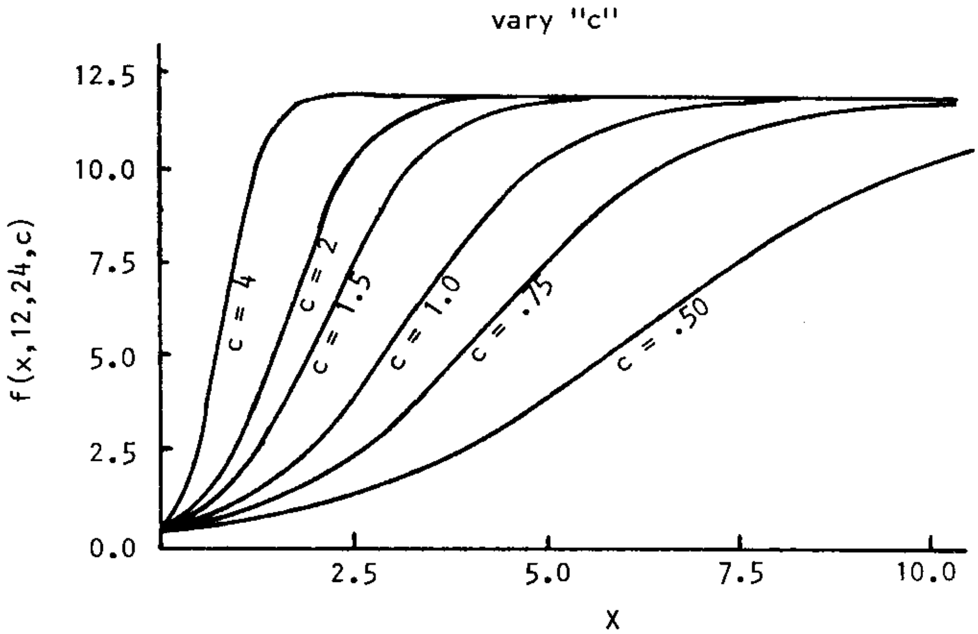
Graphs






vary "a"
f(x,a,24,.75)
12.5
10.0
7.5
5.0
2.5
0.0
2.5
5.0
7.5
10.0
X
a = 12
a = 10
a = 8
a = 6
a = 4


vary "b"
f(x,12,b,.75)
12.5
10.0
7.5
5.0
2.5
0.0
2.5
5.0
7.5
10.0
X
b = 1
b = 3
b = 6
b = 12
b = 24



FORTRAN CODE

```
      FUNCTION FL(X,A,B,C)
      FL = A/(1.+B*EXP(-C*X))
      RETURN
      END
```

ALLOMETRIC FUNCTION

Functional Form

$$f(x,a,b) = ax^{b}$$

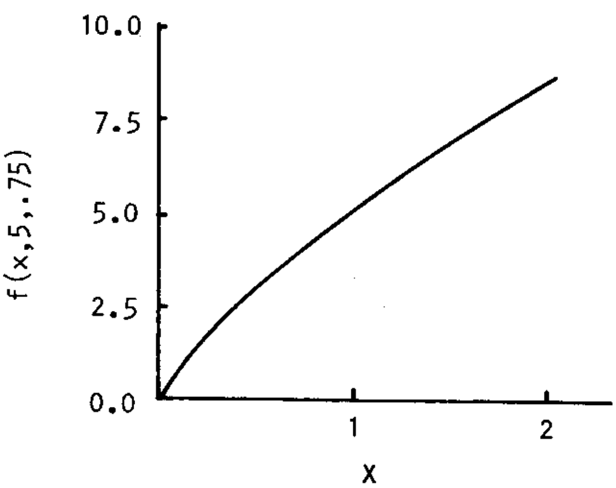
Derivative

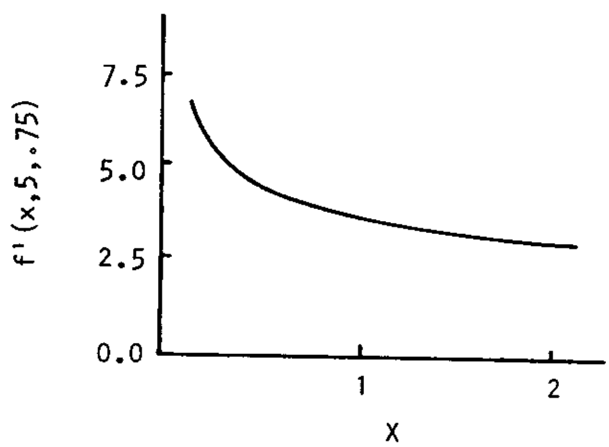
$$f'(x,a,b) = abx^{(b-1)}$$

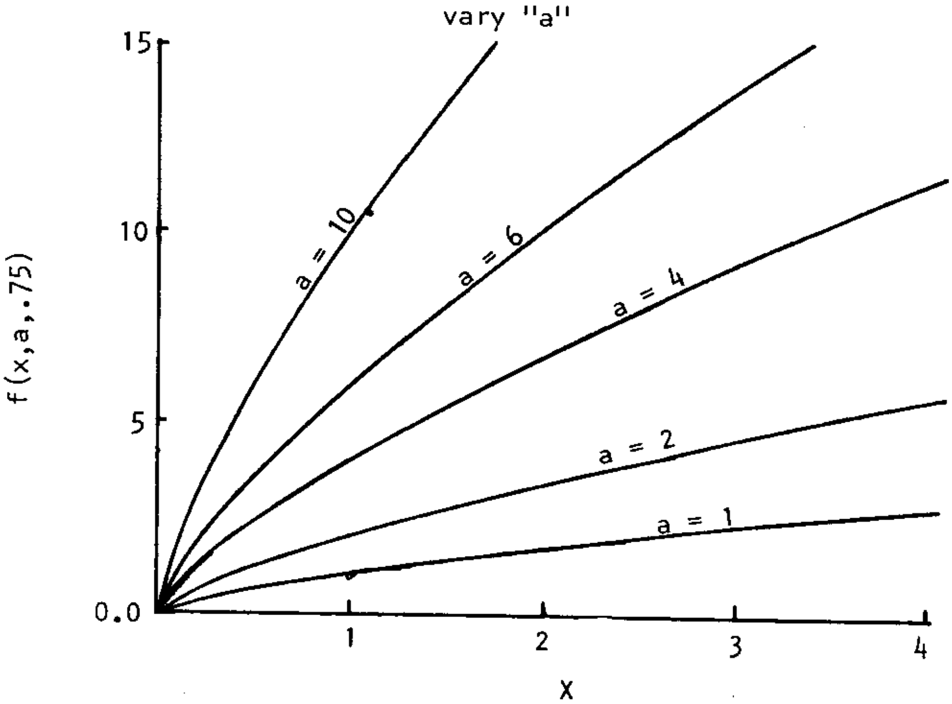
Parameter Definitions

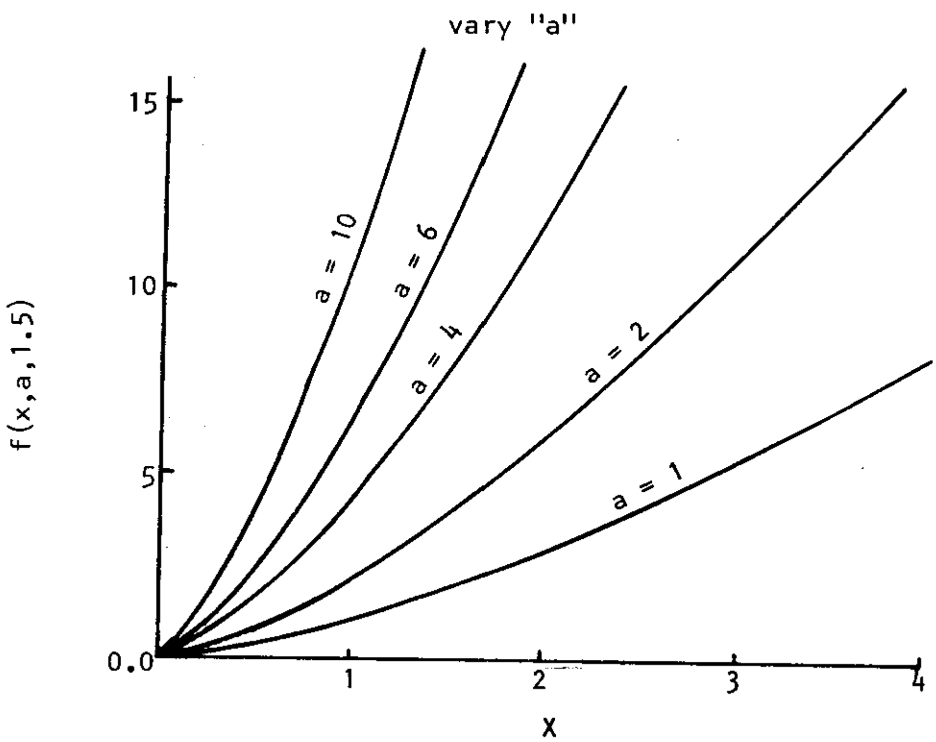
a = the value of f(x) when x = 1.0
b = control parameter for the shape of the curve

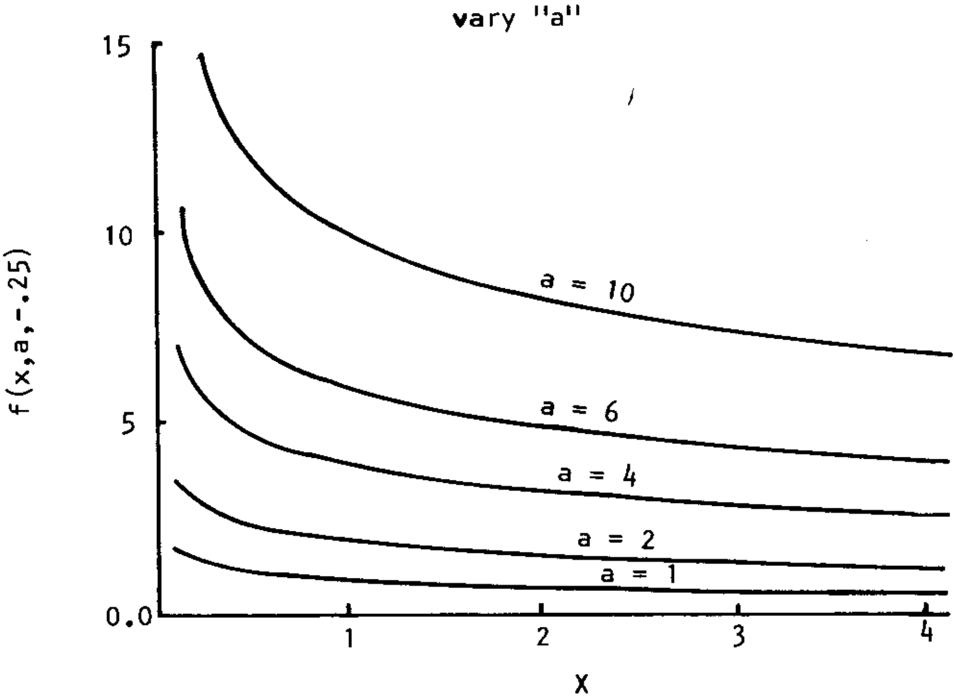
Graphs

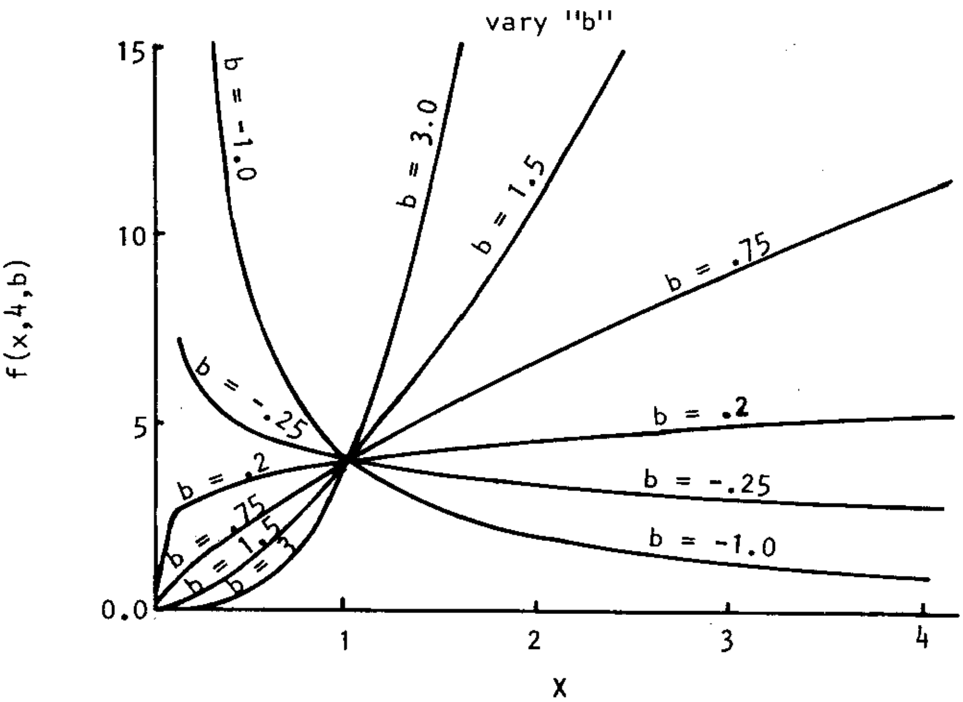





vary "a"
15
10
5
0.0
f(x,a,.75)
a = 10
a = 6
a = 4
a = 2
a = 1
1
2
3
4
X


vary "a"
15
10
5
0.0
f(x,a,1.5)
a = 10
a = 6
a = 4
a = 2
a = 1
1
2
3
4
X





FORTRAN CODE

```
      FUNCTION AF(X,A,B)
      IF (X.LE.0.) GO TO 1
      AF = A*X**B
      RETURN
    1 AF = 0.
      RETURN
      END
C...  IF X≤0. THIS FUNCTION IS GENERALLY MULTIPLE VALUED AND COMPLEX. THE
C...  ROUTINE RETURNS THE VALUE 0 IN THIS CASE.
```

GENERALIZED GOMPERTZ EQUATION

Functional Form

$$f(x,a,b,c,d) = ab^{-cb^{-dx}}$$

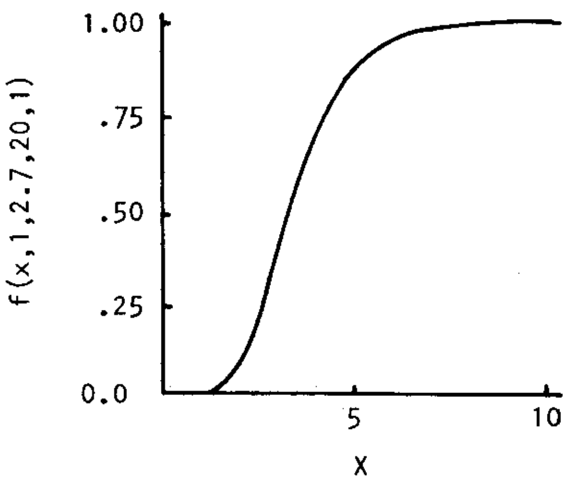
Derivative

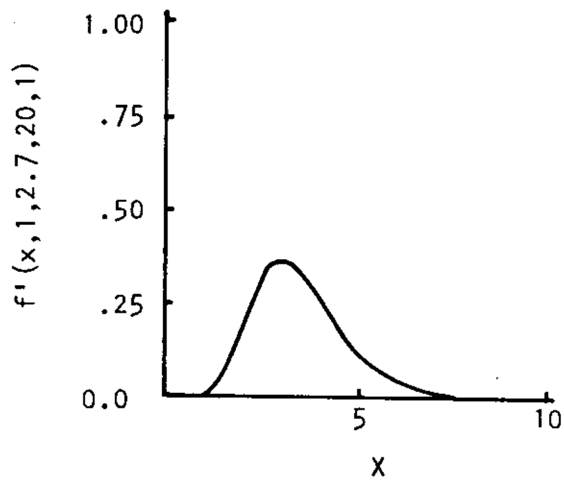
$$f'(x,a,b,c,d) = acdb^{-cb^{-dx}} \cdot (\ln_e b)^2 \cdot b^{-dx}$$

Parameter Definitions

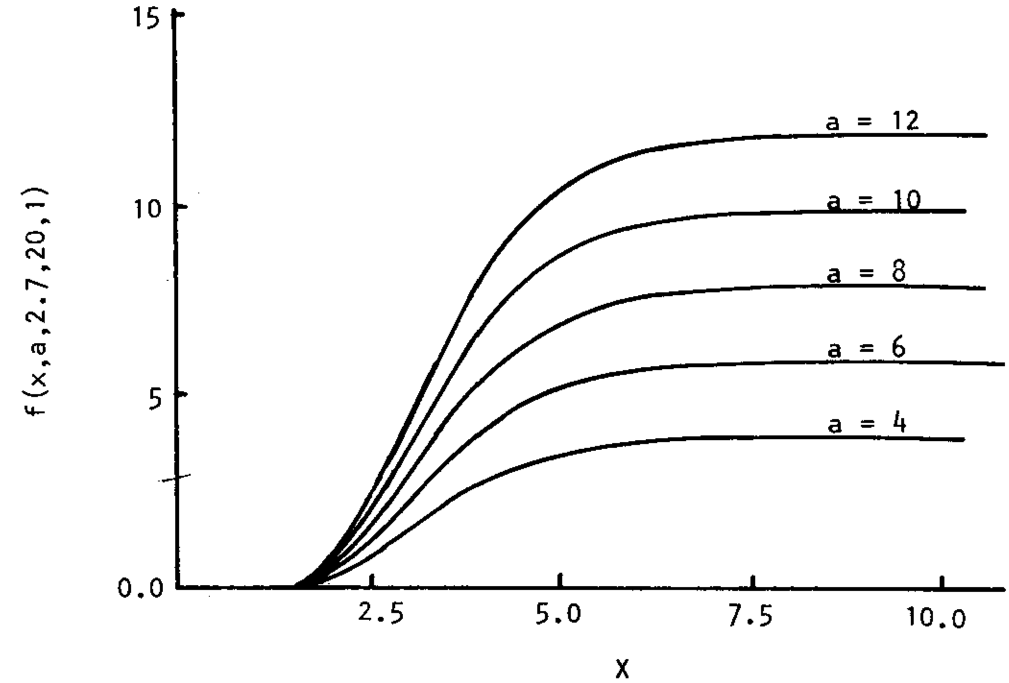
a = the maximum value of f(x)
b = control parameter that changes value of f(x) where the inflection point is located [$f(x) \approx \frac{a}{b}$ at the inflection point for values of "b" between 2 and 6]
c = control parameter that moves the "x" location of the inflection point
d = control parameter that changes the slope of the curve at inflection point

Graphs

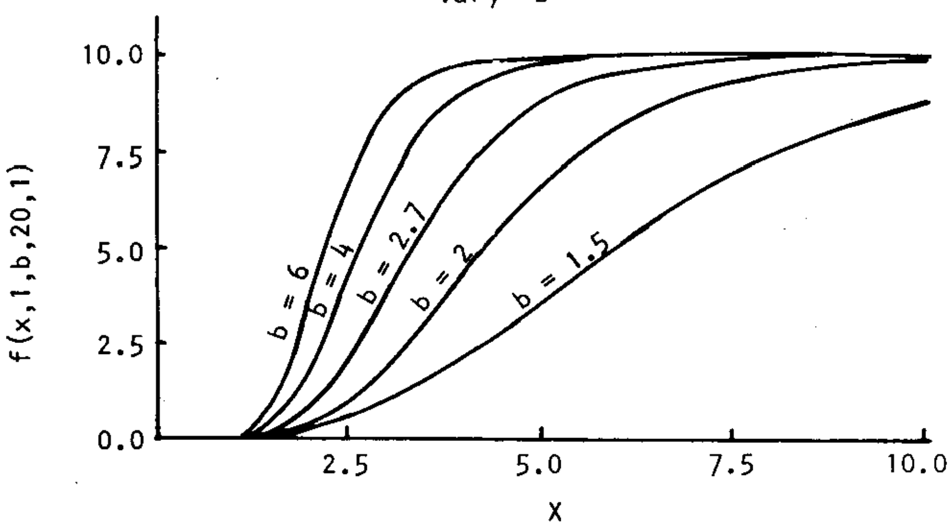




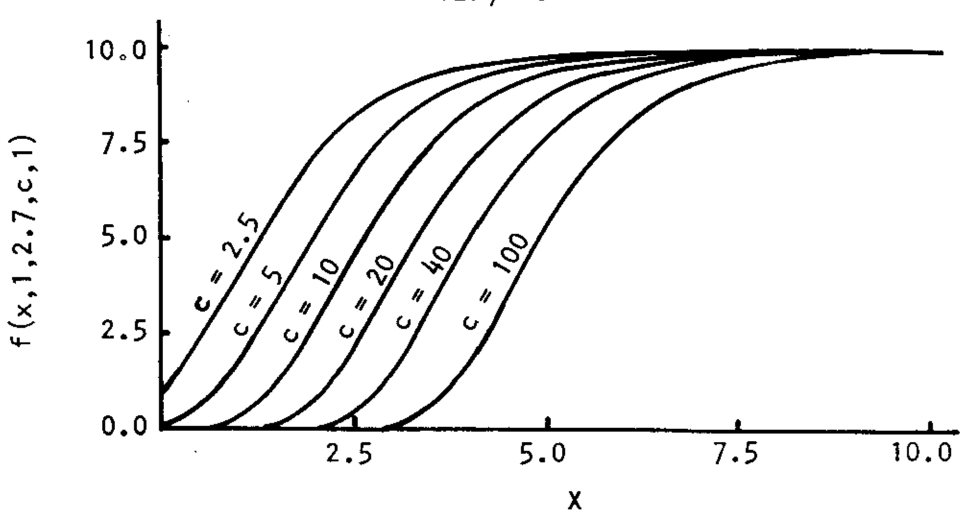
vary "a"

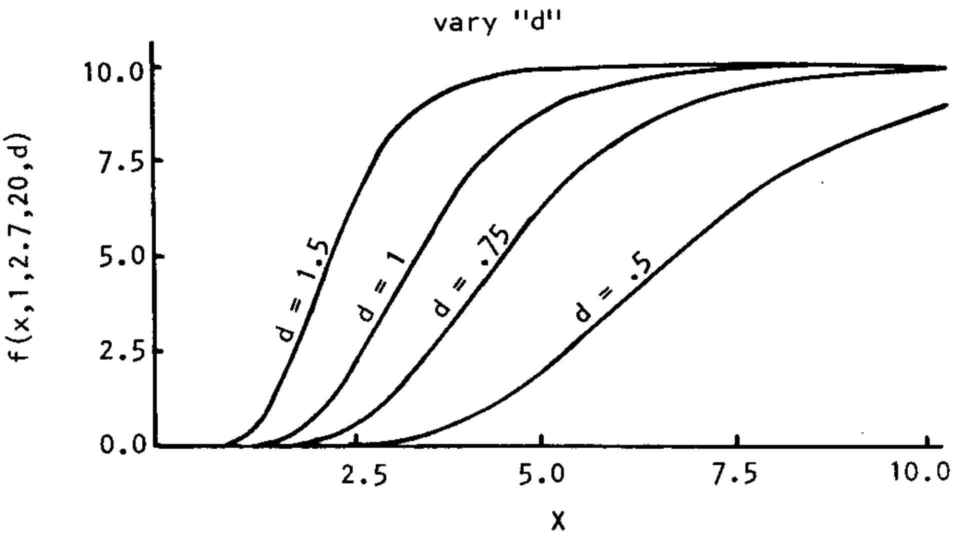


vary "b"



vary "c"





FORTRAN CODE

```
      FUNCTION GGEF(X,A,B,C,D)
      IF (B.LE.0.) GO TO 1
      GGEF=C*B**(-D*X)
      GGEF=A*B**(-GGEF)
      RETURN
    1 GGEF=0.
      RETURN
      END.
C...  THIS FUNCTION IS GENERALLY MULTIPLE VALUED AND COMPLEX
C...  IF B≤0. THE ROUTINE RETURNS THE VALUE 0.
```

QUADRATIC EQUATION

Functional Form

$$f(x,a,b,c) = a + bx + cx^2$$

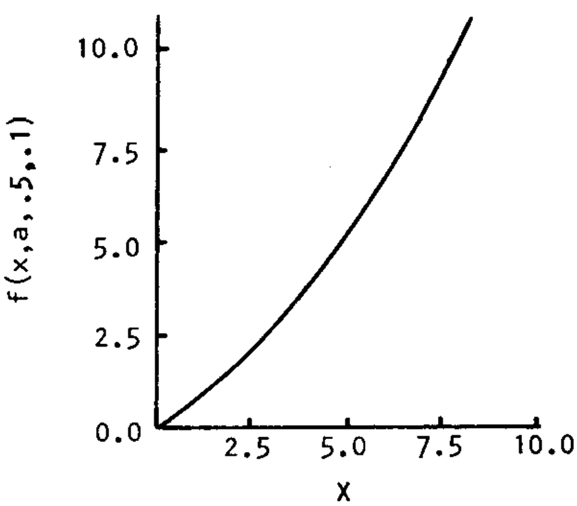
Derivative

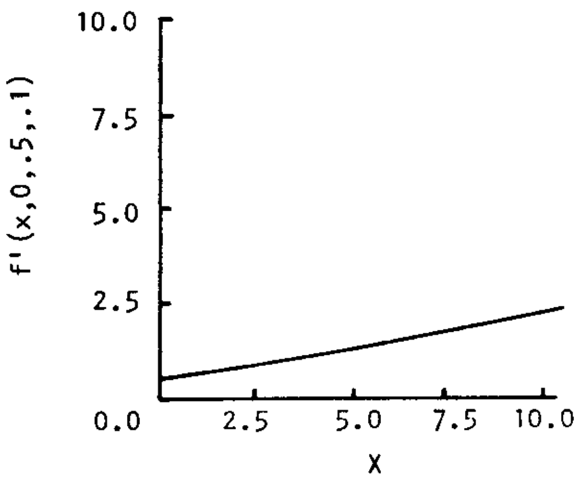
$$f'(x,a,b,c) = b + 2cx$$

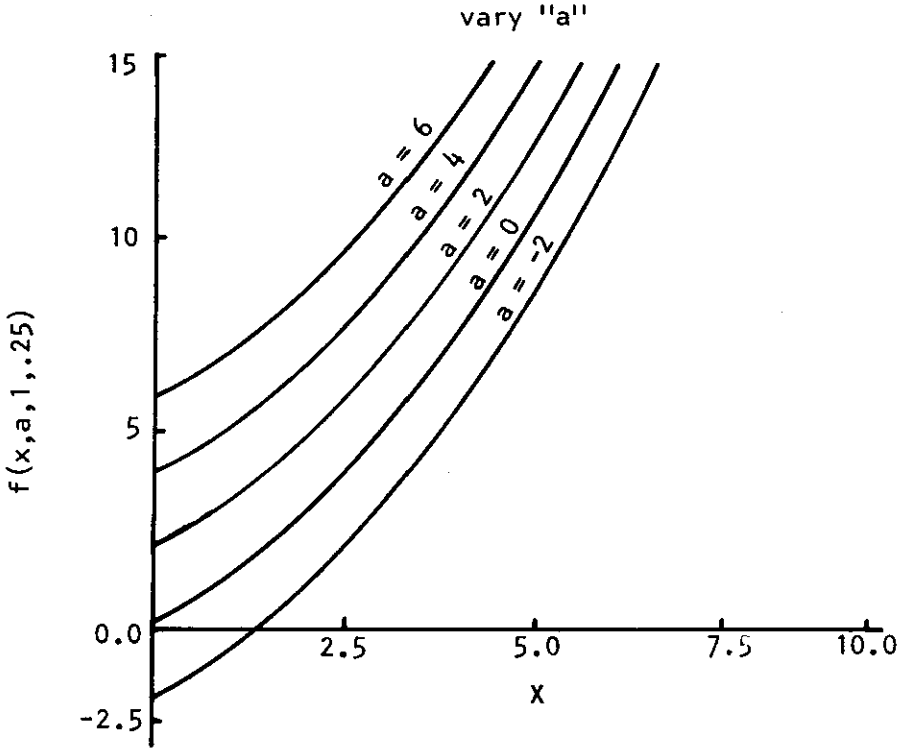
Parameter Definitions

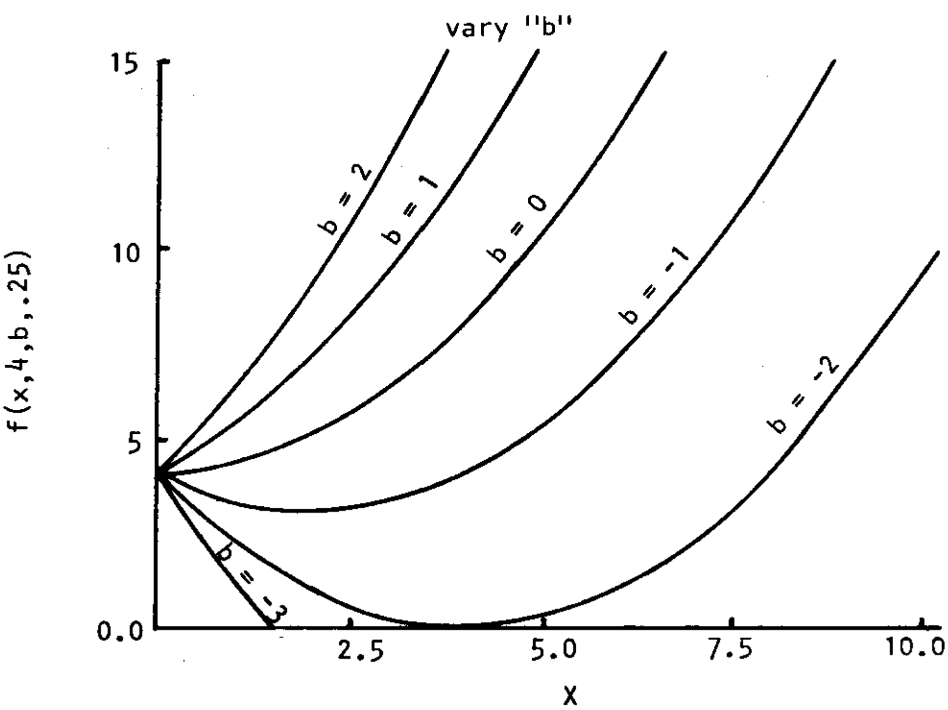
a = the value of f(x) when x = 0.0
b = parameter that controls the shape of the curve
c = parameter that controls the shape of the curve

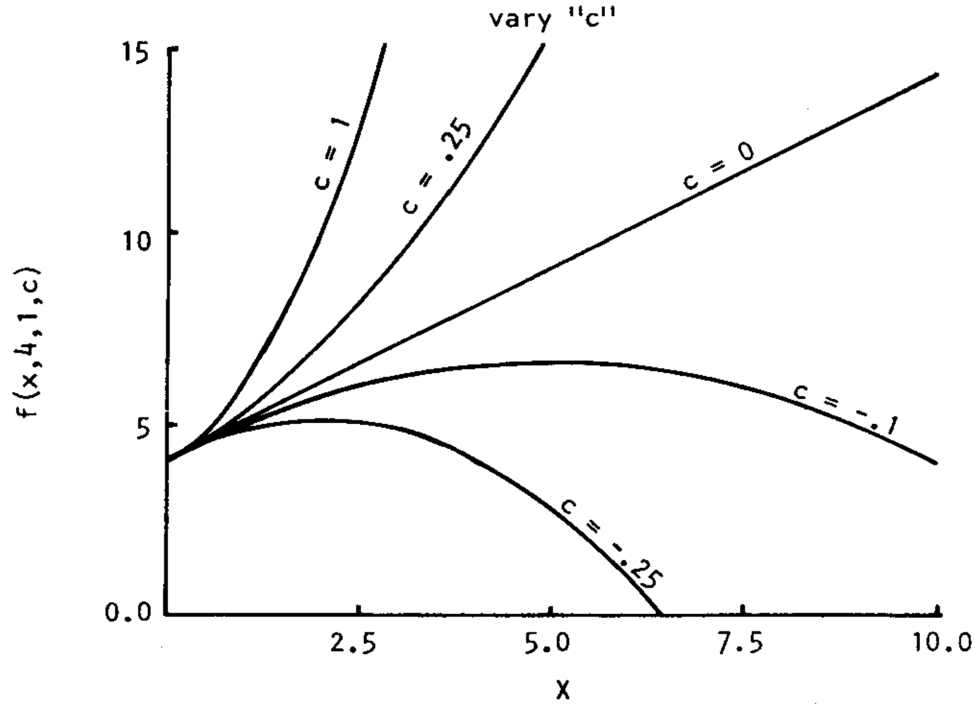
Graphs






vary "a"
15
10
5
0.0
-2.5
2.5
5.0
7.5
10.0
X
f(x,a,1,.25)
a = 6
a = 4
a = 2
a = 0
a = -2


vary "b"
15
10
5
0.0
2.5
5.0
7.5
10.0
X
f(x,4,b,.25)
b = 2
b = 1
b = 0
b = -1
b = -2
b = -3



FORTRAN CODE

```
FUNCTION QEF(X,A,B,C)
QEF=A+X*(B+C*X)
RETURN
END
```

SINE FUNCTION

Functional Form

$$f(x,a,b,c) = a \cdot \sin\left[\frac{2\pi}{b}(x - c)\right]$$

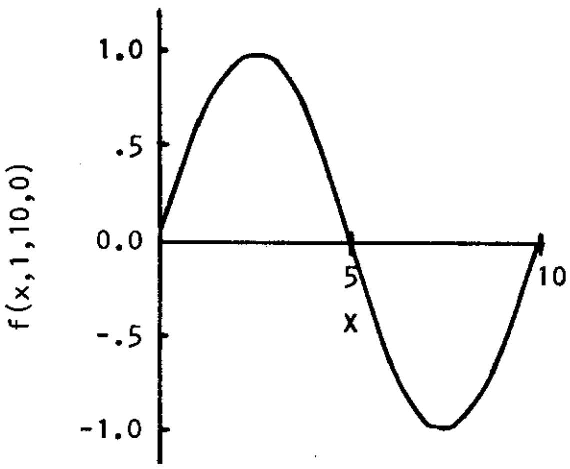
Derivative

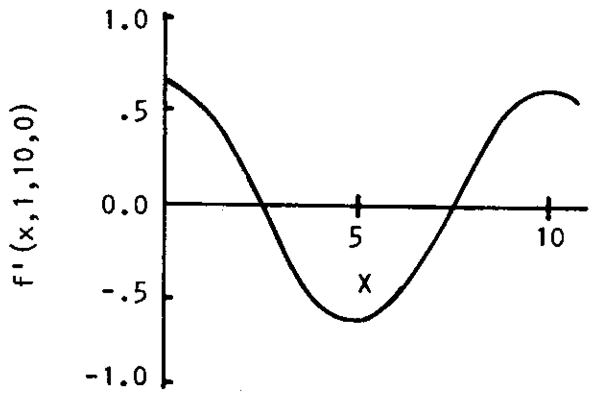
$$f'(x,a,b,c) = \frac{2\pi a}{b} \cos\left[\frac{2\pi}{b}(x - c)\right]$$

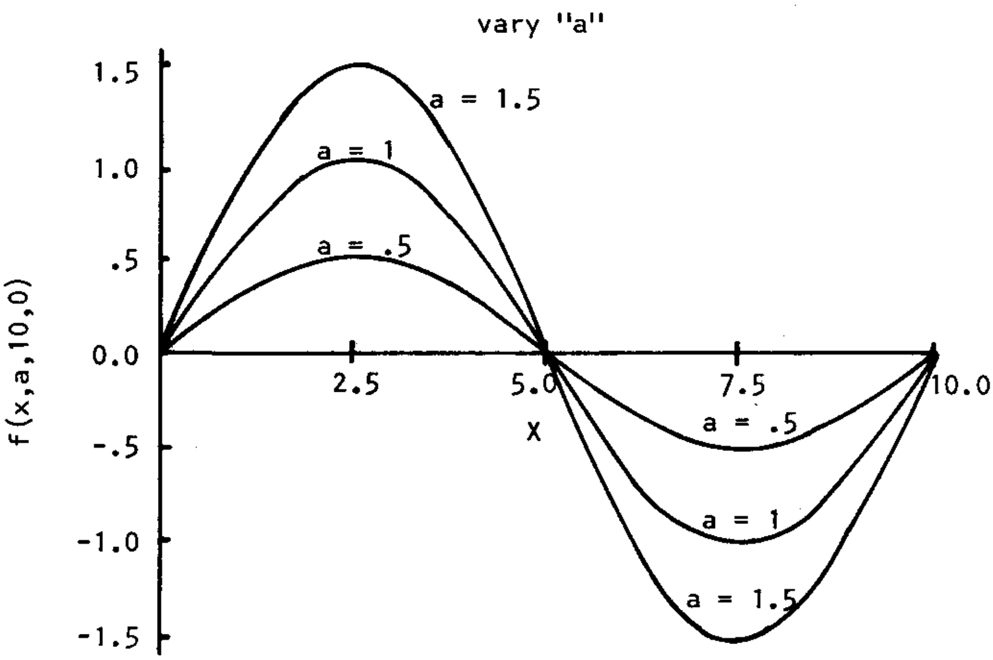
Parameter Definitions

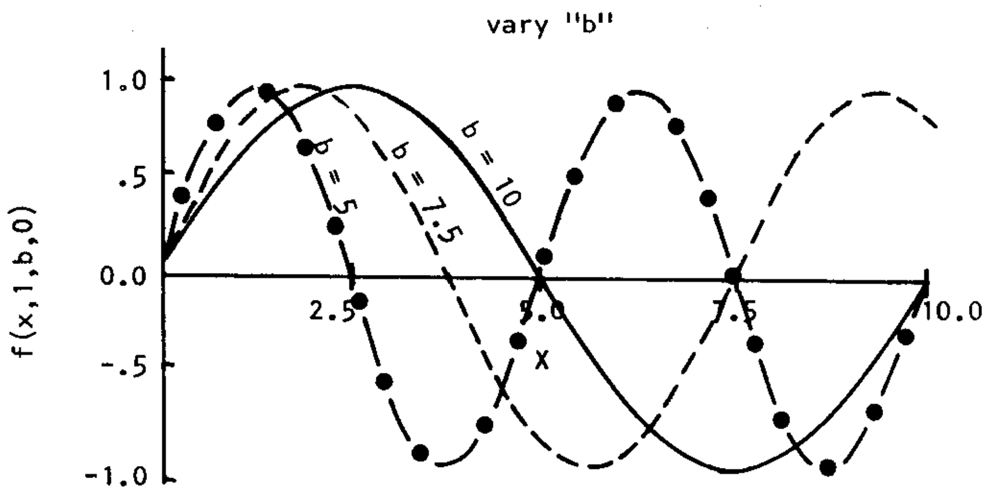
a = the amplitude of the sine wave divided by two
b = the wavelength of the sine wave
c = the parameter that moves the sine wave along the "x" axis

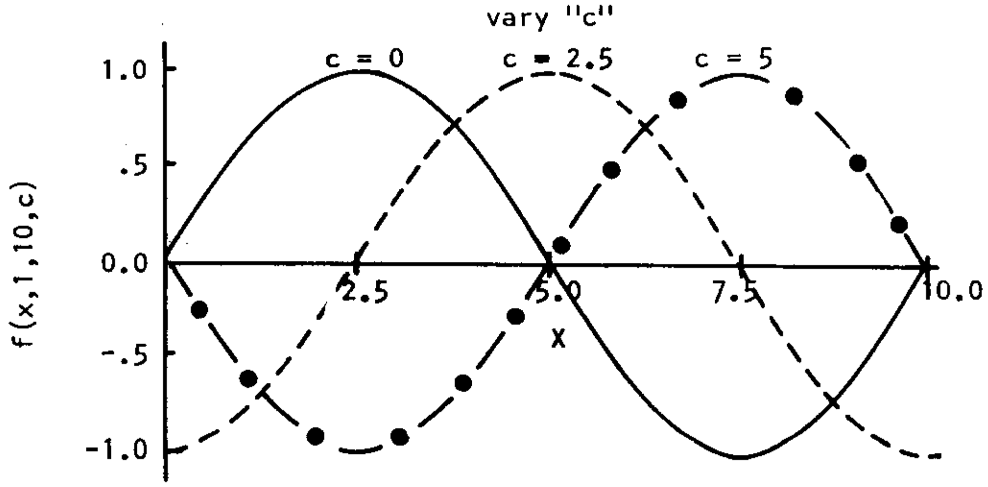
Graphs






vary "a"
f(x,a,10,0)
1.5
1.0
.5
0.0
-.5
-1.0
-1.5
a = 1.5
a = 1
a = .5
2.5
5.0
7.5
10.0
x


vary "b"
f(x,1,b,0)
1.0
.5
0.0
-.5
-1.0
b = 5
b = 7.5
b = 10
2.5
5.0
7.5
10.0
x


vary "c"
f(x,1,10,c)
1.0
.5
0.0
-.5
-1.0
c = 0
c = 2.5
c = 5
2.5
5.0
7.5
10.0
x

FORTRAN CODE

```
      FUNCTION SF(X,A,B,C)
      SF=A*SIN(6.28319/B*(X-C))
      RETURN
      END
C...  SIN IS A SYSTEM SUPPLIED FUNCTION TO
C...  COMPUTE THE SINE OF AN ANGLE IN RADIANS.
C...  6.28319 IS AN APPROXIMATION TO 2PI.
```

COSINE FUNCTION

Functional Form

$$f(x,a,b,c) = a \cdot \cos\left(\frac{2\pi}{b}(x - c)\right)$$

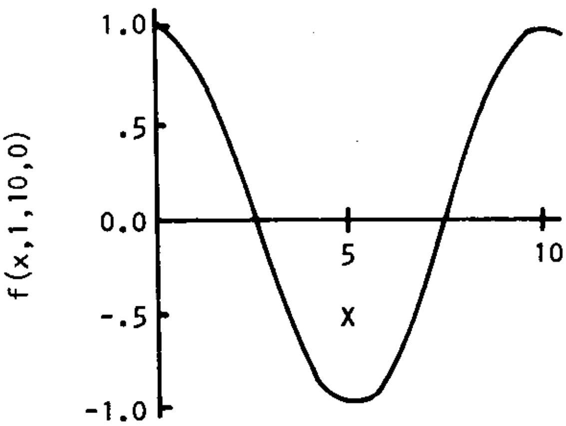
Derivative

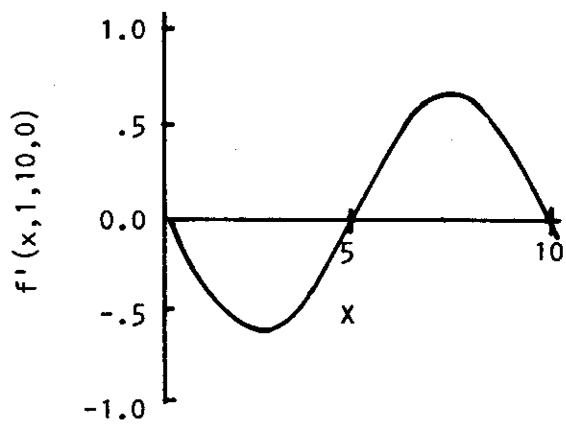
$$f'(x,a,b,c) = -\frac{2\pi a}{b} \sin\left(\frac{2\pi}{b}(x - c)\right)$$

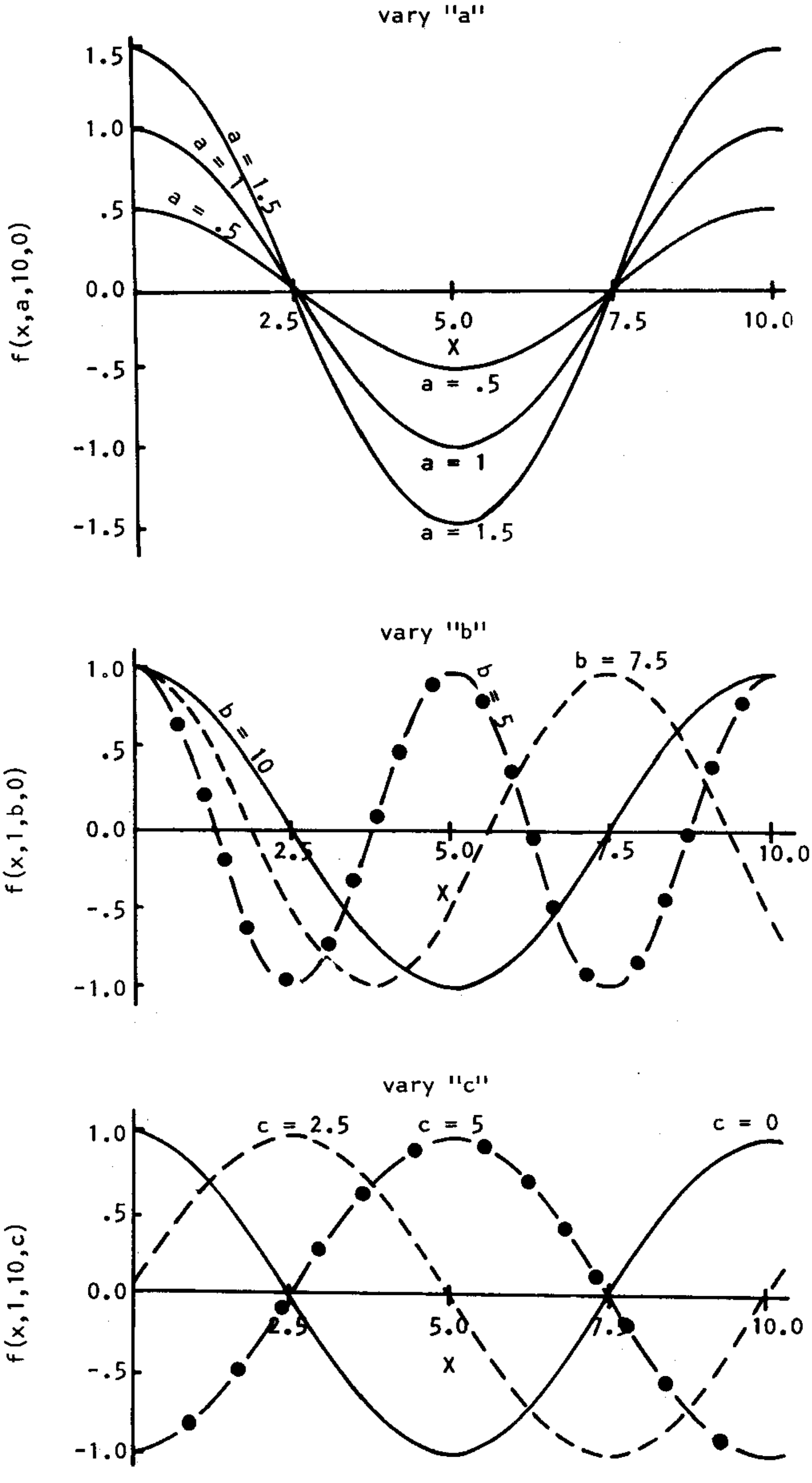
Parameter Definitions

a = the amplitude of the cosine wave divided by two
b = the wavelength of the cosine wave
c = the parameter that moves the cosine wave along the "x" axis

Graphs






vary "a"
1.5
1.0
.5
0.0
-.5
-1.0
-1.5
f(x,a,10,0)
2.5
5.0
7.5
10.0
X
a = 1.5
a = 1
a = .5
a = .5
a = 1
a = 1.5
vary "b"
1.0
.5
0.0
-.5
-1.0
f(x,1,b,0)
2.5
5.0
7.5
10.0
X
b = 10
b = 5
b = 7.5
vary "c"
1.0
.5
0.0
-.5
-1.0
f(x,1,10,c)
2.5
5.0
7.5
10.0
X
c = 2.5
c = 5
c = 0

FORTRAN CODE

```
      FUNCTION CF(X,A,B,C)
      CF=A*COS(6.28319/B*(X-C))
      RETURN
      END
C...  COS IS A SYSTEM SUPPLIED FUNCTION TO
C...  COMPUTE THE COSINE OF AN ANGLE IN RADIANS.
C...  6.28319 IS AN APPROXIMATION TO 2PI.
```

LINEAR FUNCTION

Functional Form

$$f(x,a,b) = a + bx$$

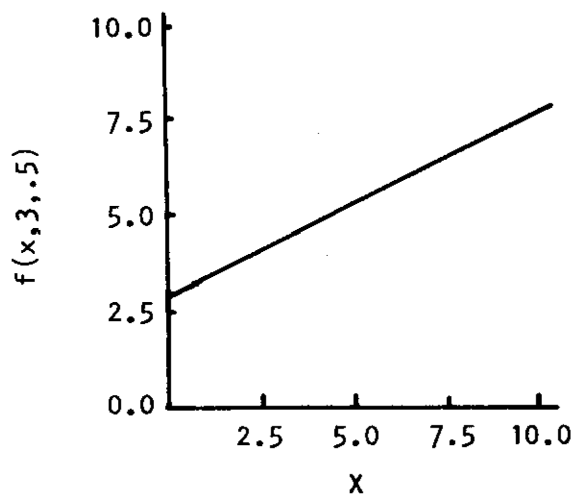
Derivative

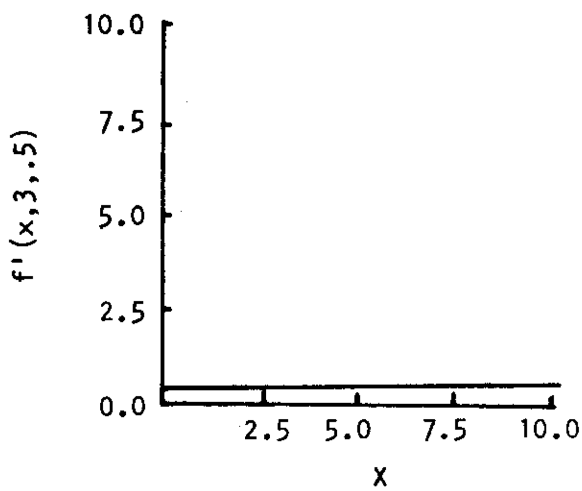
$$f'(x,a,b) = b$$

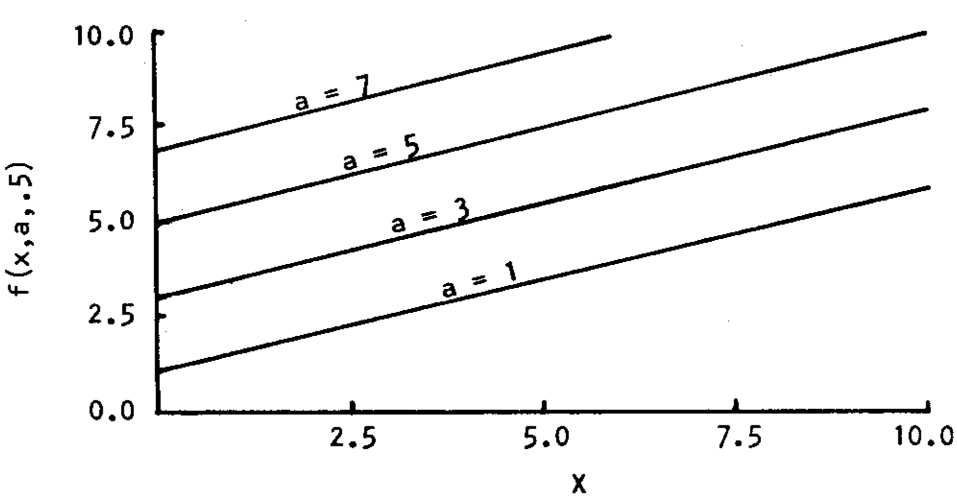
Parameter Definitions

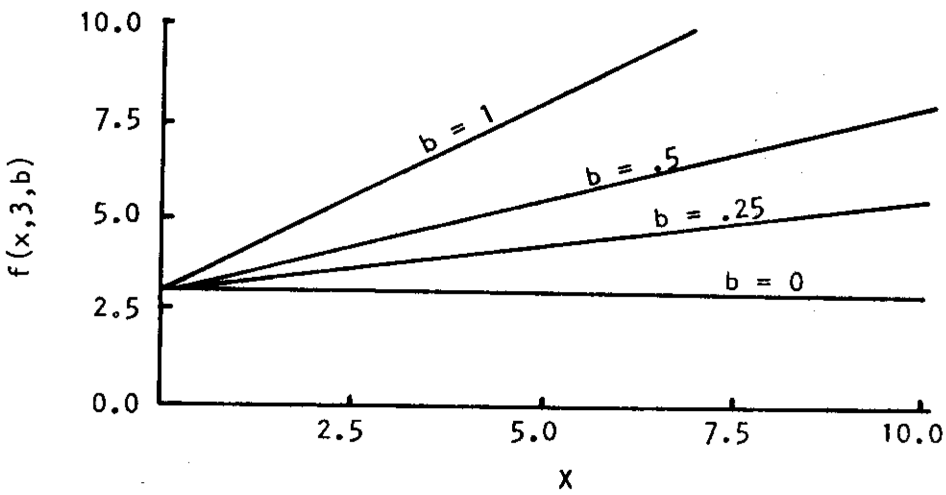
a = the value of $f(x)$ when $x = 0.0$
b = the slope of the curve

Graphs









FORTRAN CODE

```
      FUNCTION ZFL(X,A,B)
      ZFL=A+B*X
      RETURN
      END
```

ANOTHER USEFUL LINEAR FUNCTION AND ITS CODE IS

```
      FUNCTION CLI(X,A,B,X1,X2)
      IF (X1.EQ.X2) GO TO 1
      CLI=(A-B)/(X1-X2)*(X-X2)+B
      RETURN
    1 CLI=(A+B)/2.
      RETURN
      END
C...  THIS FUNCTION SUBROUTINE COMPUTES A
C...  LINEAR INTERPOLATION (EXTRAPOLATION) THROUGH
C...  THE POINTS (X1,A) AND (X2,B) UNLESS X1=X2.
C...  IN THIS LATTER CASE, (A+B)/2 IS RETURNED.
```

PIECEWISE LINEAR FUNCTION

Functional Form

$$f(x,a,b,x_1,x_2) = \begin{cases} a & \text{if } x < x_1 \\ \left(\dfrac{(b-a)}{(x_2 - x_1)}\right) \cdot (x - x_1) + a & \text{if } x_2 \geq x \geq x_1 \\ b & \text{if } x > x_2 \end{cases}$$

Derivative

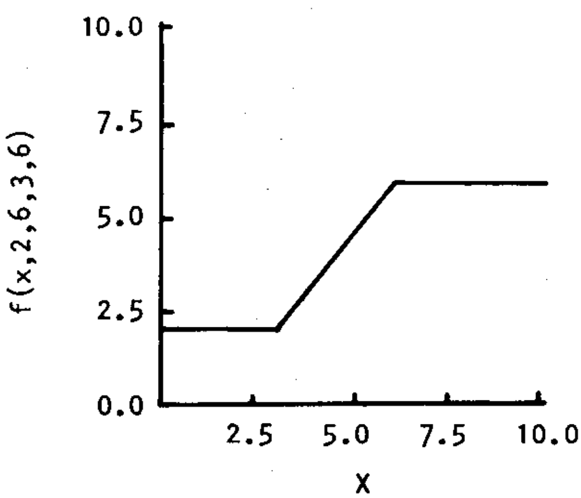
$$f'(x,a,b,x_1,x_2) = \begin{cases} 0 & \text{if } x < x_1 \\ \dfrac{b-a}{x_2 - x_1} & \text{if } x_1 < x < x_2 \\ 0 & \text{if } x > x_2 \end{cases}$$

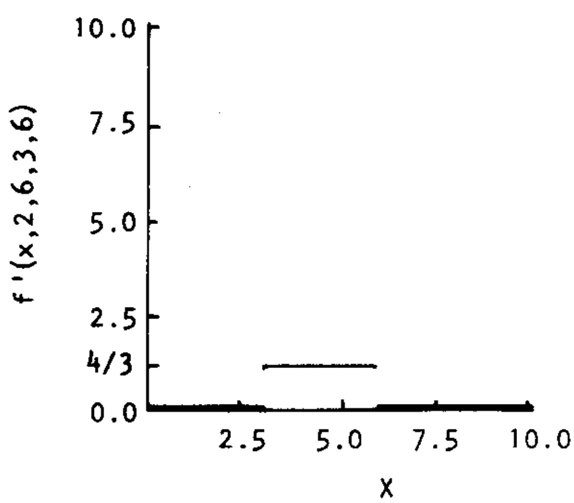
(The derivative is not defined at x_1 and x_2.)

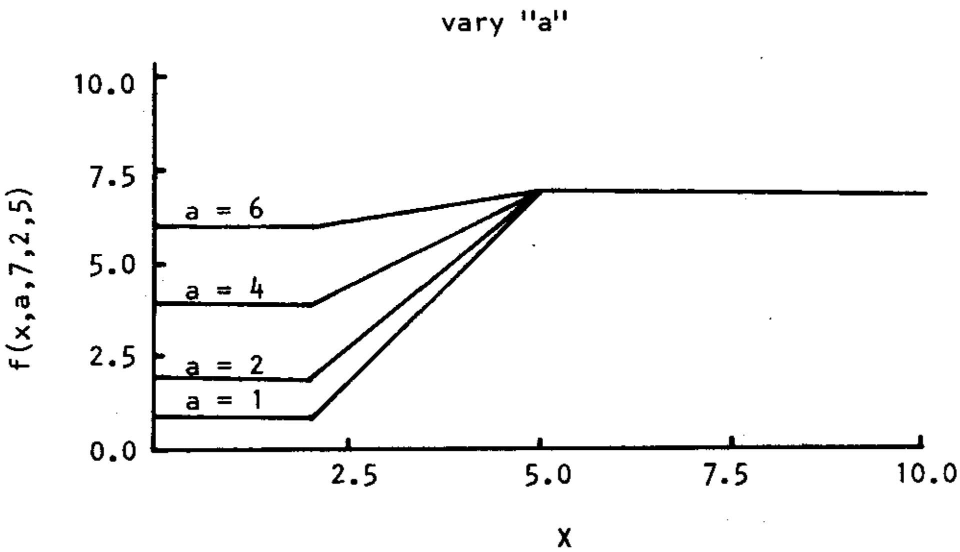
Parameter Definitions

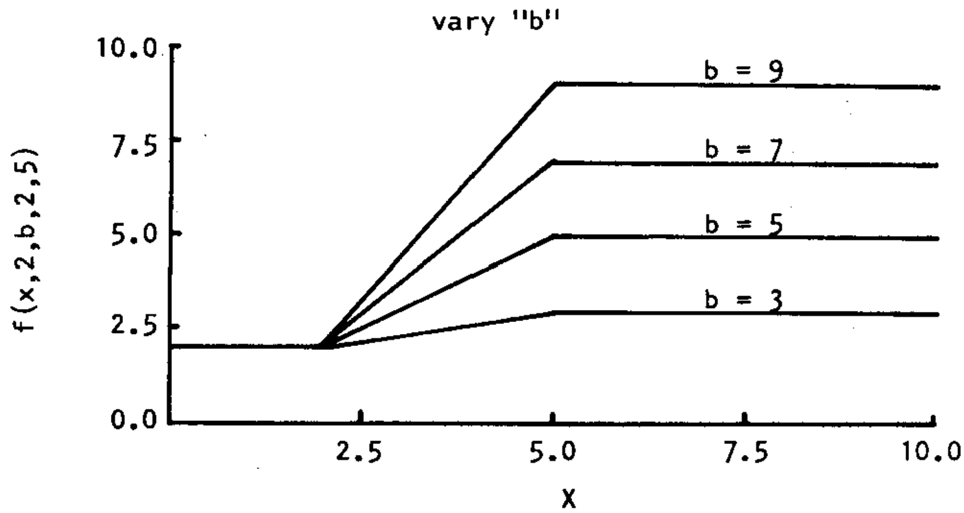
a = the value of $f(x)$ to the left of the step
b = the value of $f(x)$ to the right of the step
x_1 = the value of "x" at the left end of the step
x_2 = the value of "x" at the right end of the step

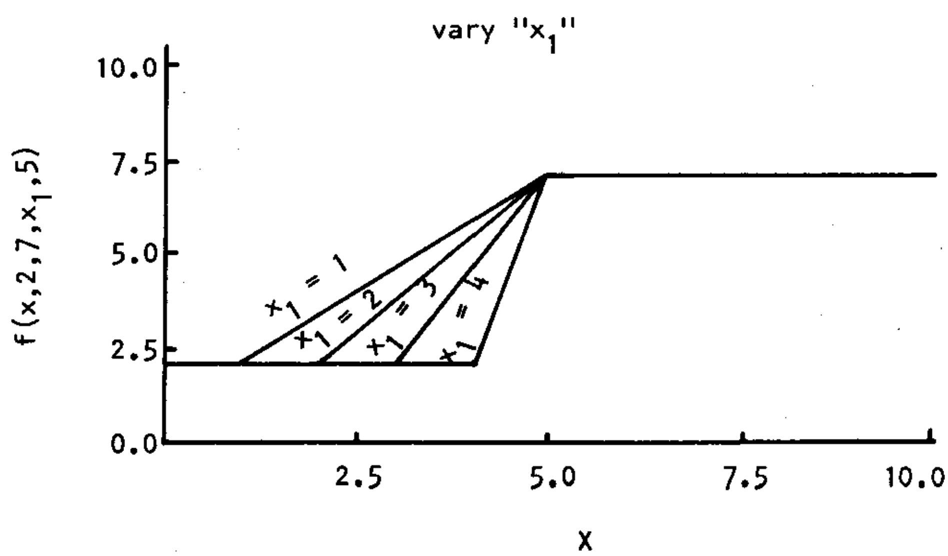
Graphs

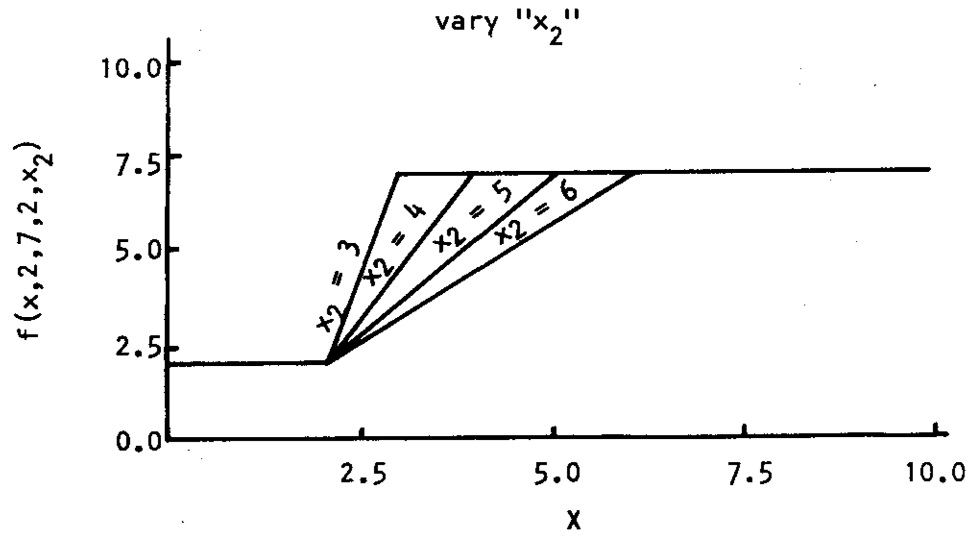





vary "a"
10.0
7.5
5.0
2.5
0.0
a = 6
a = 4
a = 2
a = 1
f(x,a,7,2,5)
2.5
5.0
7.5
10.0
X


vary "b"
10.0
7.5
5.0
2.5
0.0
b = 9
b = 7
b = 5
b = 3
f(x,2,b,2,5)
2.5
5.0
7.5
10.0
X


vary "x1"
10.0
7.5
5.0
2.5
0.0
x1 = 1
x1 = 2
x1 = 3
x1 = 4
f(x,2,7,x1,5)
2.5
5.0
7.5
10.0
X



FORTRAN CODE

```
      FUNCTION TRILF(X,A,B,X1,X2)
      IF (X.LE.X1) GO TO 1
      IF (X.GE.X2) GO TO 2
      TRILF=(B-A)/(X2-X1)*(X-X1)+A
      RETURN
    1 TRILF=A
      RETURN
    2 TRILF=B
      RETURN
      END
C...  IF X1=X2, THE VALUE RETURNED IS A IF X≤X1
C...  AND B OTHERWISE.
```

ANOTHER PIECEWISE LINEAR FUNCTION OF CONSIDERABLE UTILITY IS THE TABLE FUNCTIO
OF FORRESTER WHICH DOES A LINEAR INTERPOLATION BETWEEN TUBULAR ENTRIES.

FORRESTER, JAY W., WORLD DYNAMICS AND PRINCIPLES OF SYSTEMS

```
      FUNCTION TABLE(TBL,X,X1,X2,DX)
      DIMENSION TBL(1)
      IF (DX.EQ.0.) GO TO 1
      IF (X.LE.X1) GO TO 1
      IF (X.GE.X2) GO TO 2
      K=(X-X1)/DX
      I=K+1
      J=I+1
      TABLE=(TBL(J)-TBL(I))/DX*(X-X1-K*DX)+TBL(I)
      RETURN
    1 TABLE=TBL(1)
      RETURN
    2 K=(X2-X1)/DX+1.
      TABLE=TBL(K)
      RETURN
      END
C...  THE TABLE TBL IS LINEARLY INTERPOLATED BY THIS
C...  FUNCTION OVER THE RANGE FROM X1 TO X2 IN (EQUAL)
C...  STEPS OF DX. FOR X.LE.X1, TABLE=TBL(1) AND
C...  FOR X.GE.X2, TABLE=TBL((X2-X1)/DX+1.). IF DX=0,
C...  TBL(1) IS RETURNED.
```

RATIONAL FUNCTION I

Functional Form

$$f(x,a,b,c,d) = \frac{a}{(1 + b/x)} + \frac{c}{(1 + d/x)}$$

Derivative

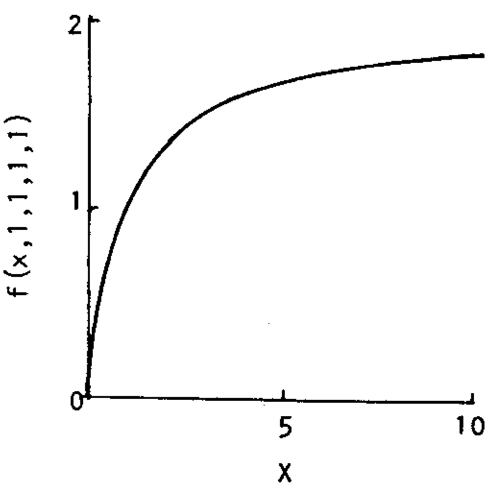
$$f'(x,a,b,c,d) = \frac{ab}{(1 + b/x)^2 x^2} + \frac{cd}{(1 + d/x)^2 x^2}$$

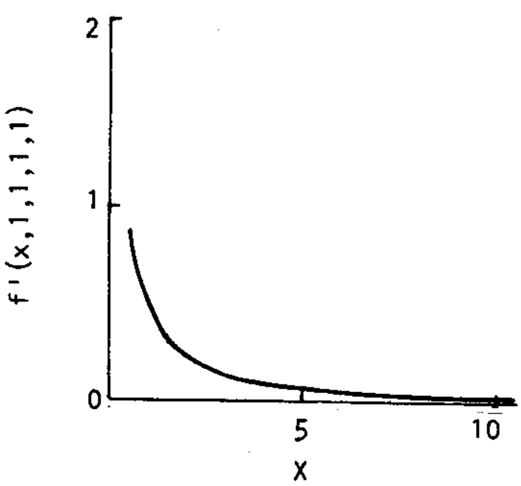
Parameter Definitions

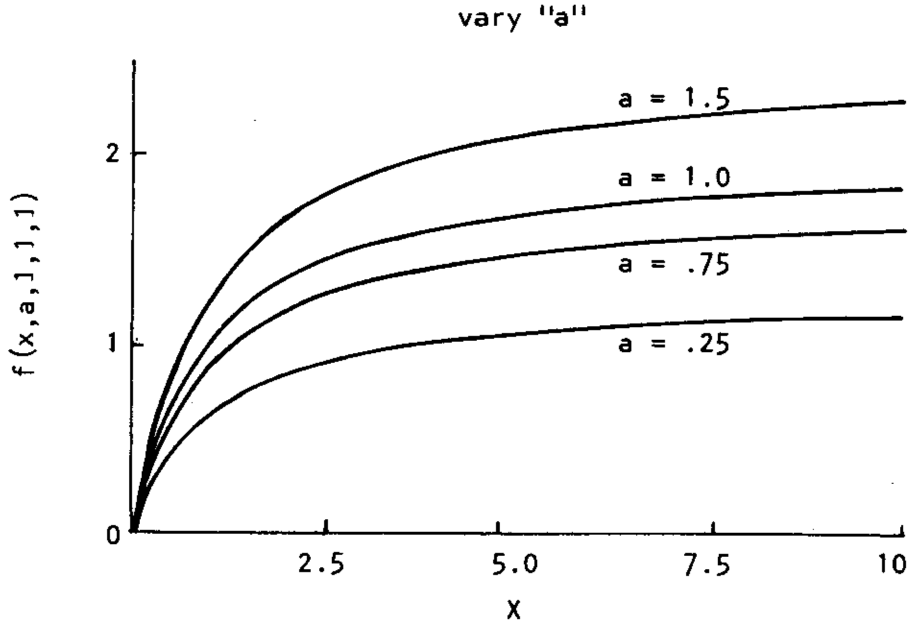
a and c = the parameters which control the maximum value of the function [a + c = the maximum value of f(x)]

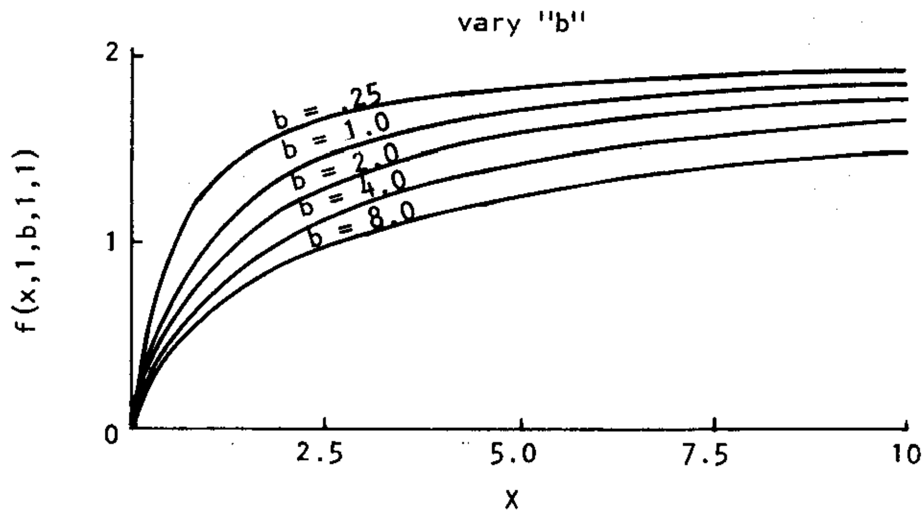
b and d = the parameters which control the rate that the function approaches its maximum value

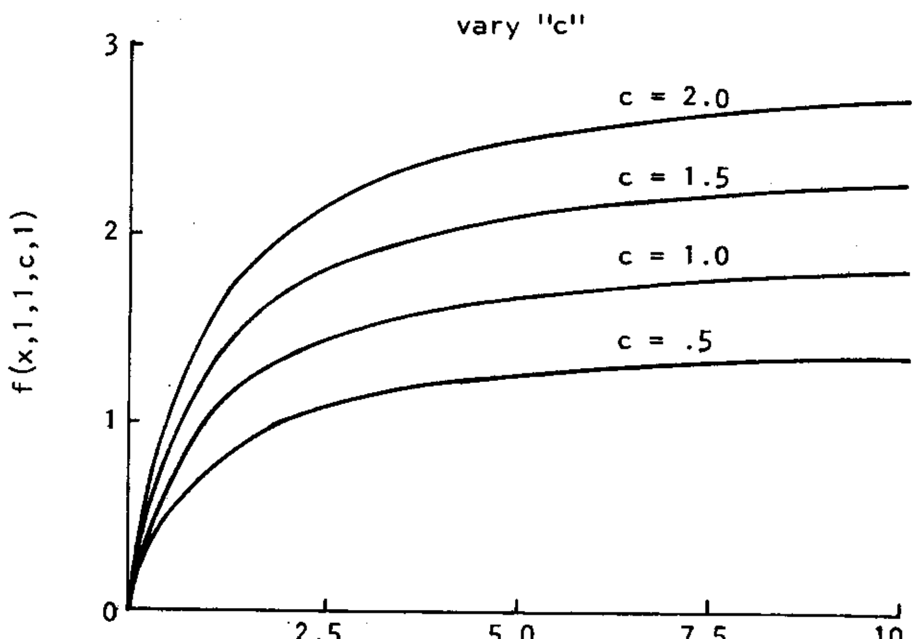
Graphs

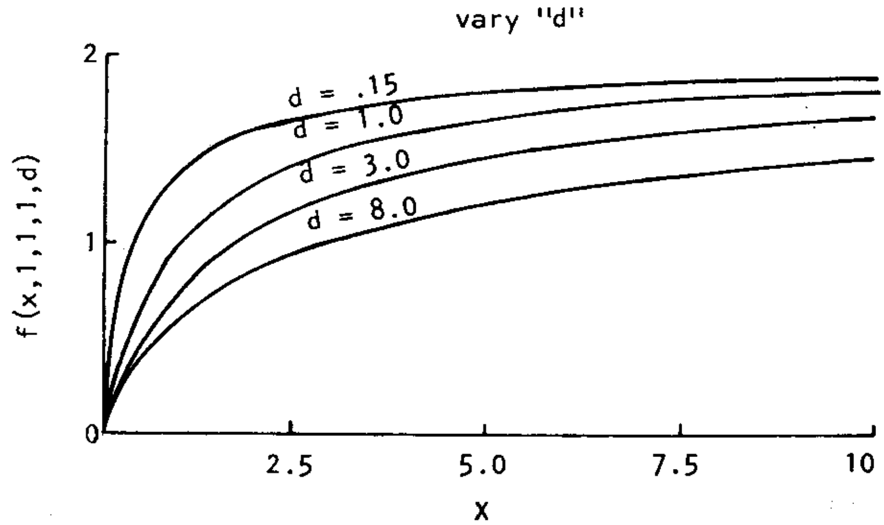





vary "a"
a = 1.5
a = 1.0
a = .75
a = .25
f(x,a,1,1,1)
0
1
2
2.5
5.0
7.5
10
X


vary "b"
b = .25
b = 1.0
b = 2.0
b = 4.0
b = 8.0
f(x,1,b,1,1)
0
1
2
2.5
5.0
7.5
10
X


vary "c"
c = 2.0
c = 1.5
c = 1.0
c = .5
f(x,1,1,c,1)
0
1
2
3
2.5
5.0
7.5
10



FORTRAN CODE

```
      FUNCTION RFI(X,A,B,C,D)
      IF (X.EQ.0.) GO TO 1
      TMP=1.+B/X
      IF (TMP.EQ.0.) GO TO 1
      RFI=A/TMP
      TMP=1.+D/X
      IF (TMP.EQ.0.) GO TO 1
      RFI=RFI+C/TMP
      RETURN
    1 RFI=0.
      RETURN
      END
C...  IF X, 1+B/X OR 1+D/X IS ZERO, THE ROUTINE
C...  RETURNS THE VALUE 0.
```

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Sesták, Z., J. Catský, and P. G. Jarvis [Ed.] 1971. Plant photosynthetic production. Manual of methods. Dr. W. Junk, N. V. Pub., The Hague. p. 392-411.