DISSERTATION

MULTIPLICITIES AND EQUIVARIANT COHOMOLOGY

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY REBECCA E. LYNN ENTITLED MULTIPLICITIES AND EQUIVARIANT COHOMOLOGY BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

MULTIPLICITIES AND EQUIVARIANT COHOMOLOGY

The aim of this paper is to address the following problem: how to relate the algebraic definitions and computations of multiplicity from commutative algebra to computations done in the cohomology theory of group actions on manifolds. Specifically, this paper is concerned with applications of commutative algebra to the study of cohomology rings arising from group actions on manifolds, in the way that Quillen initiated in [Qui71a, Qui71b]. This paper synthesizes two distinct areas of pure mathematics (commutative algebra and cohomology theory) and two ways of computing multiplicities in order to link them, a la Quillen [Qui71a] and Maiorana [Mai76]. In order to accomplish this task, a discussion of commutative algebra will be followed by a discussion of cohomology theory. A link between commutative algebra and cohomology theory will be presented, followed by its application to a significant example.

In commutative algebra, we discuss graded rings, Poincaré Series, dimension, and multiplicities. Whereas the theory for multiplicities has been developed for local rings, we give an exposition of the theory for graded rings. Several definitions for dimension will be presented, and it will be proven that all of these distinct definitions are equal. The basic properties of multiplicities will be introduced, and a brief discussion of a classical multiplicity in commutative algebra, the Samuel multiplicity, will be presented. Then, Maiorana's C-multiplicity will be defined, and a relationship between all of these multiplicities will be observed.

In cohomology theory, we address smooth actions of finite groups on manifolds. As a part of this study in cohomology theory, we will consider group actions on topological spaces and the Borel construction (equivariant cohomology), completing this part of the paper with a discussion of smooth (or differentiable) actions, setting some notation necessary for our discussion of Maiorana's results in [Mai76], which inspires some of our main theorems, but which we do not actually rely on in this dissertation.

Following the treatments of commutative algebra and cohomology theory, we present Quillen's main result of [Qui71a] without proof, linking these two distinct areas of pure mathematics. Quillen's work results in a formula for finding the multiplicity of the equivariant cohomology of a compact G-manifold with G a compact Lie group. We apply these results to the compact G-manifold U/S, where G (a compact Lie group) is embedded in a unitary group $U = U(n)$ and $S = S(n)$ is the "diagonal" p-torus of rank n in $U(n)$, resulting in a nice topological formula for computing multiplicities. Finally, we end the paper with a proposal for future research.

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Chapter 1

TOPICS FROM COMMUTATIVE ALGEBRA

In this chapter, we develop the theory for multiplicities in the graded case. The material in this chapter follows the exposition of [Eis95] and [Ser00], who develop the theory for local rings. Many of the proofs follow directly from Serre's exposition with some minor notational adjustments, in which case the proof is not provided here. However, when there are more significant differences, we sometimes provide the proof. In addition, we define Maiorana's C-multiplicity and relate the C-multiplicity to more classical multiplicities. It is assumed that the reader has a firm understanding of basic concepts in module theory over commutative rings. Additional sources include [Gro61], [Tha07], [DF99], [Eve91], and [AM69].

1.1 Graded Rings and Modules

We will claim that the cohomology ring of a group G with coefficients in Λ, $H^*(G,\Lambda) = \bigoplus_{n\geq 0} H^n(G,\Lambda)$, forms a graded ring under particular conditions. For the purposes of this paper, we present only the basics of graded rings and modules in this section.

1.1.1 Basic Definitions for Any Graded Ring

Definition 1.1.1. *Let S be a ring, not necessarily commutative, with identity. S is* graded *if there exists a family of abelian subgroups S_n, $n \in \mathbb{Z}$, of S such that $S = \bigoplus_{n\in\mathbb{Z}} S_n$ and $S_n \cdot S_m \subseteq S_{n+m}$ for all $n, m \in \mathbb{Z}$. A graded ring S is* nonnegatively graded *if $S_n = 0$ for all $n < 0$. A non-zero element $x \in S_n$ is a* homogeneous element *of S of degree n. We define zero to be homogeneous of every degree.*

If $S = \bigoplus_{n \in \mathbb{Z}} S_n$ is a graded ring, then S_0 is a subring of S and S_n is an S_0-module for all n.

Definition 1.1.2. *Let $S = \bigoplus_{n \in \mathbb{Z}} S_n$ be a graded ring. If $x \in S$, $x =! \sum x^{(i)}$ where $x^{(i)} \in S_i$ and $x^{(j)} = 0$ for all but finitely many j. $x^{(i)}$ is the* homogeneous compontent *of x of degree i.*

Definition 1.1.3. *An ideal of a graded ring S is a* homogeneous (or graded) ideal *if and only if $\mathcal{I} = \bigoplus_{n \in \mathbb{Z}} \mathcal{I}_n$ where $\mathcal{I}_n = \mathcal{I} \bigcap S_n$.*

The terms "homogeneous" and "graded" will be used interchangeably when applied to ideals in this paper.

Proposition 1.1.4. *An ideal $\mathcal{I}$ is homogeneous if and only if for all $x \in \mathcal{I}$, the homogeneous components of x are all in $\mathcal{I}$. That is, $\mathcal{I}$ is homogeneous if and only if for all $x \in \mathcal{I}$, x can be written uniquely as $\sum x^{(i)}$ where $x^{(i)} \in S_i$, and $x^{(i)} \in \mathcal{I}$.*

For an example of a nonhomogeneous ideal, consider the polynomial ring $k[x]$ where x has degree 1. Then

$$k \oplus kx \oplus kx^2 \oplus kx^3 \oplus \cdots$$

is a graded ring given this grading. Let $\mathcal{I} = (x + 1)$. Then $x + 1 \in \mathcal{I}$. Consider $x + 1$ written with the zeroth homogeneous component and the first homogeneous component: $(x + 1)^{(0)} + (x + 1)^{(1)}$. Since $(x + 1)^{(0)} = 1$ is not in $\mathcal{I}$, then $\mathcal{I}$ is not a homogeneous ideal. (Note that $(x + 1)^{(1)} = x$ is not in $\mathcal{I}$ either.) On the other hand, (x) is a homogeneous ideal in the graded ring $k \oplus kx \oplus kx^2 \oplus kx^3 \oplus \cdots$.

Proposition 1.1.5. *If $\mathcal{I}$ is a homogeneous ideal of a graded ring S, then $S/\mathcal{I}$ is a graded ring, using the definition $(S/\mathcal{I})_n \doteq S_n/(\mathcal{I} \cap S_n) \doteq S_n/\mathcal{I}_n$.*

Note that any commutative ring A may be regarded as a graded ring "concentrated in degree zero;" i.e., $A_0 = A$ and $A_i = 0$ for all $i \neq 0$.

1.1.2 Nonnegatively Graded Rings and Graded Modules

Definition 1.1.6. *Suppose $R = R_0 \oplus R_1 \oplus \cdots$ is a nonnegatively graded ring. Then a* graded module *over R is a module M with a decomposition*

$$M = \bigoplus_{i=-\infty}^{\infty} M_i \text{ as abelian groups}$$

such that $R_i M_j \subset M_{i+j}$ for all i, j.

We can define a graded vector space over the field k (regarded as a graded ring concentrated in degree zero) similarly.

One way of forming graded rings and modules, given an ideal $\mathcal{I}$ in a ring A is as follows.

Definition 1.1.7. *Let $\mathcal{I}$ be an ideal of a ring A, not necessarily graded. We define the* associated graded ring *of A with respect to $\mathcal{I}$, denoted $\mathfrak{g}_{\mathcal{I}}A$, to be the graded ring*

$$\mathfrak{g}_{\mathcal{I}}A := A/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \cdots \text{ where } (\mathfrak{g}_{\mathcal{I}}A)_n \doteq \mathcal{I}^n/\mathcal{I}^{n+1}.$$

For $a \in \mathcal{I}^m/\mathcal{I}^{m+1}$ and $b \in \mathcal{I}^n/\mathcal{I}^{n+1}$ with representatives a' and b' of a in $\mathcal{I}^m$ and b in $\mathcal{I}^n$, respectively, we define $ab \in \mathcal{I}^{m+n}/\mathcal{I}^{m+n+1}$ to be the image of $a'b'$. Notice that this is well-defined, modulo $\mathcal{I}^{m+n+1}$. This defines multiplication in $\mathfrak{g}_{\mathcal{I}}A$.

Similarly, if M is a A-module, not necessarily graded, we can define the *associated graded module* as follows.

Definition 1.1.8. *If M is a A-module and $\mathcal{I}$ is an ideal in A, not necessarily graded, the* associated graded module *of M with respect to $\mathcal{I}$, denoted $\mathfrak{g}_{\mathcal{I}}M$, is the graded module*

$$\mathfrak{g}_{\mathcal{I}}M := M/\mathcal{I}M \oplus \mathcal{I}M/\mathcal{I}^2M \oplus \cdots .$$

Note that $\mathfrak{g}_{\mathcal{I}}M$ is a graded $\mathfrak{g}_{\mathcal{I}}A$-module.

If R is a nonnegatively graded ring and M is a graded R-module, we can alter M by "shifting" (or "twisting") its grading d steps.

Definition 1.1.9. *Define $M(d)$ to be the* shifted graded module *(or, dth twist of M) such that*

$$M(d)_e = M_{d+e}.$$

Definition 1.1.10. *For R a nonnegatively graded ring, an R-module homomorphism of* degree zero *does not change degree; that is, if M and N are graded R-modules, then $\phi : M \to N$ is an R-module homomorphism of degree zero if and only if $\phi : M_i \to N_i$ for all i.*

1.2 Results from Grothendieck

We summarize some useful definitions and results from Grothendieck [Gro61] in this section. Assume that R is a nonnegatively graded ring, but R_0 is not necessarily a field. A graded R-module M need not be nonnegativley graded.

Definition 1.2.1. *Define the* "superfluous" *or* "irrelevant" ideal *of R*

$$R_+ \doteq \bigoplus_{n\geq 1} R_n.$$

Note that if R_0 is a field, then R_+ is the unique graded (homogeneous) maximal ideal in R.

Definition 1.2.2. *If $d > 0$, define*

$$R_{(d)} \doteq \bigoplus_{n\geq 0} R_{nd}.$$

This is also a graded ring.

Definition 1.2.3. *If $d > 0$ and i is an integer such that $0 \leq i \leq d-1$, then define*

$$M_{(d,i)} \doteq \bigoplus_{n\in\mathbb{Z}} M_{nd+i}.$$

Note that $M_{(d,i)}$ is an $R_{(d)}$-module.

Lemma 1.2.4. *[Gro61, Lem. 2.1.3, Cor. 2.1.4, Cor. 2.1.5]. Let R be a nonnegatively graded ring.*

a. *$E \subseteq R_+$ (consisting of homogeneous elements) generates R_+ as an R-module if and only if E generates R as an R_0-algebra.*

b. *R_+ is a finitely generated ideal in R if and only if R is a finitely generated R_0-algebra.*

c. *R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.*

Proposition 1.2.5. *[Gro61, Lem. 2.1.6]. Suppose that R is a nonnegatively graded ring and that M is a finitely generated graded R-module.*

a. *For all $n \in \mathbb{Z}$, M_n is a finitely generated R_0-module, and there exists an $n_0 \in \mathbb{Z}$ such that $M_i = 0$ for $i \leq n_0$.*

b. *There exists an $n_1 \in \mathbb{Z}$ and a positive integer i such that for every $n \geq n_1$, $M_{n+i} = R_i M_n$.*

c. *For every (d, i), where d is a positive integer and i is an integer, $0 \leq i \leq d-1$, $M_{(d,i)}$ is a finitely generated $R_{(d)}$-module.*

d. *For every positive integer d, $R_{(d)}$ is a finitely generated R_0-module.*

e. *There exists a positive integer i such that $R_{mi} = (R_i)^m$, for every $m \geq 0$.*

f. *For every positive integer n, there exists a nonnegative integer m_0 such that $R_m \subseteq R_+^n$, for $m \geq m_0$.*

Corollary 1.2.6. *[Gro61, Cor. 2.1.7]. If R is Noetherian, then so is $R_{(d)}$ for all $d > 0$.*

1.3 The Category $\mathcal{C}(R)$

From this point forward, R will be a nonnegatively graded commutative ring, with $R_0 = k$, a fixed field. We begin by defining the category $\mathcal{C}(R)$ and the direct sum for modules in this category.

Definition 1.3.1. *For R a nonnegatively graded commutative ring with $R_0 = k$, a fixed field, the* category $\mathcal{C}(R)$ *has as its objects the graded R-modules that are finitely generated as graded R-modules, with all nonzero homogeneous elements having nonnegative degree. The morphisms of $\mathcal{C}(R)$ are the R-module homomorphisms of degree zero.*

Definition 1.3.2. *Let $M, N \in \mathcal{C}(R)$. Then $M \oplus N \in \mathcal{C}(R)$ is the graded R-module defined by*

$$(M \oplus N)_i \doteq M_i \oplus N_i.$$

We will not have to define the arbitrary graded tensor product $M \otimes_R N$, which is defined by Grothendieck [Gro61], but we do define $M \otimes_k N$.

Definition 1.3.3. *Let $M, N \in \mathcal{C}(R)$ and let k be the field R_0. Then $M \otimes_k N$ is the graded vector space over k defined by*

$$(M \otimes_k N)_i \doteq \bigoplus_{0 \leq j \leq i} M_j \otimes_k N_{i-j}.$$

1.3.1 $\mathcal{C}(R)$ when R is Noetherian

We will assume that R is finitely generated as a k-algebra (thereby implying that R is Noetherian) by a set of homogeneous elements of positive degree. Note that from the usual proofs in the non-graded case, for all $M \in \mathcal{C}(R)$, every submodule and quotient of M is in $\mathcal{C}(R)$ as well.

The following three propositions are proven in this paper and will be utilized later in the paper as we develop our theory for graded rings.

Let R be a nonnegatively graded ring, such that $R_0 = k$, a field. The superfluous ideal

$$\mathfrak{m} = R_+ = \bigoplus_{n \geq 1} R_n$$

is the unique graded homogeneous maximal ideal in R. Consequently, for $R = \oplus R_n$ with $R_0 = k$ and $R_n = 0$ for $n < 0$, we see that $R/\mathfrak{m} = k$ as graded rings concentrated in degree zero. Note also that k is a graded submodule of R.

Proposition 1.3.4. *Suppose that $M \in \mathcal{C}(R)$. For every $n \geq 0$, $M/\mathfrak{m}^n M$ is a finite dimensional graded vector space over k.*

Proof. By the Noetherian hypotheses, $\mathfrak{m}^i M$ is a finitely generated graded R-module for all i, so that for all $j \geq i$, $\mathfrak{m}^i M/\mathfrak{m}^j M$ is a finitely generated graded R-module. Thus, $\mathfrak{m}^i M/\mathfrak{m}^{i+1} M$ is a finitely generated $R/\mathfrak{m}$-module, so it is a finite dimensional vector space over k for all i. Notice that $M/\mathfrak{m}^n M$ has a finite filtration

$$0 \subseteq \mathfrak{m}^{n-1} M/\mathfrak{m}^n M \subseteq \cdots \subseteq \mathfrak{m} M/\mathfrak{m}^n M \subseteq M/\mathfrak{m}^n M$$

such that successive quotients are finite dimensional graded vector spaces over k. Therefore, $M/\mathfrak{m}^n M$ is a finite dimensional graded vector space over k. $\square$

For the following proposition, we need some notation. If $\alpha = (\alpha_1, \ldots, \alpha_t)$ is a *multi-index* of nonnegative integers, we say the degree of α is $\alpha_1 + \cdots + \alpha_t$. If $w_1, \ldots, w_t \in \mathfrak{m}$ are homogeneous elements, the graded subring of R generated by k and $w_1, \ldots, w_t$ will be denoted by $k \langle w_1, \ldots, w_t \rangle$. The superfluous ideal in $k \langle w_1, \ldots, w_t \rangle$ will be denoted by $\mathfrak{m}(\mathbf{w})$, and the monomial w^α denotes $w_1^{\alpha_1} \cdots w_t^{\alpha_t}$.

Proposition 1.3.5. *Suppose that $\mathcal{I} \subseteq R$ is an ideal in R generated as an R-module by homogeneous elements $w_1, \ldots, w_t \in \mathfrak{m}$. Then, if $M \in \mathcal{C}(R)$,*

a. For every $n \geq 0$, $\mathcal{I}^n M = \mathfrak{m}(\mathbf{w})^n M$.

b. $M/\mathcal{I} M \in \mathcal{C}(k)$ if and only if $M/\mathfrak{m}(\mathbf{w}) M \in \mathcal{C}(k)$.

Proof. For part (a), since $\mathfrak{m}(\mathbf{w}) \subseteq \mathcal{I}$, then $\mathfrak{m}(\mathbf{w})^n M \subseteq \mathcal{I}^n M$ for all $n \geq 0$. On the other hand, suppose that $x \in \mathcal{I}^n M$. Then there exist $r_\alpha \in R$ and $x_\alpha \in M$ such that

$$\begin{aligned} x &= \sum_{\deg(\alpha)=n} r_\alpha w^\alpha x_\alpha \\ &= \sum_{\deg(\alpha)=n} w^\alpha (r_\alpha x_\alpha), \text{ where } r_\alpha x_\alpha \in M \text{ for all } \alpha. \end{aligned}$$

Thus $x \in \mathfrak{m}(\mathbf{w})^n M$.

Part (b) follows naturally from part (a). □

Proposition 1.3.6. *Suppose that $\mathcal{I} \subseteq R$ is an ideal in R generated by homogeneous elements $w_1, \ldots, w_t \in \mathfrak{m}$. Then, if $M \in \mathcal{C}(R)$, the following are equivalent:*

a. $M \in \mathcal{C}(k\langle w_1, \ldots, w_t \rangle)$.

b. $M/\mathcal{I}M \in \mathcal{C}(k)$.

c. $M/\mathcal{I}^n M \in \mathcal{C}(k)$, for every $n \geq 0$.

d. There exists an $n_0 > 0$ such that $\mathfrak{m}^{n_0} M \subseteq \mathcal{I}M$.

e. There exists an $n_0 > 0$ such that $\mathfrak{m}^{n_0 j} M \subseteq \mathcal{I}^j M$ for every $j \geq 1$.

Proof. For (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c), apply Proposition 1.3.4 to $k\langle w_1, \ldots, w_t \rangle$ and $\mathfrak{m}(\mathbf{w})$. Then, $M/\mathfrak{m}(\mathbf{w})^n M$ is a finite dimensional graded vector space over k. Applying Proposition 1.3.5, we get that $M/\mathcal{I}M \in \mathcal{C}(k)$ and $M/\mathcal{I}^n M \in \mathcal{C}(k)$ for all $n \geq 0$.

For (b) $\Rightarrow$ (a), suppose that $M/\mathcal{I}M \in \mathcal{C}(k)$. Then $M/\mathcal{I}M$ is a finite dimensional vector space over k and there exist $x_1, \ldots, x_s$, homogeneous elements of M such that for $\tilde{x}_i \doteq x_i + \mathcal{I}M$, $\tilde{x}_1, \ldots, \tilde{x}_s$ is a basis for $M/\mathcal{I}M$. Suppose there exists a nonzero homogeneous element $m_0 \in M$ such that m_0 has least positive degree with respect to not being in the span of $x_1, \ldots, x_s$ as a $k\langle w_1, \ldots, w_t \rangle$-module. Then

$$m_0 + \mathcal{I}M = \sum_{i=1}^{s} a_i \tilde{x}_i,\ a_i \in k \text{ for all } i$$

$$\Rightarrow m_0 - \sum_{i=1}^{s} a_i x_i \in \mathcal{I}M.$$

So there exist elements $z_1, \ldots, z_t \in M$ such that

$$m_0 - \sum_{i=1}^{s} a_i x_i = \sum_{j=1}^{t} w_j z_j.$$

The degree of $w_j \in R$ is positive for all j, so the degree of z_j is strictly less than the degree of m_0 for all j. Hence, z_j is in the $k\langle w_1, \ldots, w_t\rangle$-span of $x_1, \ldots, x_s$ and m_0 is in the $k\langle w_1 \ldots, w_t\rangle$-span of $x_1, \ldots, x_s$ also.

For (d) $\Rightarrow$ (b), there exists $n_0 > 0$ such that $\mathfrak{m}^{n_0}M \subseteq \mathcal{I}M$. Hence, $M/\mathcal{I}M \subseteq M/\mathfrak{m}^{n_0}M$ and $M/\mathcal{I}M$ is finite dimensional over k by Proposition 1.3.4.

For (b) $\Rightarrow$ (d), let $M/\mathcal{I}M \in \mathcal{C}(k)$. Suppose that $x_1, \ldots, x_s$ are homogeneous elements of M such that, for $\tilde{x}_i = x_i + \mathcal{I}M$, $\tilde{x}_1, \ldots \tilde{x}_s$ form a basis of $M/\mathcal{I}M$ as a k-vector space. Let $n_0 - 1 = \max\{\deg(x_i) \mid i = 1, \ldots, s\}$. Then, $(M/\mathcal{I}M)_i = 0$ for $i \geq n_0$. Hence, if $z \in \mathfrak{m}^{n_0}M$ is a homogeneous element, then $\deg(z) \geq n_0$, so $z \in \mathcal{I}M$ and $\mathfrak{m}^{n_0}M \subseteq \mathcal{I}M$.

For (d) $\Rightarrow$ (e), suppose that $n_0 > 0$ such that $\mathfrak{m}^{n_0}M \subseteq \mathcal{I}M$. Fix $j \geq 1$. Assume that $\mathfrak{m}^{n_0 j}M \subseteq \mathcal{I}^j M$. Then

$$\begin{aligned}
\mathfrak{m}^{n_0(j+1)}M &= \mathfrak{m}^{n_0 j}\left(\mathfrak{m}^{n_0}M\right) \\
&\subseteq \mathfrak{m}^{n_0 j}\mathcal{I}M \\
&= \mathcal{I}\mathfrak{m}^{n_0 j}M \\
&\subseteq \mathcal{I}\mathcal{I}^j M \\
&= \mathcal{I}^{j+1}M.
\end{aligned}$$

□

1.3.2 Filtrations of Graded Rings

Definition 1.3.7. *A* filtration of a graded ring R *is a sequence of homogeneous ideals*

$$\cdots \subseteq \mathcal{F}^2(R) \subseteq \mathcal{F}(R) \subseteq \mathcal{F}^0(R) = R$$

satisfying $\mathcal{F}^i(R)\mathcal{F}^j(R) \subseteq \mathcal{F}^{i+j}(R)$ *for all* i, j. *We say that* R *is a* filtered ring *if a filtration exists. Similarly, a* filtration of a graded R-module M *over the filtered* *ring R (where R has a specific filtration $\mathcal{F}^\bullet(R)$ given) is a sequence of graded submodules*

$$\cdots \subseteq \mathcal{F}^2(M) \subseteq \mathcal{F}(M) \subseteq \mathcal{F}^0(M) = M$$

satisfying $\mathcal{F}^i(R)\mathcal{F}^j(M) \subseteq \mathcal{F}^{i+j}(M)$ for all i, j, and we say that M is a filtered module if a filtration exists.

Filtrations are most often used in the case where the $\mathcal{F}^j(R)$ are the jth powers of a single graded ideal, $\mathcal{I}$. This is called the *$\mathcal{I}$-adic filtration* of R. We can generalize to the case of modules where the $\mathcal{I}$-adic filtration of a module M is

$$\cdots \subseteq \mathcal{I}^2 M \subseteq \mathcal{I}^1 M \subseteq \mathcal{I}^0 M = M.$$

Another filtration unique to graded rings is the "standard" filtration defined by $R^{(i)} \doteq \bigoplus_{j \geq i} R_j$. The standard filtration over the standard filtered ring R of a graded module $M = \bigoplus_{i=-\infty}^{\infty} M_i$ is given by

$$\cdots \subseteq M^{(i+1)} \subseteq M^{(i)} \subseteq \cdots$$

where $M^{(i)} \doteq \bigoplus_{j \geq i} M_j$.

Another example of a filtration uses the functor Hom. Let the graded ring R have a filtration, M a graded R-module, and N a graded R-module with filtration. Then, the submodules $\mathrm{Hom}_R(M, \mathcal{F}^n(N))$ of $\mathrm{Hom}_R(M, N)$ define a filtration on $\mathrm{Hom}_R(M, N)$.

Theorem 1.3.8. *(Artin-Rees). [Ser00] Suppose that $M \in \mathcal{C}(R)$, L is a graded submodule of M, and $\mathcal{I}$ is a graded proper ideal in R. Then, there exists an $m_0 \geq 0$ such that*

$$L \cap \mathcal{I}^{m+m_0} M = \mathcal{I}^m \left(L \cap \mathcal{I}^{m_0} M \right),$$

for every $m \geq 0$.

The proof of Theorem 1.3.8 is the same as in the non-graded case.

Definition 1.3.9. *Suppose $M \in \mathcal{C}(R)$. Let*

$$\cdots \subseteq \mathcal{F}^n(M) \subseteq \mathcal{F}^{n-1}(M) \subseteq \cdots \subseteq \mathcal{F}^1(M) \subseteq \mathcal{F}^0(M) = M$$

*be a filtration of M by submodules $\mathcal{F}^i(M) \in \mathcal{C}(R)$, and let $\mathcal{I}$ be a graded proper ideal in R. The filtration $\mathcal{F}(M)$ is $\mathcal{I}$-*bonne *if $\mathcal{F}^{n+1}(M) \supseteq \mathcal{I}\mathcal{F}^n(M)$ for every $n \geq 0$, and if $\mathcal{F}^{i+1}(M) = \mathcal{I}\mathcal{F}^i(M)$, for $i >> 0$.*

Example 1.3.10. *Examples of $\mathcal{I}$-bonne filtrations.*

a. The $\mathcal{I}$-adic filtration

$$\cdots \mathcal{I}^n M \subseteq \mathcal{I}^{n-1} M \subseteq \cdots \subseteq \mathcal{I}^1 M \subseteq \mathcal{I}^0 M = M$$

is $\mathcal{I}$-bonne.

b. Theorem 1.3.8 gives us that the filtration $\mathcal{F}^n(L) = L \cap \mathcal{I}^n M$ is $\mathcal{I}$-bonne if L is a graded submodule of M.

Throughout the rest of this chapter, we will often follow Serre's exposition in [Ser00] with some modifications in the graded case. We now define an *ideal of definition* in the graded case, which is similar to Serre's definition in the local case.

Definition 1.3.11. *[Ser00] If $M \in \mathcal{C}(R)$, then an* ideal of definition *for M is a graded ideal $\mathcal{I} \subseteq \mathfrak{m}$ (i.e., a graded proper ideal) such that $M/\mathcal{I}M$ is a finite dimensional vector space over k.*

Proposition 1.3.6 gives some equivalent conditions for an ideal to be an ideal of definition which are useful.

Theorem 1.3.12. *(See [Ser00].) Suppose that*

$$0 \to L \to M \to N \to 0$$

is a short exact sequence in $\mathcal{C}(R)$, and that $\mathcal{I}$ is an ideal of definition for M. Then $\mathcal{I}$ is also an ideal of definition for N and L.

Proof. There exists a short exact sequence for all $n \geq 0$

$$0 \to L/(L \cap \mathcal{I}^n M) \to M/\mathcal{I}^n M \to N/\mathcal{I}^n N \to 0.$$

Since $\mathcal{I}$ is an ideal of definition for M, we have that $\dim_k M/\mathcal{I}^n M < \infty$. Thus, $\dim_k L/(L \cap \mathcal{I}^n M) < \infty$ and $\dim_k N/\mathcal{I}^n N < \infty$ for all $n \geq 0$. This gives us that $\mathcal{I}$ is an ideal of definition for N.

By Artin-Rees (Theorem 1.3.8), there exists $m_0 \geq 0$ such that

$$L \cap \mathcal{I}^{m_0+m} M = \mathcal{I}^m (L \cap \mathcal{I}^{m_0} M) \subseteq \mathcal{I}^m L$$

for every $m \geq 0$. So, there exists a surjection

$$L/(\mathcal{I}^m (L \cap \mathcal{I}^{m_0} M)) \to L/\mathcal{I}^m L$$

for all $m \geq 0$. Therefore, $\dim_k L/\mathcal{I}^n L < \infty$ for all $n \geq 0$, and $\mathcal{I}$ is an ideal of definition for L. $\square$

Corollary 1.3.13. *If $M \in \mathcal{C}(R)$ and*

$$\cdots \mathcal{F}^n(M) \subseteq \mathcal{F}^{n-1}(M) \subseteq \cdots \subseteq \mathcal{F}^1(M) \subseteq \mathcal{F}^0(M) = M$$

is a filtration of M by graded R-submodules, and $\mathcal{I}$ is an ideal of definition for M, then $\mathcal{I}$ is an ideal of definition for $\mathcal{F}^i(M)$ and for $\mathcal{F}^{i-1}(M)/\mathcal{F}^i(M)$ for every i.

Proof. Consider the short exact sequence

$$0 \to \mathcal{F}^{i+1}(M) \to \mathcal{F}^i(M) \to \mathcal{F}^i(M)/\mathcal{F}^{i+1}(M) \to 0$$

for every i. We will proceed using induction on i.

For $i = 0$, we have the short exact sequence

$$0 \to \mathcal{F}^1(M) \to M \to M/\mathcal{F}^1(M) \to 0.$$

By Theorem 1.3.12, since $\mathcal{I}$ is an ideal of definition for M, $\mathcal{I}$ is also an ideal of definition for $\mathcal{F}^1(M)$ and for $M/\mathcal{F}^1(M)$.

Suppose that $\mathcal{I}$ is an ideal of definition for $\mathcal{F}^{i-1}(M)$ and for $\mathcal{F}^{i-2}(M)/\mathcal{F}^{i-1}(M)$. Consider the short exact sequence

$$0 \to \mathcal{F}^{i}(M) \to \mathcal{F}^{i-1}(M) \to \mathcal{F}^{i-1}(M)/\mathcal{F}^{i}(M) \to 0.$$

By Theorem 1.3.12, we have that $\mathcal{I}$ is an ideal of definition for $\mathcal{F}^{i}(M)$ and for $\mathcal{F}^{i-1}(M)/\mathcal{F}^{i}(M)$ for every i. $\square$

If $w_1, \ldots, w_t$ are homogeneous elements of the superfluous ideal in R, $\mathfrak{m}$, that generate an ideal $\mathcal{I}$ in R, we define

$$\mathfrak{g}_{\mathfrak{m}(\mathbf{w})} k\langle w_1, \ldots, w_t\rangle \doteq A(\mathbf{w}) = \bigoplus_{n \geq 0} \mathfrak{m}(\mathbf{w})^n/\mathfrak{m}(\mathbf{w})^{n+1},$$

the *associated graded ring* for $\mathfrak{m}(\mathbf{w}) \subseteq k\langle w_1, \ldots, w_t\rangle$. Now, $A(\mathbf{w})$ is really a bi-graded ring, but we disregard the grading that $A(\mathbf{w})$ inherits from the grading that $k\langle w_1, \ldots, w_t\rangle$ has as a subring of R. That is, when we say "an element σ of $A(\mathbf{w})$ of degree s" we mean "$\sigma \in \mathfrak{m}(\mathbf{w})^s/\mathfrak{m}(\mathbf{w})^{s+1}$." Note that $A(\mathbf{w})$ is generated as a graded k-algebra by a k-basis for $\mathfrak{m}(\mathbf{w})/\mathfrak{m}(\mathbf{w})^2$, so $A(\mathbf{w})$ is finitely generated as a k-algebra by elements of degree one.

We define the graded $A(\mathbf{w})$-module

$$\mathfrak{g}_{\mathcal{I}} M \doteq \bigoplus_{n \geq 0} \mathcal{I}^n M/\mathcal{I}^{n+1} M,$$

the *associated graded module* of M with respect to $\mathcal{I}$. Note that $\mathfrak{g}_{\mathcal{I}} M$ is also a graded $\mathfrak{g}_{\mathcal{I}} R$-module. More generally, suppose that

$$\cdots \subseteq \mathcal{F}^{n}(M) \subseteq \mathcal{F}^{n-1}(M) \subseteq \cdots \subseteq \mathcal{F}^{1}(M) \subseteq \mathcal{F}^{0}(M) = M$$

is a filtration of M by submodules $\mathcal{F}^{i}(M) \in \mathcal{C}(R)$. The associated graded module for the filtration is

$$\mathfrak{g}\mathcal{F}(M) \doteq \bigoplus_{n \geq 0} \mathcal{F}^{n}(M)/\mathcal{F}^{n+1}(M).$$

We disregard the grading that these associated graded modules inherit from M. That is, we consider $M(\mathcal{I})$ as a graded $A(\mathbf{w})$-module with the set of elements of degree s being $\mathcal{I}^s M/\mathcal{I}^{s+1} M$.

Proposition 1.3.14. *Suppose that $M \in \mathcal{C}(R)$ and $\mathcal{I}$ is an ideal of definition for M. Let $\mathcal{F}(M)$ be an $\mathcal{I}$-bonne filtration of M. Suppose also that $w_1, \ldots, w_t$ are homogeneous elements that generate $\mathcal{I}$ as an ideal in R. Then,*

a. $\mathcal{I}^n M \subseteq \mathcal{F}^n(M)$ for every $n \geq 0$.

b. $\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M)$ is a finite dimensional vector space over k for every $n \geq 0$.

c. $M \in \mathcal{C}(k\langle w_1, \ldots, w_t\rangle)$, via restriction to the subring $k\langle w_1, \ldots, w_t\rangle$ of R, and $\mathcal{F}(M)$ is also an $\mathfrak{m}(\mathbf{w})$-bonne filtration of M as a $k\langle w_1, \ldots, w_t\rangle$-module.

d. The associated graded module

$$\mathfrak{g}\mathcal{F}(M) \doteq \sum_{n\geq 0} \mathcal{F}^n(M)/\mathcal{F}^{n+1}(M)$$

is a finitely generated graded $A(\mathbf{w})$-module.

Proof. For part (a), $\mathcal{F}(M)$ is an $\mathcal{I}$-bonne filtration of M, so $\mathcal{I}\mathcal{F}^s(M) \subseteq \mathcal{F}^{s+1}(M)$ for $s \geq 0$. Since $\mathcal{F}^0(M) = M$, we have

$$\mathcal{I}M = \mathcal{I}\mathcal{F}^0(M) \subseteq \mathcal{F}^1(M)$$

and

$$\mathcal{I}^2 M \subseteq \mathcal{I}\mathcal{F}^1(M) \subseteq \mathcal{F}^2(M).$$

Therefore, by induction, $\mathcal{I}^n M \subseteq \mathcal{F}^n(M)$ for all $n \geq 0$.

For (b), there exists a short exact sequence of vector spaces over k for all $n \geq 0$,

$$0 \to \mathcal{F}^n(M)/\mathcal{F}^{n+1}(M) \to M/\mathcal{F}^n(M) \to M/\mathcal{F}^{n+1}(M) \to 0.$$

From part (a), $\mathcal{I}^n M \subseteq \mathcal{F}^n(M)$ for all $n \geq 0$, so there exists a surjection

$$M/\mathcal{I}^n M \to M/\mathcal{F}^n(M) \to 0.$$

From Proposition 1.3.6, $M/\mathcal{F}^n(M) \in \mathcal{C}(k)$ (and $M/\mathcal{F}^{n+1}(M) \in \mathcal{C}(k)$) for $n \geq 0$, so $\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M) \in \mathcal{C}(k)$ for $n \geq 0$.

At this time we sketch proofs for parts (c) and (d). Part (c) follows from Propositions 1.3.5 and 1.3.6, Corollary 1.3.13, and the definition of $\mathcal{I}$-bonne, Definition 1.3.9. For part (d), Let n_0 be such that $\mathcal{I}\mathcal{F}^n(M) = \mathfrak{m}(\mathbf{w})\mathcal{F}^n(M) = \mathcal{F}^{n+1}(M)$ for $n \geq n_0$. Then $\mathfrak{g}\mathcal{F}(M)$ is generated by a k-basis for

$$\sum_{i=0}^{n_0} \mathcal{F}^i(M)/\mathcal{F}^{i+1}(M),$$

as an $A(\mathbf{w})$-module. □

1.4 Integer-Valued Functions and Polynomial Functions

Definition 1.4.1. *The polynomials $Q_i(x)$, $i = 0, 1, \ldots$*

$$Q_0(x) = 1,$$

$$Q_1(x) = x,$$

$$\vdots$$

$$Q_i(x) = \binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!},$$

$$\vdots$$

are called the binomial polynomials *and make up a basis of $\mathbb{Q}[x]$ as a vector space over $\mathbb{Q}$.*

Definition 1.4.2. Δ *denotes the* standard difference operator*; that is, for $f(x) \in \mathbb{Q}[x]$, $\Delta f(x) = f(x+1) - f(x)$.*

Proposition 1.4.3. *For the binomial polynomials $Q_i(x)$ as defined above, we have that $\Delta Q_i = Q_{i-1}$ for $i > 0$.*

Proof. Let $Q_i(n) = \binom{n}{i}$. Then, for all $n \in \mathbb{Z}$, we have

$$\begin{aligned} \Delta Q_i(n) &= Q_i(n+1) - Q_i(n) \\ &= \tbinom{n+1}{i} - \tbinom{n}{i} = \tbinom{n}{i-1} \\ &= Q_{i-1}(n). \end{aligned}$$

Since $\Delta Q_i(n) = Q_{i-1}(n)$ for all $n \in \mathbb{Z}$, then $\Delta Q_i(x) = Q_{i-1}(x)$ in $\mathbb{Q}[x]$. □

Lemma 1.4.4. *[Ser00] For $f \in \mathbb{Q}[x]$, the following are equivalent:*

a. f is a $\mathbb{Z}$-linear combination of the binomial polynomials.

b. $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

c. $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$ such that $n >> 0$.

d. Δf has property (a) and there is at least one integer n such that $f(n) \in \mathbb{Z}$.

A polynomial f having the properties of Lemma 1.4.4 is called an *integer-valued polynomial*, and we write $e_i(f)$ for the coefficients of Q_i in the decomposition of f, so that

$$f = \sum e_i(f) Q_i,$$

where $e_i(f) \in \mathbb{Z}$ for all i, and the sum is finite. Clearly, $e_i(f) = e_{i-1}(\Delta f)$ for $i > 0$ (since $Q_i = \Delta Q_{i+1}$ by Proposition 1.4.3). In fact, if $\deg f \leq i$, then we see (by induction on i) that $e_i(f) = \Delta^i f$, a constant polynomial, and we have

$$f(x) = e_i(f)\frac{x^i}{i!} + g(x), \tag{1.4.0.1}$$

with $\deg g < i$. It is also clear that if $\deg f = i$, then

$$\lim_{n\to\infty}\left[e_i(f) + \frac{g(n)i!}{n^i}\right] = \lim_{n\to\infty}\frac{f(n)i!}{n^i},$$

and since $\deg g < i$, we conclude that

$$e_i(f) = \lim_{n\to\infty}\frac{f(n)i!}{n^i}.$$

Thus, $e_i(f) > 0$ if and only if $f(n) > 0$ for all $n >> 0$.

Definition 1.4.5. *For $n_0 \in \mathbb{Z}$ and $[n_0, \infty) = \{z \in \mathbb{Z} \mid z \geq n_0\}$, let the function $f : [n_0, \infty) \to \mathbb{Z}$. (If $n_0 = -\infty$, then $f : \mathbb{Z} \to \mathbb{Z}$.) We say f is a* polynomial function of n *if and only if there exists $P_f \in \mathbb{Z}[x]$ such that*

$$P_f(n) = f(n) \text{ for all } n >> 0.$$

We then say $f \sim P_f$, and f has degree d if and only if P_f has degree d.

Clearly, P_f is uniquely defined by f, and P_f is integer-valued. If f is a polynomial function of n with associated polynomial P_f, we shall write $e_k(f)$ in place of $e_k(P_f)$. Serre [Ser00] refers to polynomial functions as "polynomial-like" functions. Now, we give some properties of polynomial functions that we will use later. The straightforward proofs are left to the reader.

Lemma 1.4.6. *The following are equivalent for f as in Definition 1.4.5:*

a. *f is a polynomial function of n of degree d;*

b. *Δf is a polynomial function of n of degree $d-1$;*

c. *There exists $r \geq 0$ such that $\Delta^r f(n) = 0$ for all $n >> 0$, and d is the largest r such that $\Delta^r f \not\equiv 0$.*

Furthermore, if any of the above equivalent conditions hold,

d. *$\Delta^d f = e_d(f) =$ (the leading coefficient of P_f)$d!$ if $d = \deg P_f$.*

Lemma 1.4.7. *If f and g are polynomial functions of n and $c \in \mathbb{Z}$ is an integer constant, then the following hold:*

a. *If $f, g : [n_0, \infty) \to \mathbb{Z}$, then $f + cg$ and fg are polynomial functions of n.*

b. *If $f : [n_0, \infty) \to [m_0, \infty)$ is increasing strictly for $n >> 0$ and $g : [m_0, \infty) \to \mathbb{Z}$, then $g \circ f$ is a polynomial function of n.*

Lemma 1.4.8. *For n_0 a positive integer, let $f : [n_0, \infty) \to \mathbb{Z}$ be a polynomial function of n with degree d, and let $g(n) = f(n) + f(n-1) + \cdots + f(n_0)$ for $n >> 0$. Then,*

a. *g is a polynomial function of n;*

b. *$\Delta g(n) = f(n+1)$ for $n >> 0$;*

c. *$\deg g = d + 1$.*

Lemma 1.4.9. *If f is a polynomial function of n of degree d and g is a polynomial function of n of degree less than d, then $f+g$ is a polynomial function of n of degree d and the leading coefficient of P_{f+g} is the same as that for P_f.*

Proof. Suppose $P_f(t), P_g(t) \in \mathbb{Z}[t]$ such that $f(n) \sim P_f(n)$ and $g(n) \sim P_g(n)$ for $n >> 0$ where $\deg P_f = d$ and $\deg P_g < d$. Denote $P_f(t) = a_d t^d + \cdots + a_0$ and $P_g(t) = b_e t^e + \cdots + b_0$ where $e < d$ and $a_i, b_j \in \mathbb{Z}$ for every i, j. Then for $n >> 0$,

$$\begin{aligned} f(n) + g(n) &= P_f(n) + P_g(n) \\ &= \left(a_d t^d + \cdots + a_0\right) + \left(b_e t^e + \cdots + b_0\right) \\ &= a_d t^d + \cdots + a_{e+1} t^{e+1} + \left(a_e + b_e\right) t^e + \cdots + \left(a_0 + b_0\right) \end{aligned}$$

Therefore, $\deg P_{f+g} = \deg(f+g) = d = \deg f$ and the leading coefficient of P_{f+g} is a_d, which is the leading coefficient of P_f. □

1.5 Poincaré Series of Graded Modules

As we have seen in Section 1.3, if $M \in \mathcal{C}(R)$ (recall the Noetherian hypothesis on R), then M_i is a finite dimensional vector space over k for every i.

Definition 1.5.1. *For $M \in \mathcal{C}(R)$, we define*

$$\mathrm{PS}(M,t) = \sum_{i=0}^{\infty} (\dim_k M_i)\, t^i,$$

the Poincaré series for M. In fact, $\mathrm{PS}(M,t)$ is defined in the same way for every nonnegatively graded k-vector space M such that $\dim_k M_i < \infty$ for all i.

Proposition 1.5.2. *For $M, N \in \mathcal{C}(R)$, the following properties hold:*

a. $\mathrm{PS}(M(-r),t) = t^r\, \mathrm{PS}(M,t)$ *for all* $r \geq 0$.

b. $\mathrm{PS}(M \oplus N) = \mathrm{PS}(M) + \mathrm{PS}(N)$.

c. $\mathrm{PS}(M \otimes_k N) = \mathrm{PS}(M)\, \mathrm{PS}(N)$.

Proof. Let $M, N \in \mathcal{C}(R)$.

a. Applying the defintion of PS(M) and shifting (or twisting), we see

$$\begin{aligned}\mathrm{PS}(M(-r),t) &= \sum_{i=0}^{\infty}(\dim_k M(-r)_i)\,t^i = \sum_{i=0}^{\infty}(\dim_k M_{i-r})\,t^i \\ &= \sum_{i=0}^{\infty}(\dim_k M_i)\,t^{r+i} = t^r\sum_{i}^{\infty}(\dim_k M_i)\,t^i = t^r\,\mathrm{PS}(M,t).\end{aligned}$$

b. Using definitions, we have

$$\begin{aligned}\mathrm{PS}(M\oplus N,t) &= \sum_{i=0}^{\infty}(\dim_k(M_i\oplus N_i))\,t^i = \sum_{i=0}^{\infty}(\dim_k M_i + \dim_k N_i)\,t^i \\ &= \sum_{i=0}^{infty}\dim_k M_i t^i + \sum_{i=0}^{\infty}\dim_k N_i t^i = \mathrm{PS}(M,t)+\mathrm{PS}(N,t).\end{aligned}$$

c. Again, applying the definitions, we have

$$\begin{aligned}\mathrm{PS}(M\otimes_k N,t) &= \sum_{i=0}^{\infty}\dim_k\left(\bigoplus_{0\le j\le i}(M_j\otimes_k N_{i-j})\right)t^i \\ &= \sum_{i}^{\infty}\sum_{j}^{i}(\dim_k(M_j\otimes_k N_{i-j}))\,t^i \\ &= \sum_{i=0}^{\infty}\sum_{j}^{i}(\dim_k M_j)\,(\dim_k N_{i-j})\,t^i \\ &= \sum_{i=0}^{\infty}(\dim_k M_i)\,t^i\cdot\sum_{i=0}^{\infty}(\dim_k N_i)\,t^i = \mathrm{PS}(M,t)\,\mathrm{PS}(N,t).\end{aligned}$$

□

Example 1.5.3. *Examples of Poincaré Series.*

a. Let $M = k[x]$*, where* $\deg x = d$*. As a graded ring, we write*

$$k[x] = k\oplus kx\oplus kx^2\oplus\cdots.$$

Since $\dim_k k = 1$*, we have*

$$\mathrm{PS}(k[x],t) = 1+t^d+t^{2d}+\cdots,$$

so

$$\mathrm{PS}(k[x],t) = \sum_{i=0}^{\infty}(t^d)^i = \frac{1}{1-t^d}.$$

b. Let $M = k[x_1, \ldots, x_n]$*, where* $\deg x_i = d_i$*. As a graded vector space,*

$$k[x_1, \ldots x_n] \cong k[x_1] \otimes_k \cdots \otimes_k k[x_n].$$

Since $\mathrm{PS}(k[x_j], t) = \frac{1}{1-t^{d_j}}$ *for all* $1 \leq j \leq n$*, and using Proposition 1.5.2, we have*

$$\mathrm{PS}(k[x_1, \ldots, x_n], t) = \mathrm{PS}(k[x_1], t)\,\mathrm{PS}(k[x_2], t) \cdots \mathrm{PS}(k[x_n], t) = \frac{1}{\prod_{j=1}^{n}(1-t^{d_j})}.$$

In the following proposition, we see that the Poincaré series is a rational function of t; that is, the Poincaré series can be written as a quotient of polynomials where the denominator has a particular form.

Proposition 1.5.4. *[Smo72, Thm. 4.2]. Suppose that* $M \in \mathcal{C}(R)$*,* $M \neq 0$*. If* R *is generated as a graded* k*-algebra by* $x_1, \ldots, x_n$ *of positive degrees* $d_1, \ldots, d_n$*, then*

$$\mathrm{PS}(M, t) = \frac{q(t)}{\prod_{i=1}^{n}(1-t^{d_i})},$$

where $q(t) \in \mathbb{Z}[t]$*.*

Therefore, we see that, given the hypotheses of Proposition 1.5.4, $\mathrm{PS}(M, t)$ has a pole of order ≥ 0 at $t = 1$.

Definition 1.5.5. *If* M *satisfies the hypotheses of Proposition 1.5.4, we define* $\ell(M)$ *to be* the order of the pole at $t = 1$ of $\mathrm{PS}(M, t)$*. For convenience,* $\ell(0) \doteq -\infty$*.*

In [Mai76], Maiorana defines his C-multiplicity.

Definition 1.5.6. *([Mai76]) For* $M \neq 0$*, define the* C*-multiplicity as*

$$C(M) \doteq \lim_{t \to 1}(1-t)^{\ell(M)}\,\mathrm{PS}(M, t) \in \mathbb{Q}.$$

If $M = 0$*, define* $C(M) \doteq 0$*.*

$C(M)$, Maiorana's C-multiplicity, is usually not an integer but is a rational number and is called $\deg(M)$ (the degree of M) by Benson [BCB95], among others.

$\mathrm{PS}(M,t)$ expanded as a Laurent series in $(1-t)$ is

$$\mathrm{PS}(M,t) = C(M)(1-t)^{-\ell(M)} + \text{ ``higher order terms''}$$

where $C(M) \neq 0$, if $M \neq 0$.

The reader may verify the following two lemmas from Maiorana.

Lemma 1.5.7. *[Mai76, Pg. 254]. Suppose that* $0 \to L \to M \to N \to 0$ *is a short exact sequence in* $\mathcal{C}(R)$*. Then* $\ell(M) = \max\{\ell(L), \ell(N)\}$*. Also, the following properties hold:*

a. If $\ell(L) < \ell(M)$*, then* $C(M) = C(N)$*.*

b. If $\ell(N) < \ell(M)$*, then* $C(M) = C(L)$*.*

c. If $\ell(L) = \ell(N) = \ell(M)$*, then* $C(M) = C(L) + C(N)$*.*

Definition 1.5.8. *For* $M, N \in \mathcal{C}(R)$*, we say that* $\mathrm{PS}(M,t) \leq \mathrm{PS}(N,t)$ *if and only if* $\mathrm{PS}(M,t)$ *is less than* $\mathrm{PS}(N,t)$ *coefficientwise—that is,* $\dim_k M_i \leq \dim_k N_i$ *for all* i*.*

Lemma 1.5.9. *[Mai76, Pg. 254-255] Let* $L, M, N \in \mathcal{C}(R)$*. Then, the following properties hold:*

a. $\ell(M(-r)) = \ell(M)$ *and* $C(M(-r)) = C(M)$*.*

b. $\ell(M \otimes_k N) = \ell(M) + \ell(N)$*.*

c. If N *is such that* $N_n = 0$ *for* $n >> 0$*, and* $\mathrm{PS}(L,t) \leq \mathrm{PS}(M \otimes_k N, t)$*, then* $\ell(L) \leq \ell(M)$*.*

d. $C(M \otimes_k N) = C(M)C(N)$*.*

Now that we have several properties established for the Poincaré series, we define the following invariants from Smoke [Smo72] for future use.

Definition 1.5.10. *[Smo72] Let $M \in \mathcal{C}(R)$, $M \neq 0$.*

a. *Define $d(M)$ to be the least d such that there exist positive integers $\xi_1, \ldots, \xi_d$ with*
$$\left(\prod_{i=1}^{d} \left(1 - t^{\xi_i}\right)\right) PS(M,t) \in \mathbb{Z}[t].$$
We define $d(M) = 0$ if and only if $PS(M,t)$ is a polynomial in t.

b. *Define $s(M)$ to be the least s such that there exist homogeneous elements $y_1, \ldots, y_s \in \mathfrak{m}$ with M finitely generated over $k\langle y_1, \ldots, y_s\rangle \subseteq R$. We define $s(M) = 0$ if and only if M is a finite dimensional vector space over k.*

c. *For convenience, we define $d(0) = s(0) = -\infty$.*

1.6 Dimension and Multiplicities for Graded Rings

We begin this section by reviewing some basic definitions and theorems from commutative algebra in the graded context, followed by a definition of Krull dimension. Then we will discuss a classical example of multiplicity, the Samuel multiplicity, for graded rings. We will end this section with some properties of Maiorana's C-multiplicity. The material of this section draws from the works of [Eis95], [Ser00], [Smo72] and [Mai76].

1.6.1 Background

Recall that R is a nonnegatively graded Noetherian ring with $R_0 = k$, a field, as usual. Let M be a (finitely generated) graded R-module. As we have seen, the Noetherian hypothesis means that R is a finitely generated k-algebra.

Definition 1.6.1. *The* annihilator *of M in R is*

$$Ann(M) = \{r \in R \mid rm = 0, \text{ for all } m \in M\}.$$

Proposition 1.6.2. *[Eis95] $Ann(M)$ is a homogeneous, or graded, ideal in R.*

Definition 1.6.3. *Let $M \in \mathcal{C}(R)$.*

a. The homogeneous spectrum of M *is*

$$\mathcal{V}(M) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a homogeneous prime ideal in } R, Ann(M) \subseteq \mathfrak{p}\}.$$

b. Proj$(M) \doteq \mathcal{V}(M) - \{\mathfrak{m}\}$.

Definition 1.6.4. *A prime ideal $\mathfrak{p}$ is* associated to M *if $\mathfrak{p}$ is the annihilator of an element of M (i.e., there exists $x \in M$ such that $\mathfrak{p} = ann(x)$). The set of all associated primes to M is called $Ass(M)$.*

Lemma 1.6.5. *[Eis95] For $M \in \mathcal{C}(R)$ where $M \neq 0$, $Ass(M) \neq \emptyset$; moreover, every associated prime is homogeneous and contains $Ann(M)$. In fact, if $\mathfrak{p} \in Ass(M)$, then $\mathfrak{p} = ann(x)$ for some homogeneous element $x \in M$.*

Let $M \neq 0$ and let x be a homogeneous element of M of degree $d(x) \geq 0$, such that $ann(x) = \mathfrak{p}$ is a homogeneous prime ideal in R. Then, there exists an injection of graded R-modules

$$\phi : (R/\mathfrak{p})\,(-d(x)) \to M$$

defined by $r + \mathfrak{p} \mapsto rx$. Conversely, if $\mathfrak{p}$ is a homogeneous prime ideal of R such that there exists an injection of graded R-modules $\iota : (R/\mathfrak{p})\,(-d) \to M$, for some $d \geq 0$, then $\mathfrak{p} = ann\,(\iota(1 + \mathfrak{p}))$ is an associated prime of M.

Definition 1.6.6. *A* minimal prime of M *is a minimal element, with respect to inclusion, of the set of all prime ideals of R that contain $Ann(M)$.*

Forgetting the grading on M and R, form the localizations

$$R_{\mathfrak{p}} \doteq S^{-1}R \text{ and } M_{\mathfrak{p}} \doteq S^{-1}M, \text{ where } S = R - \mathfrak{p}.$$

Now consider the following theorem from commutative algebra, the proof of which carries over to the category $\mathcal{C}(R)$ with slight modification.

Theorem 1.6.7. *[Ser00] Let $M \in \mathcal{C}(R)$.*

a. *There exists a finite filtration of M by graded submodules*

$$0 = \mathcal{F}^{N+1}(M) \subseteq \mathcal{F}^{N}(M) \subseteq \cdots \subseteq \mathcal{F}^{i+1}(M) \subseteq \mathcal{F}^{i}(M) \subseteq \cdots \subseteq \mathcal{F}^{0}(M) = M,$$

a set of homogeneous prime ideals $\mathfrak{p}_i$ in R and integers $d_i \geq 0$, for $0 \leq i \leq N$, with $\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M) \cong (R/\mathfrak{p}_i)(-d_i)$ as graded R-modules, for every i.

b. *For any filtration $\mathcal{F}$ of M satisfying the conditions of (a) above, define $\mathcal{S}_{\mathcal{F}} \doteq \{\mathfrak{p}_i | 1 \leq i \leq N\}$. Then, $Ass(M) \subseteq \mathcal{S}_{\mathcal{F}} \subseteq \mathcal{V}(M)$, and all three of these sets have the same minimal elements. Thus, every minimal prime of M is homogeneous. In addition, there are only a finite number of minimal primes, and there are only a finite number of associated primes.*

c. *The set of minimal elements in $Ass(M)$ is equal to the set of minimal primes of M.*

d. *Let $\mathcal{F}$ be any filtration of M satisfying the conditions of (a) above. Let $\mathfrak{p}$ be a fixed minimal prime of M. Then, $M_{\mathfrak{p}}$ has finite length $l(M_{\mathfrak{p}})$ as an $R_{\mathfrak{p}}$-module, and the number of times that $(R/\mathfrak{p})(-d)$ appears as (isomorphic to) a successive quotient $\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)$, for all d, is equal to $l(M_{\mathfrak{p}})$. Hence, this number is independent of the filtration.*

1.6.2 Dimension

The length of the chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ involving $n+1$ distinct homogeneous prime ideals is n.

Definition 1.6.8. *The* Krull dimension, *(or just* dimension*) of a graded ring R, written* $\mathrm{Dim}(R)$*, is the supremum of the lengths of chains of distinct homogeneous prime ideals in R.*

That is, $\mathrm{Dim}(R)$ is the greatest D such that there exists a strictly increasing chain

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_D$$

of homogeneous prime ideals in R. Similarly, the *Krull dimension of a graded R-module M*, denoted $\mathrm{Dim}(M)$, is the greatest D such that there exists a strictly increasing chain

$$\mathrm{Ann}(M) \subseteq \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_D$$

of homogeneous prime ideals in R. Note that the Krull dimension of M is the same as that of the graded ring $R/Ann(M)$ since, for all homogeneous ideals I in R, the homogeneous prime ideals in R, $\mathfrak{p}$, containing I are in one-to-one correspondence with the homogeneous prime ideals in R/I. We define $\mathrm{Dim}(0) \doteq -\infty$.

Example 1.6.9. *Examples of Dimension*

a. *Any field k has Krull dimension 0. The graded polynomial ring $k[x]$ (where x has any degree you like) has dimension 1. The chain of homogeneous primes $(x) \supset 0$ is of length 1. Since (x) is the only homogeneous prime ideal, there are no chains of homogeneous primes of greater length.*

b. *More generally, consider a graded polynomial ring $R = k[x_1, \ldots, x_n]$, where $\deg x_i = d_i$, and the sequence of homogeneous prime ideals*

$$(x_1, \ldots, x_n) \supset (x_1, \ldots, x_{n-1}) \supset \cdots \supset (x_1) \supset 0.$$

The theorem below states that there is no chain of greater length.

Theorem 1.6.10. *Suppose k is a field. Let $x_1, \ldots, x_n$ be indeterminants of degrees $d_1, \ldots, d_n \geq 0$, where $\deg x_i = d_i$. Then, $\mathrm{Dim}(k[x_1, \ldots, x_n]) = n$.*

Recall the invariants $d(M)$ and $s(M)$ as defined in Section 1.5. The following proposition will prove to be valuable later in this section.

Proposition 1.6.11. *[Smo72, Thm. 5.5, Prop. 6.2]. Let $M \in \mathcal{C}(R)$, and let $d(M)$ and $s(M)$ be as defined in Section 1.5.*

a. $d(M) = s(M) = \mathrm{Dim}(M) < \infty$.

b. If $d(M) = s(M) = \mathrm{Dim}(M) = D$, and $y_1, \ldots, y_D \in \mathfrak{m}$ are homogeneous elements such that M is finitely generated over $k\langle y_1, \ldots, y_D\rangle$, then $y_1, \ldots, y_D$ are algebraically independent over k.

Part (b) in Proposition 1.6.11 is not necessarily true in the non-graded case. The next corollary follows from Proposition 1.6.11.

Corollary 1.6.12. *If R is a Noetherian graded ring, then the Krull dimension of R is finite. Similarly, for a finitely generated graded R-module M, the Krull dimension of M is finite.*

1.6.3 Hilbert Polynomial

Definition 1.6.13. *Let M be a module. A finite* chain *of submodules of M is a sequence of submodules with strict inclusions*

$$0 = \mathcal{F}^n(M) \subset \cdots \subset \mathcal{F}^1(M) \subset \mathcal{F}^0(M) = M.$$

This chain is said to have length *n.*

Definition 1.6.14. *Let M be a module. A chain of submodules of M is a* composition series *if $\mathcal{F}^i M/\mathcal{F}^{i+1}M$ is a (nonzero) simple module (no proper submodules) for all i.*

Definition 1.6.15. *For M a module, we define* length M *to be the smallest length of a composition series of M. If M has no finite composition series, we define* length $M \doteq \infty$.

Definition 1.6.16. *A commutative ring A is* Artinian *if and only if A has the descending chain condition on ideals (i.e., every descending chain of ideals in A eventually terminates).*

The following lemmas are well-known.

Lemma 1.6.17. *Let the ring A be commutative. Then, the following are equivalent:*

a. A is Artinian.

b. A has finite length (as a module over itself).

c. A is Noetherian, and every prime ideal of A is maximal.

Lemma 1.6.18. *Suppose A is an Artinian commutative ring. If M is a finitely generated A-module, then M has a (finite) composition series and any two composition series for M have the same finite length.*

Theorem 1.6.19. *Let A be an Artinian ring. If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated modules over A, then*

$$0 = \operatorname{length}_A(L) - \operatorname{length}_A(M) + \operatorname{length}_A(N).$$

Corollary 1.6.20. *Given a long exact sequence of finitely generated modules, $M_{(i)}$, over the Artinian ring A,*

$$0 \to M_{(1)} \to \cdots \to M_{(n-1)} \to M_{(n)} \to 0$$

we have that

$$0 = \sum_{i=1}^{n} (-1)^i \operatorname{length}_A M_{(i)}.$$

Returning to the graded category and applying Lemma 1.6.18, we have the following well-known lemma.

Lemma 1.6.21. *For S a nonnegatively graded ring, with S_0 Artinian, then any finitely generated module, M, over S_0 has a composition series*

$$0 = \mathcal{F}^n(M) \subsetneq \cdots \subsetneq \mathcal{F}^2 M \subsetneq \mathcal{F}^1(M) \subsetneq \mathcal{F}^0(M) = M,$$

which has the defining property that the successive quotients are nonzero simple S_0-modules (no nonzero proper submodules). The length of the composition series above is $\operatorname{length}_{S_0}(M) = n$. *Furthermore, any two such composition series have the same length.*

For example, let $S_0 = k$, a field. The only simple vector space is a one-dimensional vector space. For every finite dimensional vector space, V, we get a descending series where we get the quotients by adding one basis vector at a time. The dimension of the vector space is the sum of the dimensions of the quotients. Therefore, $\operatorname{length}_k(V)$ as a k-module is $\dim_k V$.

For S such that S_0 is not necessarily a field but is an Artinian ring, let M be a finitely generated graded S-module. Each M_n is a finitely generated S_0-module by Proposition 1.2.5; therefore, each M_n has finite length, denoted $\operatorname{length}_{S_0}(M_n)$. If S_0 is a field, then

$$\operatorname{length}_{S_0}(M_n) = \dim_{S_0}(M_n), \text{ for all } n.$$

Consider the nonnegatively graded ring S such that S_0 is an Artinian ring and S is finitely generated as an S_0-algebra by r elements in S_1. Suppose that Z is any nonnegatively graded S-module that is finitely generated as an S-module. Then Z_n has finite length as an S_0-module for every n. When S_0 is a field, note that the length of Z_n is the dimension of Z_n as a vector space over S_0.

Definition 1.6.22. *[Ser00] For S and Z as defined in the previous paragraph, $\chi(Z,n) = \text{length}_{S_0}(Z_n)$, where $\text{length}_{S_0}(Z_n)$ is the length of Z_n as an S_0-module. We call $n \mapsto \chi(Z,n)$ the* Hilbert function.

Theorem 1.6.23. *[Ser00, Sec. II.B.3, Thm. 2].(Hilbert's Theorem). Let S be a nonnegatively graded ring such that S_0 is an Artinian ring and S is finitely generated as an S_0-algebra by r elements in S_1. Let Z be any nonnegatively graded S-module that is finitely generated as an S-module. Then $\chi(Z,n)$ is a polynomial function of n of degree no greater than $r-1$.*

Suppose S and Z are as in Theorem 1.6.23. The *Hilbert polynomial of Z*, P_χ is the polynomial associated to the Hilbert function, $n \mapsto \chi(Z,n)$. Note that $\text{length}_{S_0}(Z) < \infty$ if and only if $P_\chi(n) \equiv 0$ since $\text{length}_{S_0}(Z) < \infty$ implies $Z_n = 0$ for $n >> 0$. By Theorem 1.6.23, $\deg P_\chi \leq r-1$ for $r \geq 1$.

Thus, $\Delta^{r-1}P_\chi \doteq e_{r-1}(P_\chi)$ is a nonnegative constant (using the notation from Section 1.4) since $P_\chi(n) \geq 0$ for $n >> 0$ as it measures the length of Z over S_0, which is never negative. Note that if $\deg P_\chi = r-1$, then

$$\Delta^{r-1}P_\chi = (\text{the leading coefficient of } P_\chi)(r-1)!,$$

using Lemma 1.4.6; and if $\deg P_\chi < r-1$, then $\Delta^{r-1}P_\chi = 0$. If $P_\chi(n) \not\equiv 0$, then its leading coefficient must be positive since, for $n >> 0$, the polynomial is dominated by the term with the leading coefficient.

Now we have an upper bound for $\Delta^{r-1}P_\chi \doteq e_{r-1}(P_\chi)$.

Theorem 1.6.24. *[Ser00, Sec. II.B.3, Thm. 2']. Let S and Z be as defined in Theorem 1.6.23. Suppose that Z_0 generates Z as a graded S-module. Then $\Delta^{r-1}P_\chi \leq \text{length}(Z_0)$.*

1.6.4 Samuel Polynomial

In this section, we define a classical polynomial function from commutative algebra and consider some of its properties. Let R be a nonnegatively graded Noetherian ring with $R_0 = k$, a fixed field, as usual. Let $\mathcal{I}$ be an ideal of definition for $M \in \mathcal{C}(R)$. Therefore, by Proposition 1.3.6, we have that $M/\mathcal{I}^i M$ is a finite dimensional graded vector space over k for every $i \geq 1$. We assume that $\mathrm{Ann}(M) \subseteq \mathcal{I}$ from this point forward.

Definition 1.6.25. *Let $\mathcal{I}$ be an ideal of definition for $M \in \mathcal{C}(R)$. The* Samuel function for the $\mathcal{I}$-adic filtration *is*

$$p(M, \mathcal{I}, n) = \dim_k(M/\mathcal{I}^{n+1}M),$$

for $n \geq 0$.

Using Proposition 1.3.14, we see that there are Samuel functions for any $\mathcal{I}$-bonne filtration of M, and we see that Definition 1.6.25 is a special case of the following definition.

Definition 1.6.26. *The* Samuel function for an $\mathcal{I}$-bonne filtration *of M, $\mathcal{F}(M)$, is*

$$p(\mathcal{F}(M), n) = \dim_k(M/\mathcal{F}^{n+1}(M)).$$

Theorem 1.6.27. *(Samuel's Theorem) Suppose that $M \in \mathcal{C}(R)$. Let $\mathcal{I}$ be an ideal of definition for M and let $\mathcal{F}(M)$ be an $\mathcal{I}$-bonne filtration of M. Then $p(\mathcal{F}(M), n)$ is a polynomial function of n for $n >> 0$ with degree not more than r, where $\mathcal{I}$ is generated by $x_1, \ldots, x_r$.*

Proof. We apply Proposition 1.3.14. Let $A(\mathbf{x}) = \bigoplus_{i \geq 0} \mathfrak{m}(\mathbf{x})^i/\mathfrak{m}(\mathbf{x})^{i+1}$, the associated graded ring for $\mathfrak{m}(\mathbf{x}) \subseteq k\langle x_1, \ldots, x_r \rangle$, with grading such that $A(\mathbf{x})_n = \mathfrak{m}(\mathbf{x})^n/\mathfrak{m}(\mathbf{x})^{n+1}$. Define

$$Z = \sum_{n \geq 0} \mathcal{F}^n(M)/\mathcal{F}^{n+1}(M).$$

Using Proposition 1.3.14(d), we know that $A = A(\mathbf{x})$ and Z satisfy the hypotheses of Hilbert's Theorem. (Here, $A_0 = k$ and $\text{length}_{A_0}(--) = \dim_k(--)$.)

Thus,

$$\chi(Z, n) = \dim_k \left(\mathcal{F}^n(M)/\mathcal{F}^{n+1}(M)\right)$$

is a polynomial function of n for $n >> 0$ of degree no greater than $r - 1$. Also,

$$\begin{aligned}
\Delta p(\mathcal{F}(M), n) &\doteq p(\mathcal{F}(M), n+1) - p(\mathcal{F}(M), n) \\
&= \dim_k \left(M/\mathcal{F}^{n+2}(M)\right) - \dim_k \left(M/\mathcal{F}^{n+1}(M)\right) \\
&= \dim_k \left(\mathcal{F}^{n+1}(M)/\mathcal{F}^{n+2}(M)\right) \\
&= \chi(Z, n+1).
\end{aligned}$$

Since $\chi(Z, n+1)$ is a polynomial function of n of degree no greater than $r-1$, using Lemma 1.4.7 and Theorem 1.6.23, $p(\mathcal{F}(M), n)$ is a polynomial function of n for $n >> 0$ of degree less than or equal to r by Lemma 1.4.6. □

Definition 1.6.28. *If $M \in \mathcal{C}(R)$ and $\mathcal{I}$ is an ideal of definition for M, then $d_1(M)$ is the degree of the polynomial function of n that calculates $p(M, \mathcal{I}, n)$ for $n >> 0$. We define $d_1(0) \doteq -\infty$.*

Now we will show how the Samuel functions for any $\mathcal{I}$-bonne filtration can be compared to the Samuel functions for the $\mathcal{I}$-adic filtration. This comparison will allow us to conclude that $d_1(M)$ is independent of the choice of the ideal of definition $\mathcal{I}$. The following theorems follow from Serre [Ser00].

Theorem 1.6.29. *[Ser00] Suppose that $M \in \mathcal{C}(R)$, $M \neq 0$, and $\mathcal{I}$ is an ideal of definition for M containing Ann(M). Let $\mathcal{F}(M)$ be an $\mathcal{I}$-bonne filtration of M.*

a. For $n >> 0$, $p(M, \mathcal{I}, n) = p(\mathcal{F}(M), n) + R(n)$, where $R(n)$ is a polynomial function of n whose leading coefficient is positive if $R(n) \not\equiv 0$, and whose degree is strictly less than the degree of the polynomial function determined by $p(M, \mathcal{I}, n)$.

b. If $\mathcal{I}/\operatorname{Ann}(M)$ is finitely generated by r homogeneous elements as an ideal in $R/\operatorname{Ann}(M)$, then for $n >> 0$, $p(M, \mathcal{I}, n)$ is a polynomial function of degree less than or equal to r, and $\Delta^r p(M, \mathcal{I}, n) \leq \dim_k(M/\mathcal{I}M)$.

Lemma 1.6.30. *[Ser00] Suppose that*

$$0 \to L \to M \to N \to 0$$

is a short exact sequence in $\mathcal{C}(R)$*, and that* $\mathcal{I}$ *is an ideal of definition for* M*. Then,* $\mathcal{I}$ *is an ideal of definition for* N *and* L*, and, for* $n >> 0$*,*

$$p(M, \mathcal{I}, n) + q(n) = p(N, \mathcal{I}, n) + p(L, \mathcal{I}, n),$$

where $q(n)$ *is a polynomial function of* n*, whose leading coefficient is positive if* $q \not\equiv 0$*, and whose degree is strictly less than* $d_1(L)$*.*

Proof. By Theorem 1.3.12, we have that $\mathcal{I}$ is an ideal of definition for N and L.

Consider the filtration $\mathcal{F}^n(L) = L \cap \mathcal{I}^n M$. For all $n \geq 0$, there exists a short exact sequence

$$0 \to L/\mathcal{F}^n(L) \to M/\mathcal{I}^n M \to N/\mathcal{I}^n N \to 0.$$

(Note that it is necessary to use $L/\mathcal{F}^n(L)$ instead of $L/I^n L$ in order to have an exact sequence.) Hence,

$$p(M, \mathcal{I}, n) = p(N, \mathcal{I}, n) + p(\mathcal{F}(L), n).$$

Using the Artin-Rees Theorem (Theorem 1.3.8), $\mathcal{F}(L)$ is an $\mathcal{I}$-bonne filtration of L. So, by Theorem 1.6.29, for $n >> 0$,

$$p(L, \mathcal{I}, n) = p(\mathcal{F}(L), n) + q(n),$$

where $q(n)$ is a polynomial function of n whose leading coefficient is positive if $q \not\equiv 0$ and whose degree is strictly less than the degree of the polynomial function determined by $p(L, \mathcal{I}, n)$, giving

$$p(M, \mathcal{I}, n) = p(N, \mathcal{I}, n) + p(L, \mathcal{I}, n) - q(n).$$

□

In the following theorem, we see that the degree of the Samuel polynomial does not depend on $\mathcal{I}$.

Theorem 1.6.31. *Suppose that $\mathcal{I}$ is an ideal of definition for $M \in \mathcal{C}(R)$. Then, for $n >> 0$, the degrees of the polynomial functions defined by $p(M, \mathcal{I}, n)$ and $p(M, \mathfrak{m}, n)$ are equal.*

Proof. Let $P_{\mathcal{I}}$ and $P_{\mathfrak{m}}$ be the associated polynomials for the polynomial functions of n, $p(M, \mathcal{I}, n)$ and $p(M, \mathfrak{m}, n)$, respectively, and let $d_{\mathcal{I}} = \deg P_{\mathcal{I}}(n)$ and $d_{\mathfrak{m}} = \deg P_{\mathfrak{m}}(n)$.

First we show that $d_{\mathfrak{m}} \leq d_{\mathcal{I}}$. Since $\mathcal{I} \subseteq \mathfrak{m}$, we know that $\mathcal{I}^n \subseteq \mathfrak{m}^n$ for all $n \geq 1$. Hence, $\mathcal{I}^n M \subseteq \mathfrak{m}^n M$ for all $n \geq 1$, and for $\phi_n : M/\mathcal{I}^n M \to M/\mathfrak{m}^n M$, ϕ is a surjective map for all $n \geq 1$. Thus, for $n >> 0$,

$$\begin{array}{rcl} \dim_k (M/\mathfrak{m}^n M) & \leq & \dim_k (M/\mathcal{I}^n M) \\ \Rightarrow p(M, \mathfrak{m}, n) & \leq & p(M, \mathcal{I}, n) \\ \Rightarrow P_{\mathfrak{m}}(n) & \leq & P_{\mathcal{I}}(n). \end{array}$$

Suppose that $d_{\mathfrak{m}} > d_{\mathcal{I}}$. Then, since

$$x_n \doteq \frac{P_{\mathfrak{m}}(n)}{n^{d_{\mathcal{I}}}} \leq \frac{P_{\mathcal{I}}(n)}{n^{d_{\mathcal{I}}}} \doteq y_n$$

for $n >> 0$, and the left sequence x_n diverges, so must the right sequence, y_n, contradicting that y_n converges to $a_{d_{\mathcal{I}}}$, where $a_{d_{\mathcal{I}}}$ is the leading coefficient for $P_{\mathcal{I}}(n)$. Thus, $d_{\mathfrak{m}} \leq d_{\mathcal{I}}$.

Now we show that $d_{\mathfrak{m}} \geq d_{\mathcal{I}}$. By Proposition 1.3.6 (e), there exists $n_0 > 0$ such that $\mathfrak{m}^{n_0 n} M \subseteq \mathcal{I}^n M$ for all $n \geq 1$. Hence,

$$p(M, \mathcal{I}, n) \leq p(M, \mathfrak{m}, n_0 n)$$

for all $n \geq 1$. We proceed as before, but divide by $n^{d_{\mathfrak{m}}}$, concluding that $d_{\mathfrak{m}} \geq d_{\mathcal{I}}$.

Therefore, $d_{\mathfrak{m}} = d_{\mathcal{I}}$. $\square$

1.6.5 Equality of $Dim(M)$, $s_1(M)$, and $d_1(M)$

For $\mathcal{I}$ an ideal of definiton for $M \in \mathcal{C}(R)$, the following definition will help us find a relationship between the Krull dimension of M and the degree of the polynomial function of n that calculates the Samuel function $p(M, \mathcal{I}, n)$ for the $\mathcal{I}$-adic filtration.

Definition 1.6.32. *If $M \in \mathcal{C}(R)$ with $M \neq 0$, let $s_1(M)$ be the least s such that there exist homogeneous elements $y_1, \ldots, y_s \in \mathfrak{m}$ such that $M/(y_1, \ldots, y_s)M$ is a finite dimensional graded vector space over k. Note that $s_1(M) = 0$ if and only if M is a finite dimensional graded vector space over k. We define $s_1(0) \doteq -\infty$.*

The relationship between the degree of the Hilbert polynomial and the Krull dimension is the following:

Theorem 1.6.33. *[Ser00, Sec. III.B.2, Thm. 1]. If $M \in \mathcal{C}(R)$, then* $\mathrm{Dim}(M) = d_1(M) = s_1(M)$.

We end this section with a proof of this important theorem. However, in order to prove the theorem, we need a few tools. Our first tool is Nakayama's lemma, modified for our context.

Lemma 1.6.34. *[Ser00] Let $M \in \mathcal{C}(R)$, and let I be any ideal of R contained in $\mathfrak{m}$, the superfluous ideal of R. If $IM = M$, then $M = 0$.*

Definition 1.6.35. *Let $M \in \mathcal{C}(R)$ and let $x \in \mathfrak{m}$ be a homogeneous element. Then we define $(x : 0)_M \doteq {}_xM \doteq \{m \in M \mid xm = 0\}$; this is called the* conductor of x to 0 in M.

Note that $(x : 0)_M$ is a graded R-module.

The following lemma is an adaptation of a lemma in Serre (see [Ser00, Sec. III.B.2, Lem. 2]).

Lemma 1.6.36. *Let M be in $\mathcal{C}(R)$, $\mathcal{I}$ be an ideal of definition for M, and $x \in \mathfrak{m}$ be a homogeneous element.*

a. $\mathcal{I}$ is also an ideal of definition for M/xM and for $(x:0)_M$.

b. For $n >> 0$, $p((x:0)_M, \mathcal{I}, n) - p(M/xM, \mathcal{I}, n)$ is a polynomial function of n of degree less than $d_1(M)$.

c. $s_1(M) \leq s_1(M/xM) + 1$.

Proof. For part (a), we see that there are exact sequences of graded R-modules

$$0 \to (x:0)_M \to M \to xM \to 0$$

and

$$0 \to xM \to M \to M/xM \to 0.$$

By Theorem 1.3.12, $\mathcal{I}$ is an ideal of definition of $(x:0)_M$ and M/xM (and xM).

For part (b), using the sequences above and by Lemma 1.6.30, there exist $q_1, q_2 \in \mathbb{Z}[t]$, polynomials with nonnegative leading terms and with

$$\deg q_1(n) \leq \deg p\left((x:0)_M, \mathcal{I}, n\right) - 1$$

and

$$\deg q_2(n) \leq \deg p\left(xM, \mathcal{I}, n\right) - 1$$

such that

$$p(M, \mathcal{I}, n) = p\left((x:0)_M, \mathcal{I}, n\right) + p(xM, \mathcal{I}, n) - q_1(n)$$

and

$$p(M, \mathcal{I}, n) = p\left(xM, \mathcal{I}, n\right) + p(M/xM, \mathcal{I}, n) - q_2(n).$$

So

$$p\left((x:0)_M, \mathcal{I}, n\right) - q_1(n) = p(M/xM, \mathcal{I}, n) - q_2(n)$$

and

$$p\left((x:0)_M,\mathcal{I},n\right)-p(M/xM,\mathcal{I},n)=q_1(n)-q_2(n).$$

Notice that

$$\deg\left(p(M,\mathcal{I},n)\right)=\max\{\deg\left(p\left((x:0)_M,\mathcal{I},n\right)\right),\deg\left(p(xM,\mathcal{I},n)\right)\}.$$

Since $\deg\left(q_1(n)-q_2(n)\right)\leq\deg\left(p(M,\mathcal{I},n)\right)-1$, we know that

$$\deg\left(p\left((x:0)_M,\mathcal{I},n\right)-p(M/xM,\mathcal{I},n)\right)=\deg(q_1-q_2)<d_1(M).$$

Part (c) is a direct application of the definition of $s_1(M)$. □

Recall from Definition 1.6.3 that $\mathcal{V}(M)$ is the set of homogeneous prime ideals in R containing $\operatorname{Ann}(M)$. As stated in Theorem 1.6.7, M has a finite number of associated primes, and the associated primes are all homogeneous prime ideals in R. Certain associated primes are the minimal primes of M.

Definition 1.6.37. *A subset of the minimal primes of M, $\mathcal{D}(M)$, is defined as follows:*

$$\mathfrak{p}\in\mathcal{D}(M)\text{ if and only if }\operatorname{Dim}(R/\mathfrak{p})=\operatorname{Dim}(M).$$

The following lemma is a modification from Serre [Ser00].

Lemma 1.6.38. *[Ser00, Sec. III.B.2, Lem. 2.b]. If $M\in\mathcal{C}(R)$ and $x\in\mathfrak{m}$ is a homogeneous element such that $x\notin\mathfrak{p}$, for every $\mathfrak{p}\in\mathcal{D}(M)$, then $\operatorname{Dim}(M/xM)\leq\operatorname{Dim}(M)-1$.*

The previous two lemmas give us the desired theorem about dimension.

Theorem 1.6.39. *[Ser00] If $M\in\mathcal{C}(R)$, then $\operatorname{Dim}(M)=d_1(M)=s_1(M)$.*

Proof. We begin by showing $\operatorname{Dim}(M)\leq d_1(M)$, using induction on $d_1(M)$. If $d_1(M)=0$, then the polynomial function of n, $p(M,\mathfrak{m},n)$, has degree 0, so $\dim_k(M/\mathfrak{m}^nM)$ is constant for all $n>>0$. Hence, $\mathfrak{m}^nM=\mathfrak{m}^{n+1}M$ for some

n. By Lemma 1.6.34, $\mathfrak{m}^n M = 0$. Thus, M is Artinian as we see by refining the filtration of Proposition 1.3.4, and $\mathrm{Dim}(M) = 0$ using Lemma 1.6.17 applied to $R/Ann(M)$, noting that $\mathfrak{m}^n \subseteq Ann(M)$.

Now we sketch the proof for $\mathrm{Dim}(M) \leq d_1(M)$ for $d_1(M) \geq 1$. Considering a chain $\mathfrak{p}_0 \subset \cdots \mathfrak{p}_m$ of length m of homogeneous prime ideals in R containing $\mathrm{Ann}(M)$, let $\mathfrak{p}_0$ be the minimal prime ideal of the chain. There exists a submodule N of M and an integer d_i such that $N \cong R/\mathfrak{p}_0(-d_i)$, and $\mathfrak{p}_0$ is the only associated prime of N. We show $m \leq d_1(M)$ by induction on m.

The result is trivial for $m = 0$. Consider $m \geq 1$. There is a homogeneous element $y \in \mathfrak{p}_1$ such that $y \notin \mathfrak{p}_0$. Hence, y is not a zero divisor on N and

$$0 \to N \to yN \to N/yN \to 0$$

is a short exact sequence. By Lemma 1.6.38,

$$\mathrm{Dim}(N/yN) \leq \mathrm{Dim}(N) - 1 < \mathrm{Dim}(N)$$

and by Lemma 1.6.36,

$$d_1(N/yN) \leq d_1(N) - 1.$$

By induction,

$$m - 1 \leq d_1(N/yN) \leq d_1(N) - 1,$$

so

$$m \leq d_1(N) \leq d_1(M) \text{ since } N \subseteq M.$$

Therefore, $\mathrm{Dim}(M) \leq d_1(M)$.

Now we show that $d_1(M) \leq s_1(M)$. Let $\mathcal{I} = (x_1, \ldots, x_r) \subseteq \mathfrak{m}$ be such that $M/\mathcal{I}M$ has finite dimension over k. Without loss of generality, we may assume that $\mathrm{Ann}(M) \subseteq \mathcal{I}$. $\mathcal{I}$ is an ideal of definition for M by definition. By Theorem 1.6.29 (b), the degree of the polynomial function of n, $p(M, \mathcal{I}, n)$, is less than or equal to r. Therefore, $d_1(M) \leq s_1(M)$.

Finally, we show that $s_1(M) \leq \mathrm{Dim}(M)$, using induction on $\mathrm{Dim}(M)$, which is finite according to the first part of our proof. Suppose $\mathrm{Dim}(M) = 0$. Since $Ann(M) \subseteq \mathfrak{m}$, $\mathfrak{m}$ is the only homogeneous prime containing $Ann(M)$. Thus, $\mathfrak{m}$ is minimal over $Ann(M)$ and an associated prime to M. Since all associated primes to M are homogeneous, $\mathfrak{m}$ is the only associated prime of M. Hence, $Ann(M)$ is $\mathfrak{m}$-primary, $Ann(M) = \mathfrak{m}$. There exists j such that $\mathfrak{m}^j \subseteq Ann(M)$, so $M = M/\mathfrak{m}^j M$. By Proposition 1.3.4, M is a finite dimensional graded vector space over k, and $s_1(M) = 0$.

Now, suppose that for $\mathrm{Dim}(M) = m - 1$ where $m \geq 1$, we have also that $s_1(M) \leq m - 1$. Consider M such that $\mathrm{Dim}(M) = m \geq 1$. Let $\mathfrak{p}_i \in \mathcal{D}(M)$ as defined in Definition 1.6.37 be the set of prime ideals of $\mathcal{V}(M)$, the homogeneous spectrum of M, such that $\mathrm{Dim}\, R/\mathfrak{p}_i = m$. This is a finite set, and these ideals are not maximal when $m \geq 1$. Hence, there exists a homogeneous element $x \in \mathfrak{m}$ such that $x \notin \mathfrak{p}_i$ for all i (by "prime avoidance": if $\mathfrak{p}_i \subsetneq \mathfrak{m}$ for all i, where $1 \leq i \leq t$, then $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t \subsetneq \mathfrak{m}$.). By Lemma 1.6.36, we have that $s_1(M) \leq s_1\left(M/xM\right) + 1$; by Lemma 1.6.38, we have that $\mathrm{Dim}\left(M/xM\right) \leq \mathrm{Dim}(M) - 1$. Since $\mathrm{Dim}\left(M/xM\right) = m - 1$, we know from the inductive hypothesis that $s_1\left(M/xM\right) \leq \mathrm{Dim}\left(M/xM\right)$, and we have that

$$s_1(M) \leq s_1\left(M/xM\right) + 1 \leq \mathrm{Dim}\left(M/xM\right) + 1 \leq \mathrm{Dim}(M).$$

$\square$

Combining Proposition 1.6.11 and Theorem 1.6.39, we have the following corollary.

Corollary 1.6.40. *For $M \in \mathcal{C}(R)$, we define $D(M) \doteq d(M) = d_1(M) = s(M) = s_1(M) = \mathrm{Dim}(M)$.*

Definition 1.6.41. *If $M \in \mathcal{C}(R)$ with $M \neq 0$, and $\mathrm{Dim}(M) = D$, then a sequence $y_1, \ldots, y_D$ of homogeneous elements of $\mathfrak{m} \subset R$ such that M is a finitely generated $k\left\langle y_1, \ldots, y_D \right\rangle$-module is called a* system of parameters for M*, as an R-module.*

Note that by Proposition 1.6.11, the y_i's are algebraically independent.

1.6.6 Samuel Multiplicity for Graded Rings

Although there are several differently-defined types of multiplicities, in this section we discuss only the classical definition of multiplicity from commutative algebra, the Samuel multiplicity. In a following section, we will define and compare Maiorana's C-multiplicity.

Using Theorems 1.6.27 and 1.6.39, we conclude that the Samuel function

$$n \mapsto \dim_k \left(M/\mathcal{I}^{n+1}M\right)$$

is a polynomial function of n of degree $D(M) \doteq \mathrm{Dim}(M)$, for $n >> 0$. From equation 1.4.0.1, recall for $M \neq 0$ that there is a positive integer

$$e(M, \mathcal{I})$$

such that for $n >> 0$,

$$p(M, \mathcal{I}, n) = \frac{e(M, \mathcal{I})n^{D(M)}}{D(M)!} + \text{ terms of lower degree in } n.$$

In addition, from the same section we conclude that

$$e(M, \mathcal{I}) = \Delta^{D(M)}\left(p(M, \mathcal{I}, n)\right),$$

and if $r > D(M)$,

$$\Delta^r\left(p(M, \mathcal{I}, n)\right) = 0$$

for $n >> 0$.

Definition 1.6.42. *If $M \in \mathcal{C}(R)$ and $\mathcal{I}$ is an ideal of definition for M, then the integer*

$$e(M, \mathcal{I}) \doteq \lim_{n \to \infty} \frac{p(M, \mathcal{I}, n)D(M)!}{n^{D(M)}}$$

is called the Samuel multiplicity of M *with respect to the ideal $\mathcal{I}$.*

The following is a corollary to the proof of Theorem 1.6.31.

Corollary 1.6.43. *Suppose that $\mathcal{I}$ is an ideal of definition for $M \in \mathcal{C}(R)$. If $n_0 - 1$ is the highest degree of a basis element for $M/\mathcal{I}M$ as a graded vector space over k, then*

$$e(M, \mathfrak{m}) \leq e(M, \mathcal{I}) \leq e(M, \mathfrak{m}) n_0^{D(M)}.$$

Proof. From the proof of Theorem 1.6.31, we have that $p(M, \mathfrak{m}, n) \leq p(M, \mathcal{I}, n)$ for all n and $p(M, \mathcal{I}, n) \leq p(M, \mathfrak{m}, n_0 n)$ for all $n \geq 1$. Then for $n >> 0$,

$$\frac{p(M, \mathfrak{m}, n) D(M)!}{n^{D(M)}} \leq \frac{p(M, \mathcal{I}, n) D(M)!}{n^{D(M)}} \leq \frac{p(M, \mathfrak{m}, n_0 n) D(M)!}{(n_0 n)^{D(M)}} n_0^{D(M)}.$$

Applying $n \to \infty$, we have our conclusion. □

1.6.7 Properties of Abstract Multiplicities

A "multiplicity," or "geometric degree," function for $\mathcal{C}(R)$ has the following properties.

A multiplicity function for $\mathcal{C}(R)$ is a correspondence

$$M \mapsto E(M) \in \mathbb{Q}$$

from objects of $\mathcal{C}(R)$ to the rational numbers such that,

1. for an exact sequence in $\mathcal{C}(R)$

$$0 \to L \to M \to N \to 0,$$

   we have the following:

   a. if $D(L) = D(M) = D(N)$, then $E(M) = E(L) + E(N)$,

   b. if $D(L) = D(M) > D(N)$, then $E(M) = E(L)$,

   c. if $D(L) < D(M) = D(N)$, then $E(M) = E(N)$.

and

2. An associativity (or linearity) formula

$$E(M) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} N_{\mathfrak{p}} E\left(R/\mathfrak{p}\right)$$

holds, where $N_{\mathfrak{p}} \in \mathbb{Q}$ for all $\mathfrak{p} \in \mathcal{D}(M)$ as defined in Definition 1.6.37.

Example 1.6.44. *For $M \in \mathcal{C}(R)$ and $\mathcal{I}$ an ideal of definition for M, the Samuel multiplicity, $e(M, \mathcal{I})$, is a multiplicity according to the above properties. That the Samuel multiplicity is a multiplicity in the non-graded case was discussed by Eisenbud [Eis95].*

In the graded case, however, we provide the proof here. For the first property, consider the short exact sequence

$$0 \to L \to M \to N \to 0.$$

From Lemma 1.6.30, we have that

$$p(M, \mathcal{I}, n) + q(n) = p(N, \mathcal{I}, n) + p(L, \mathcal{I}, n)$$

where $q(n)$ is a polynomial function of n whose leading coefficient is positive and whose degree is strictly less than $D(L)$. Suppose that $D(L) = D(M) = D(N)$. Then,

$$\frac{p(M, \mathcal{I}, n)D(M)!}{n^{D(M)}} + \frac{q(n)D(M)!}{n^{D(M)}} = \frac{p(N, \mathcal{I}, n)D(M)!}{n^{D(M)}} + \frac{p(L, \mathcal{I}, n)D(M)!}{n^{D(M)}}.$$

Applying the limit as $n \to \infty$, since $\deg q(n) < D(L) = D(M) = D(N)$, we have

$$e(M, \mathcal{I}) = e(N, \mathcal{I}) + e(L, \mathcal{I}).$$

For the cases $D(L) = D(M) > D(N)$ and $D(L) < D(M) = D(N)$, we proceed similarly with the results $e(M, \mathcal{I}) = e(L, \mathcal{I})$ and $e(M, \mathcal{I}) = e(N, \mathcal{I})$, respectively.

For the second property, using Corollary 1.3.13 and Lemma 1.6.30, we obtain the following theorem, similar to Serre in the non-graded case (see [Ser00, Sec. II.B.4, Prop. 10]).

Theorem 1.6.45. *Let $M \in \mathcal{C}(R)$ and let $\mathcal{I}$ be an ideal of definition of M generated by homogeneous elements $x_1, \ldots, x_{D(M)}$. Let $\mathcal{D}(M)$ be the subset of the set of minimal primes defined by $\mathfrak{p} \in \mathcal{D}(M)$ if and only if* $\mathrm{Dim}(M) = \mathrm{Dim}(R/\mathfrak{p})$*. Then, for every $\mathfrak{p} \in \mathcal{D}(M)$, $\mathcal{I}$ is an ideal of definition for $R/\mathfrak{p}$, and*

$$e(M, \mathcal{I}) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} N_{\mathfrak{p}} e(R/\mathfrak{p}, \mathcal{I}),$$

for certain integers $N_{\mathfrak{p}}$.

Proof. Using Theorem 1.6.7, there is a filtration of M by graded submodules

$$0 = \mathcal{F}^{N+1}(M) \subseteq \mathcal{F}^{N}(M) \subseteq \cdots \subseteq \mathcal{F}^{i+1}(M) \subseteq \mathcal{F}^{i}(M) \subseteq \cdots \subseteq \mathcal{F}^{0}(M) = M,$$

homogeneous prime ideals $\mathfrak{p}_i$ and positive integers d_i, for $0 \leq i \leq N$, with $\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M) \cong (R/\mathfrak{p}_i)(-d_i)$ as R-modules, for every i. Every minimal prime for M occurs as one of the primes $\mathfrak{p}_i$.

By Corollary 1.3.13, since $\mathcal{I}$ is an ideal of definition for M, it is also an ideal of definition for $\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)$, $0 \leq i \leq N$, and thereby $\mathcal{I}$ is an ideal of definition for $(R/\mathfrak{p}_i)(-d_i)$ for $0 \leq i \leq N$. By Proposition 1.3.14, $(R/\mathfrak{p}_i)(-d_i)$ is a finite dimensional vector space over k. Since $\mathcal{I}$ is an ideal of definition for $(R/\mathfrak{p}_i)(-d_i)$, then $\mathcal{I}$ is also an ideal of definition for $R/\mathfrak{p}$ for every $\mathfrak{p} \in \mathcal{D}(M)$.

Notice that we have the short exact sequence

$$0 \to \mathcal{F}^1(M) \to M \to \underbrace{M/\mathcal{F}^1(M)}_{(R/\mathfrak{p}_0)(-d_0)} \to 0.$$

Using Lemma 1.6.30, we conclude that, for $n >> 0$,

$$p(M, \mathcal{I}, n) = p(\mathcal{F}^1(M), \mathcal{I}, n) + p((R/\mathfrak{p}_0)(-d_0), \mathcal{I}, n) - q_0(n)$$

where $q_0(n)$ is a polynomial with positive leading coefficient and with $\deg q_0 < \mathrm{Dim}(\mathcal{F}^1(M))$.

Notice also that we have the short exact sequence

$$0 \to \mathcal{F}^{i+1}(M) \to \mathcal{F}^{i}(M) \to \underbrace{\mathcal{F}^{i}(M)/\mathcal{F}^{i+1}(M)}_{(R/\mathfrak{p}_i)(-d_i)} \to 0$$

for all i such that $0 \leq i \leq N$. Likewise, for $n >> 0$ and for $0 \leq i \leq N$, we have the equation

$$p(\mathcal{F}^{i}(M), \mathcal{I}, n) = p(\mathcal{F}^{i+1}, \mathcal{I}, n) + p((R/\mathfrak{p}_i)(-d_i), \mathcal{I}, n) - q_i(n),$$

where $q_i(n)$ is a polynomial with $\deg q_i < \mathrm{Dim}(\mathcal{F}^{i}(M))$ and with positive leading coefficient.

Since $p((R/\mathfrak{p}_i)(-d_i), \mathcal{I}, n) = p(R/\mathfrak{p}_i, \mathcal{I}, n)$, for $n >> 0$ we obtain the equation

$$p(M, \mathcal{I}, n) = \sum_{i=0}^{N} p(R/\mathfrak{p}_i, \mathcal{I}, n) - \sum_{i=0}^{N} q_i(n).$$

We see that for i when $\mathrm{Dim}(M) > \mathrm{Dim}((R/\mathfrak{p}_i)(-d_i)) = \mathrm{Dim}(R/\mathfrak{p}_\mathfrak{i})$, multiplying by $\frac{D(M)!}{n^{D(M)}}$ and applying the limit as $n \to \infty$ results in 0 for the corresponding $p(R/\mathfrak{p}_i, \mathcal{I}, n)$ terms. In addition, since $\deg q_i(n) < \mathrm{Dim}(\mathcal{F}^{i}(M)) \leq \mathrm{Dim}(M)$ for all $0 \leq i \leq N$, the $q_i(n)$ terms also go to 0 in the limit.

Therefore, we obtain the equation

$$e(M, \mathcal{I}) = \sum_{i=0}^{N} e(R/\mathfrak{p}_i, \mathcal{I}),$$

and, dropping the $R/\mathfrak{p}_i$ where $e(R/\mathfrak{p}_i, \mathcal{I}) = 0$ and letting $N_\mathfrak{p}$ be the number of times that $R/\mathfrak{p}$ occurs as a factor in the filtration, we get

$$e(M, \mathcal{I}) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} N_\mathfrak{p} e(R/\mathfrak{p}, \mathcal{I}).$$

□

1.6.8 Smoke Multiplicity

At this point, we define and state some properties of a second multiplicity—the Smoke multiplicity—for future use. We begin by reviewing some definitions of homological algebra following the treatment in [Smo72].

Now, every free module in $\mathcal{C}(R)$ is of the form $R \otimes_k V$, for some finite dimensional vector space V over k.

Definition 1.6.46. *A surjective map $\theta : L \to M$ in $\mathcal{C}(R)$ is* minimal *if L is free and $\ker \theta \subset \mathfrak{m}L$.*

Definition 1.6.47. *A* minimal resolution *of M in $\mathcal{C}(R)$ is a resolution*

$$\cdots \xrightarrow{\theta_2} R \otimes_k V_1 \xrightarrow{\theta_1} R \otimes_k V_0 \xrightarrow{\theta_0} M \to 0$$

of M by free modules in $\mathcal{C}(R)$ such that θ_0 is a minimal surjection and $\theta_i : R \otimes_k V_i \to \ker \theta_{i-1}$ is a minimal surjection for every $i > 0$.

Theorem 1.6.48. *[Smo72, Cor. 2.2]. Every module M in $\mathcal{C}(R)$ has a minimal resolution.*

Theorem 1.6.49. *[Smo72, Prop. 2.3]. Given a minimal resolution*

$$\cdots \xrightarrow{\theta_2} R \otimes_k V_1 \xrightarrow{\theta_1} R \otimes_k V_0 \xrightarrow{\theta_0} M \to 0$$

of M, there are graded vector space isomorphisms

$$\mathrm{Tor}_i^R(M, k) \cong V_i,$$

for every $i \geq 0$.

Considering the vector spaces V_i of Theorem 1.6.49, we see that since V_i is a finite-dimensional graded vector space over k for every i, we must have $\dim_k(V_i)_j = 0$ for $j >> 0$.

Corollary 1.6.50. *Given $M \in \mathcal{C}(R)$, the Poincaré series of the graded module $\mathrm{Tor}_i^R(M, k)$*

$$\mathrm{PS}(\mathrm{Tor}_i^R(M, k), t) = \sum_{j=0}^{\infty} \dim_k(\mathrm{Tor}_i^R(M, k)_j) t^j$$

is a polynomial in t for every i.

Consider a graded polynomial ring $S = k[x_1, \ldots, x_n]$ over k, where $x_1, \ldots, x_n$ are algebraically independent of positive degree over k, and $M \in \mathcal{C}(S)$. Using the Hilbert syzygy theorem (Theorem 1.13 of [Eis95]), k has a finite free resolution by finitely generated free graded S-modules F_i

$$0 \to F_n \to \cdots \to F_0 \to k = S/(x_1, \ldots, x_n) \to 0.$$

Theorem 1.6.51. *[Smo72] Let $M \in \mathcal{C}(S)$ where S is defined as above. The Poincaré series*

$$\mathrm{PS}\left(\mathrm{Tor}_i^S(M,k), t\right) = \sum_j \dim_k \left(\mathrm{Tor}_i^S(M,k)_j\right) t^j$$

is a polynomial with nonnegative integer coefficients.

On the other hand, the existence of the above finite free resolution tells us that $\mathrm{Tor}_i^S(M,k) = 0$ for $i > n$. Therefore, we may define a polynomial (the Smoke polynomial) with integer coefficients as follows:

Definition 1.6.52. *(Smoke Polynomial) For $S = k[x_1, \ldots, x_n]$ a graded polynomial ring over k, where $x_1, \ldots, x_n$ are algebraically independent of positive degree over k and $M \in \mathcal{C}(S)$, we define the polynomial*

$$\chi(S, M) \doteq \sum_i (-1)^i \, \mathrm{PS}\left(\mathrm{Tor}_i^S(M,k), t\right).$$

Evaluating this polynomial at $t = 1$ results in an integer.

Definition 1.6.53. *(Smoke Multiplicity for S) For $S = k[x_1, \ldots, x_n]$ a graded polynomial ring over k, where $x_1, \ldots, x_n$ are algebraically independent of positive degree over k and $M \in \mathcal{C}(S)$, we define the multiplicity*

$$e(S, M) \doteq \chi(S, M)(1) = \sum_i (-1)^i \dim_k \mathrm{Tor}_i^S(M,k) \in \mathbb{Z}.$$

Using these definitions, Smoke proves the following theorems.

Theorem 1.6.54. *[Smo72, Thm. 3.1]. If S is any finitely generated graded polynomial ring over k and $M \in \mathcal{C}(S)$, then $\chi(S, M) = \chi(S, k)\,\mathrm{PS}(M, t)$.*

Theorem 1.6.55. *[Smo72, Prop. 4.1] Let S be the graded polynomial ring $S = k[x_1, \ldots, x_n]$ where the degree of x_i is equal to d_i for every i. Then*

$$\chi(S, k) = \prod_{i=1}^{n}(1 - t^{d_i}).$$

Corollary 1.6.56. *If the graded polynomial ring $S = k[x_1, \ldots, x_n]$ and the degree of x_i equals d_i for every i, then for $M \in \mathcal{C}(S)$,*

$$\mathrm{PS}(M, t) = \frac{\chi(S, M)}{\prod_{i=1}^{n}(1 - t^{d_i})}.$$

Theorem 1.6.57. *[Smo72, Cor. 6.5]. If S is any finitely generated graded polynomial ring over k and M is in $\mathcal{C}(S)$, then $e(S, M) \geq 0$. Furthermore, if $M \neq 0$, then $e(S, M) > 0$.*

Now we return to the case of the general graded ring R, as defined previously, and consider $M \in \mathcal{C}(R)$. Choose a system of parameters $y_1, \ldots, y_{D(M)}$ of degrees $d_1 \leq d_2 \leq \cdots \leq d_{D(M)}$ for M. (We know we can do this by the definition of $s(M)$ in Definition 1.5.10.)

Definition 1.6.58. *Let the ideal of R generated by $y_1, \ldots, y_{D(M)}$ be defined as*

$$\mathcal{I}(\bar{y}) = (y_1, \ldots, y_{D(M)}),$$

and let the polynomial subring of R generated by $y_1, \ldots, y_{D(M)}$ be denoted as

$$S_{\bar{y}} = k[y_1, \ldots, y_{D(M)}].$$

$S_{\bar{y}}$ is a graded polynomial subring of R using Proposition 1.6.11, and M is a finitely generated module over $S_{\bar{y}}$.

Definition 1.6.59. *Let $M \in \mathcal{C}(R)$, and let $y_1, \dots, y_{D(M)}$ be a system of parameters for M as an R-module. If $M \neq 0$, the positive integer $e(\bar{y}, M) \doteq e(S_{\bar{y}}, M)$ is the Smoke multiplicity of M with respect to the system of parameters $y_1, \dots, y_{D(M)}$.*

Lemma 1.6.60. *[Duf08] Let $M \in \mathcal{C}(R)$, and let $y_1, \dots, y_{D(M)}$ be a system of parameters for M as an R-module. Then the following multiplicities are equal:*

- *the Samuel multiplicity $e(M, \mathcal{I}(\bar{y}))$, the leading coefficient in the polynomial in n that computes $\dim_k M/\mathcal{I}(\bar{y})^{n+1}M$, for $n >> 0$, multiplied by $D(M)!$*
- *the Smoke multiplicity $e(\bar{y}, M) \doteq \sum_j (-1)^j \dim_k \operatorname{Tor}_j^{S_{\bar{y}}}(M, k)$.*

Furthermore, the common number defined by these multiplicities is a nonnegative integer, and if $M \neq 0$, this number is positive.

1.6.9 Maiorana's Multiplicity

We are now ready to study in more detail the multiplicity with which this paper is concerned, Maiorana's C-multiplicity. Recall the definition of $C(M)$ in Definition 1.5.6. Let $y_1, \dots, y_{D(M)}$ be a system of parameters for M as an R-module, with $\deg y_i = d_i$.

We have shown that if $M \in \mathcal{C}(R)$, $M \neq 0$, then $D(M) = d(M) = d_1(M) = s(M) = s_1(M) = \operatorname{Dim}(M)$. Using Corollary 1.6.56, we see that

$$\operatorname{PS}(M, t) = \frac{\chi(S_{\bar{y}}, M)}{\prod_{i=1}^{D(M)} (1 - t^{d_i})}.$$

Therefore,

$$(1-t)^{D(M)} \operatorname{PS}(M, t) = \frac{\chi(S_{\bar{y}}, M)}{\prod_{i=1}^{D(M)} (1 + t + \cdots + t^{d_i - 1})}. \qquad (1.6.9.2)$$

Proposition 1.6.61. *If $M \in \mathcal{C}(R)$ with $M \neq 0$, $\ell(M)$, the order of the pole of $PS(M, t)$ at $t = 1$, is exactly $D(M) = d(M) = d_1(M) = s(M) = s_1(M) = \operatorname{Dim}(M)$.*

Proof. By Theorem 1.6.57, $\chi(S_{\bar{y}}, M)(1) = e(S_{\bar{y}}, M) > 0$. Thus $\chi(S_{\bar{y}}, M)$ has no zero at 1, and using equation 1.6.9.2, we see that $(1-t)^{D(M)}\,\mathrm{PS}(M,t)$ is a rational function whose numerator and denomenator have no zero at 1. Therefore, the order of the pole at $t = 1$, by definition, of $\mathrm{PS}(M,t)$ must be equal to $D(M)$. $\square$

Other than the fact that $C(M)$ is not always an integer, Maiorana's C-multiplicity behaves "like a multiplicity should." Lemma 1.5.7 demonstrates that C behaves like a multiplicity for properties under (1) in Section 1.6.7. Now we show that C satisfies property (2) for multiplicities.

Lemma 1.6.62. *Suppose that $M \in \mathcal{C}(R)$ and $M \neq 0$. Let*

$$0 = \mathcal{F}^{n+1}(M) \subseteq \mathcal{F}^{n}(M) \subseteq \cdots \subseteq \mathcal{F}^{0}(M) = M$$

be a sequence of graded submodules of M. Let $\mathfrak{D}(\mathcal{F})$ be the set of indices i with $D\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right) = D(M)$. Then $\mathfrak{D}(\mathcal{F})$ is nonempty and

$$C(M) = \sum_{i \in \mathfrak{D}(\mathcal{F})} C\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right).$$

Proof. To prove this lemma, we combine Proposition 1.6.61 and Lemma 1.5.7, replacing ℓ in 1.5.7 with D.

With the given filtration of M, we have that

$$PS(M,t) = \sum_{i=0}^{n} PS\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M), t\right).$$

Multiplying both sides by $(1-t)^{D(M)}$ and taking the limit as $t \to 1$, and we have

$$\lim_{t\to 1}(1-t)^{D(M)} PS(M,t) = \sum_{i=0}^{n} \lim_{t\to 1}(1-t)^{D(M)} PS\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M), t\right).$$

By definition, the left hand side is $C(M)$. We know that, for all i,

$$D(M) \geq D\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right).$$

We have two possibilities to consider when determining the right hand side.

It may be that $D(M) > D\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right)$ for some of the $i = 0, \ldots, n$. Then, after simplifying, we have $(1-t)^m$, where $m = D(M) - D\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right)$, remaining, and the limit as $t \to 1$ is zero. These $\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)$ may as well be disregarded in the right hand side.

For the remaining i, $D(M) = D\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right)$. We collect these i into a set, $\mathfrak{D}(\mathcal{F})$. The right hand side, then, becomes

$$\sum_{i \in \mathfrak{D}(\mathcal{F})} C\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right),$$

and we have the desired sum.

Now suppose that $\mathfrak{D}(\mathcal{F}) = \emptyset$. Then, as shown above,

$$D(M) > D\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right)$$

for all i, resulting in $C(M) = 0$. This occurs only when $M = 0$. Therefore, $\mathfrak{D}(\mathcal{F})$ is nonempty.

$\square$

The above lemma yields the following theorem.

Theorem 1.6.63. *Let $M \in \mathcal{C}(R)$. Let $\mathcal{D}(M)$ be the subset of the set of minimal primes defined by: $\mathfrak{p} \in \mathcal{D}(M)$ if and only if* $\mathrm{Dim}(M) = \mathrm{Dim}\left(R/\mathfrak{p}\right)$. *Then*

$$C(M) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} N_{\mathfrak{p}} C\left(R/\mathfrak{p}\right),$$

for certain integers $N_{\mathfrak{p}}$.

Proof. Using theorem 1.6.7, we have a filtration of M by graded submodules

$$0 = \mathcal{F}^{n+1}(M) \subseteq \mathcal{F}^n(M) \subseteq \cdots \subseteq \mathcal{F}^0(M) = M,$$

homogeneous prime ideals $\mathfrak{p}_i$, and positive integers d_i, for $1 \leq i \leq n$, with

$$\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right) \cong R/\mathfrak{p}_i(-d_i)$$

as R-modules, for every i. We know that every minimal prime occurs as at least one of the primes $\mathfrak{p}_i$.

The proof follows directly from Lemma 1.6.62 with simply a different way of "bookkeeping." Note first that since

$$\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right) \cong R/\mathfrak{p}_i(-d_i)$$

for every i, we have

$$\mathrm{Dim}\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right) = \mathrm{Dim}\left(R/\mathfrak{p}_i(-d_i)\right)$$

and

$$C\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right) = C\left(R/\mathfrak{p}_i(-d_i)\right)$$

for all i.

Now, instead of keeping track of each $\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)$ for which

$$\mathrm{Dim}\, M = \mathrm{Dim}\, \mathcal{F}^i(M)/\mathcal{F}^{i+1}(M),$$

as we did in Lemma 1.6.62, we keep track of the homogeneous prime ideals, $\mathfrak{p}_\mathfrak{i}$, for which

$$\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right) \cong R/\mathfrak{p}_i(-d_i)$$

and

$$\mathrm{Dim}(M) = \mathrm{Dim}(R/\mathfrak{p}_i) = \mathrm{Dim}\left(\mathcal{F}^i(M)/\mathcal{F}^{i+1}(M)\right),$$

and let $N_{\mathfrak{p}_i}$ be the number of times that homogeneous ideal is used (since there may be $j \neq i$ such that $\mathfrak{p}_i = \mathfrak{p}_j$). $\square$

Therefore, by Lemma 1.5.7 and Theorem 1.6.63, $C(M)$ does act "as a multiplicity should."

Now, using Theorem 1.6.60 and the equation

$$(1-t)^{D(M)}\,\mathrm{PS}(M,t) = \frac{\chi(S_{\bar{y}}, M)}{\prod\limits_{i=1}^{D(M)} \left(1 + t + \cdots + t^{d_i - 1}\right)},$$

we have the following corollary which relates Maiorana's C-multiplicity with Smoke's and Samuel's multiplicities.

Corollary 1.6.64. *If $M \in \mathcal{C}(R)$ and $y_1, \ldots, y_{D(M)}$ of degrees $d_1 \leq d_2 \leq \cdots \leq d_{D(M)}$ form a system of parameters for M as an R-module, then*

$$C(M) = \frac{e(\bar{y}, M)}{d_1 \cdots d_{D(M)}} = \frac{e(M, \mathcal{I}(\bar{y}))}{d_1 \cdots d_{D(M)}},$$

and $C(M) > 0$ for $M \neq 0$. Furthermore, the ratio

$$\frac{e(\bar{y}, M)}{d_1 \cdots d_{D(M)}}$$

is independent of the choice of system of parameters $y_1, \ldots, y_{D(M)}$ for M.

Chapter 2

SMOOTH ACTIONS OF FINITE GROUPS ON MANIFOLDS

In this chapter, we begin by defining topological groups, compact Lie groups, and other basic concepts. We discuss Milnor's construction of the universal fibration $G \to EG \to BG$. We then consider Borel's construction of equivariant cohomology and end the chapter with a discussion of tubes and smooth actions.

Definition 2.0.65. *A* topological group G *is a set* G *together with a group structure and topology on* G *such that the function* $(st) \mapsto st^{-1}$ *is a continuous function* $G \times G \to G$.

A topological manifold is a second countable Hausdorff space which is locally homeomorphic to Euclidean space by a collection (called an *atlas*) of homeomorphisms called *charts*. An atlas on a topological space X is a collection of pairs $\{(U_i, \phi_i)\}$ called charts, where the U_i are open sets which cover X, and for each i, $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism of U_i onto an open subset of $\mathbb{R}^n$. The composition of one chart with the inverse of another chart is a function called a *transition map* and defines a homeomorphism of an open subset of Euclidean space onto another open subset of Euclidean space. The transition maps of the atlas are functions $\phi_{i,j} = \phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$. One defines a *topological manifold* to be a space as above with an equivalence class of atlases.

Definition 2.0.66. *A* differentiable manifold *is a topological manifold equipped with an atlas whose transition maps are all differentiable. More generally, a C^n-manifold is a topological manifold with an atlas whose transition maps are all n-times continuously differentiable. A* smooth manifold, *or C^∞-manifold, is a differentiable manifold for which all of the transition maps are smooth; that is, derivatives of all orders exist.*

Definition 2.0.67. *Let G be a group. A* Lie group *is a differentiable manifold G such that the group multiplication*

$$\mu : G \times G \to G$$

and the map sending g to g^{-1} are differentiable maps.

For us, the word *differentiable* means infinitely differentiable. Throughout this paper, we use the terms differentiable and smooth interchangeably.

For A an $n \times n$ matrix with complex numbers as entries, we define A^T to be the transpose of A, $\bar{A}$ to be the complex conjugate of A, and $A^* = \bar{A}^T$, the Hermitian adjoint of A.

Definition 2.0.68. *Let k be either the field $\mathbb{R}$ or the field $\mathbb{C}$. The* orthogonal group *$O(n, k)$ is defined as*

$$O(n, k) = \{A \in GL_n(k) \mid AA^T = I\}.$$

The unitary group *$U(n, k)$ is defined as*

$$U(n, k) = \{A \in GL_n(k) \mid AA^* = I\}.$$

For convenience, we let $O(n) \doteq O(n, \mathbb{R})$, and we let $U(n) \doteq U(n, \mathbb{C})$.

Example 2.0.69. *Suppose that $n = 1$. Note that for $z \in \mathbb{C}$, $z^* = \bar{z}$. Then*

$$U(1) = \{z \in \mathbb{C} | z\bar{z} = |z|^2 = 1\} = S^1 \subset \mathbb{C} = \mathbb{R}^2,$$

where S^1 is the circle group, a closed subspace of $\mathbb{C}$. Notice that $U(1)$ is a real manifold of dimension 1.

Suppose that $n = 2$. Then we have

$$U(2) = \left\{ \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \mid \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \begin{bmatrix} \bar{z_1} & \bar{z_2} \\ \bar{z_3} & \bar{z_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq \mathbb{C}^2 = \mathbb{R}^4.$$

Multiplying the matrices in the definition of $U(2)$ results in the following equations:

$$z_1\bar{z_1} + z_2\bar{z_2} = |z_1|^2 + |z_2|^2 = 1$$

$$z_3\bar{z_3} + z_4\bar{z_4} = |z_3|^2 + |z_4|^2 = 1$$

$$z_1\bar{z_3} + z_2\bar{z_4} = 0$$

$$z_3\bar{z_1} + z_4\bar{z_2} = 0.$$

Each of the first two equations result in S^3, giving us a total dimension of 6 so far. The last two equations are duals of each other, so we have only one extra condition. Without loss of generality, we will consider the third equation. In real coordinates, we write

$$\begin{aligned} 0 &= (x_1 + iy_1)(x_3 - iy_3) + (x_2 + iy_2)(x_4 - iy_4) \\ &= (x_1x_3 + y_1y_3) + i(x_3y_1 - x_1y_3) + (x_2x_4 + y_2y_4) + i(x_4y_2 - x_2y_4) \\ &= [x_1x_3 + y_1y_3 + x_2x_4 + y_2y_4] + i\,[x_3y_1 - x_1y_3 + x_4y_2 - x_2y_4] \end{aligned}$$

resulting in the two equations

$$x_1x_3 + y_1y_3 + x_2x_4 + y_2y_4 = 0$$

and

$$x_3y_1 - x_1y_3 + x_4y_2 - x_2y_4 = 0,$$

each of which decreases the real dimension by 1. Hence, $U(2)$ has dimension $6 - 2 = 4$ over $\mathbb{R}$.

In fact, it turns out that for any n, $U(n)$ has real dimension n^2.

Theorem 2.0.70. *[Bre72] $U(n)$ is a differentiable manifold of dimension n^2 (over $\mathbb{R}$).*

Since A^T and A^* are continuous functions of A and multiplication of matrices is continuous, the orthogonal group and the unitary group are closed subgroups of $GL_n(k)$. In addition, since A^* is a continuous function of A and $AA^* = I$ defines $U(n)$, we see that $U(n)$ is bounded in the metric space of all $n \times n$ matrices over $\mathbb{C}$. Therefore, $U(n)$ is a compact (as a result of the Heine-Borel Theorem) topological space, as is $O(n)$ since it is a closed subgroup of $U(n)$.

Theorem 2.0.71. *[Pri77, Thm. 6.1.1] A compact topological group is a Lie group if and only if it is isomorphic to a closed subgroup of $U(n)$ for some n.*

For an example of a proof to the above theorem, see Price [Pri77], although it was originally proven by Pontrjagin [Pon34].

Example 2.0.72. *Examples of Compact Lie Groups*

a. *finite groups (with the discrete topology)*

b. *the circle group, $S^1 = U(1)$*

c. *orthogonal groups $O(n)$*

d. *unitary groups $U(n)$*

2.1 Group Actions on Topological Spaces

We now turn to the topological formation of cohomology of groups. This process will require some basic topological understanding, the development of the join of topological spaces and the strong topology, and an understanding of connectedness. It is assumed that the reader is familiar with these concepts. We begin our discussion with some basic definitions. Then, we will consider some properties of the contractible G-space, EG, on which G acts freely, based on Milnor's construction. We will see that (EG, p, BG) is a fiber bundle with structure and fiber G and with EG contractible. Finally, after some examples of EG given particular groups, we will conclude by stating the connection between the algebraic and topological formulations of cohomology of groups.

The material in this chapter follows the exposition of [Mil56b]. Other sources might include [Mun00], [Mun84], [Mil56a], [Hus66], [Spa66] and [Ste65].

2.1.1 Background

We begin with defining a G-space and set some related important notation.

Definition 2.1.1. *Let G be a topological group. Then, a* left G-space *is a topological space X together with a continuous function $G \times X \to X$ such that $(g, x) \mapsto gx$ and the following axioms hold:*

a. The relation $(gh)x = g(hx)$ holds for each $x \in X$ and for each $g, h \in G$.

b. For 1 the identity of G, the relation $1x = x$ holds for each $x \in X$.

Right G-spaces are defined similarly.

Definition 2.1.2. *If G is a Lie group acting on a differentiable manifold X and the function $G \times X \to X$ defining the group action is differentiable, then G* acts smoothly (or differentiably) *on X.*

Definition 2.1.3. *Let X be a G-space. For $x \in X$, the subspace*

$$Gx = \{gx \in X \mid g \in G\}$$

is called the orbit *of x under the action of G. The* isotropy group *of $x \in X$ is*

$$G_x = \{g \in G \mid gx = x\}.$$

Notice that

$$G_{gx} = gG_xg^{-1}$$

for all $g \in G$ and for all $x \in X$.

Definition 2.1.4. *Let X/G denote the set whose elements are the orbits Gx of G on X. Let $\pi : X \to X/G$ denote the canonical projection map taking $x \mapsto Gx$. Then we say that X/G under the quotient topology is called the* orbit space *of X (with respect to G).*

Definition 2.1.5. *Let G be a topological group, and let X be a G-space. Then for every subgroup H of G, define the* fixed points *of X as the set*

$$X^H \doteq \{x \in X \mid hx = x \textit{ for every } h \in H\}.$$

Proposition 2.1.6. *Let G be a group. Then the following hold:*

a. If G is a topological group acting on a Hausdorff topological space X, then G_x is closed for every $x \in X$. [tD87, Prop. 3.5]

b. If G is a compact Lie group, then every closed subgroup H is a submanifold of G; i.e., H is a compact Lie group also. [BtD85, Prop. 3.11]

c. If G is a compact Lie group acting on a Hausdorff topological space X, then G_x is a compact Lie group for every $x \in X$.

Definition 2.1.7. *For all $A \leq G$, G a group, the* conjugacy class *of A is*

$$[A] = \{B \leq G \mid B \sim A\},$$

where A is conjugate *to B if and only if there exists $g \in G$ such that $gAg^{-1} = B$. For $A, B \leq G$, we say that A is* subconjugate *to B ($A \lesssim B$) if and only if there exists $g \in G$ such that $gAg^{-1} \subseteq B$. In addition, we say that*

$$[A] \leq [B] \Leftrightarrow A \lesssim B.$$

Lemma 2.1.8. *Let X be a topological space, and let $A \leq G$ where G is a topological group acting on X. Then $x \in X^A$ if and only if $A \leq G_x$. More generally, $z \in GX^A \doteq \{gx \mid g \in G, x \in X^A\}$ if and only if $A \lesssim G_z$.*

Proof. First,

$$x \in X^A \Leftrightarrow ax = x \text{ for all } a \in A \Leftrightarrow a \in G_x \text{ for all } a \in A \Leftrightarrow A \leq G_x.$$

Next, for $z \in GX^A$, there exists $g \in G$ and $x \in X^A$ such that $z = gx$. Now, $G_z = G_{gx} = gG_xg^{-1}$. Since $A \leq G_x$, we have that $gAg^{-1} \leq gG_xg^{-1} = G_z$.

Conversely, if $A \lesssim G_z$, for $z \in X$, then there exists a $g \in G$ such that $gAg^{-1} \leq G_z$, so that $A \leq G_{g^{-1}z}$. Thus, $g^{-1}z \in X^A$, so that $z \in GX^A$. □

Proposition 2.1.9. *([tD87]) If G is a compact Lie group, X a smooth G-manifold and H any isotropy group of X, then $X_{(H)} = \{x \in X \mid [G_x] = [H]\}$ is a submanifold of X.*

From the above proposition, we can conclude that $X^G = X_{(G)}$ is a closed submanifold ([tD87]); more generally, we have the following lemma.

Lemma 2.1.10. *Let G be a compact Lie group, A be a closed subgroup of G, and X be a differentiable G-manifold. Then GX^A is a closed submanifold of X.*

Proof. : Let $A \leq G$. Then A is also a compact Lie group. Therefore, X^A is a closed submanifold of X.

There is a differentiable surjective map $\theta : G \times X^A \to GX^A$ where $(g, x) \mapsto gx$ such that $(gh)x = g(hx)$ and $ex = x$ for all $g, h \in G$ and $x \in X^A$. Therefore, since both G and X^A are compact, GX^A is a compact (and thereby closed) subspace of X for every $A \leq G$.

Now, Lemma 2.1.8 shows that $GX^A = \{x \in X \mid [G_x] \geq [A]\}$. Therefore,

$$GX^A = \bigcup_{[A] \leq [H]} X_{(H)},$$

where the "H" in the index set is a subgroup of G. Since the above union is a disjoint union, GX^A must be a submanifold of X, using Proposition 2.1.9 . □

Lemma 2.1.11. *: Let G be a compact Lie group, A be a subgroup of G, and X be a differentiable G-manifold. Then GX^A is G-invariant.*

Proof. : For all $h \in G$, $h(GX^A) = h \underset{g \in G}{\cup} gX^A = \underset{g \in G}{\cup} hgX^A$. Since $hg \in G$ for all $g \in G$, $hgX^A \in GX^A$ for all $g \in G$. Thus, $hGX^A \subseteq GX^A$ for all $h \in G$. □

2.1.2 Milnor's Construction

The results in this section were established by Milnor in [Mil56a] and [Mil56b]. We do not give the details of the construction here but refer the reader to Milnor's papers.

Let G be an arbitrary topological group. Let $EG \doteq G * G * \cdots * G * \cdots$., the infinite join of copies of G in the strong topology. (For a definition of the strong topology, see [Mil56b].)

An element of EG is $\langle g, t \rangle = (t_0 g_0, t_1 g_1, \ldots, t_k g_k, \ldots)$ where $g_i \in G$ and $t_i \in [0, 1]$ such that a finite number of $t_i \neq 0$ and $\sum_{0 \leq i} t_i = 1$. In EG, $\langle g, t \rangle = \langle \hat{g}, \hat{t} \rangle$ if and only if $t_i = \hat{t}_i$ for all i, and $g_i = \hat{g}_i$ for all i such that $t_i = \hat{t}_i > 0$. Notice that if $t_i = \hat{t}_i = 0$, g_i may not be equal to $\hat{g}_i$ but $\langle g, t \rangle = \langle \hat{g}, \hat{t} \rangle$.

If we define $EG(n)$ as the finite join $G * G * \cdots * G$ of $n + 1$ copies of G, we can think of an element of $EG(n)$ as $\langle g, t \rangle = (t_0 g_0, t_1 g_1, \ldots, t_n g_n, 0, 0, 0, \ldots)$ in EG. Clearly, $EG(n) \subset EG$ for all n. Notice also that $EG(n) \subset EG(n + 1)$ since any element in $EG(n)$ can be thought of as $\langle g, t \rangle = (t_0 g_0, t_1 g_1, \ldots, t_n g_n, 0) \in EG(n+1)$. Therefore, we get the ascending chain $\cdots \subset EG(n) \subset EG(n+1) \subset \cdots \subset EG$, and we also get $EG = \cup EG(n)$.

The right action of G on $EG(n)$ is defined by

$$\begin{aligned} \langle g, t \rangle h &= (t_0 g_0, t_1 g_1, \ldots, t_n g_n) h \\ &= (t_0 g_0 h, t_1 g_1 h, \ldots, t_n g_n h) \\ &= (t_0 (g_0 h), t_1 (g_1 h), \ldots, t_n (g_n h)) \end{aligned}$$

for $h \in G$.

In order to make EG a G-space, we need the following topology. We have two families of functions, $t^i : EG \to [0, 1]$ where $(t_0 g_0, t_1 g_1, \ldots) \mapsto t_i \in [0, 1]$ and

$g^i : (t^i)^{-1}(0,1] \to G$ where $(t_0g_0, t_1g_1, \ldots) \mapsto g_i \in G$ for $0 \leq i$. Notice that for $(t^i)^{-1}(0)$ we do not get a unique $g_i \in G$. When $s \in EG$ and $h \in G$, we have the relations $g^i(sh) = (g^i(s))h$ and $t^i(sh) = t^i(s)$ between the functions $g^i$and t^i with the action of G. Let EG have the smallest topology such that the functions t^i and g^i are continuous and let $(t^i)^{-1}(0,1] \subset EG$ have the subspace topology.

With the topology described above, one sees that EG is a G-space.

Definition 2.1.12. *A group G acts* freely *on a topological space X if and only if whenever $g \in G$, $x \in X$, and $g \cdot x = x$, then $g = e$ (the identity in G).*

Note that by using the definition of above, the G-action on EG is free.

Definition 2.1.13. *A topological space X is* contractible *if the identity map $i_X : X \to X$ is nullhomotopic (that is, homotopic to a constant map).*

It turns out that EG is the total space of a principal fiber bundle (see [Hus66] for a definition) whose base space is the space BG, the orbit space of the G-action on EG, where $b \sim \hat{b}$ if and only if $\hat{b} \in Gb$, the orbit of b. We will let $p : EG \to BG$ be this identification map, and we define the resulting bundle by $\omega_G = (EG, p, BG)$. One can show that BG is a CW-complex since EG is a CW-complex, and the G-action on EG is free; we summarize this in the following theorem.

Theorem 2.1.14. *([Mil56a]). Let the G-space EG be as defined in the Milnor construction. Then,*

a.) EG is contractible.

b.) EG is a CW-complex.

c.) (EG, p, BG) is a principal G-bundle.

In fact, (EG, p, BG) is a universal G-bundle. For more about universal bundles, see [Hus66].

The theory (see [Hus66] and [Mil56b]) says that a property of the universal G-bundle $\omega_G = (EG, p, BG)$ is that ω_G is the unique principal G-bundle (up to equivalence) $E \to B$ such that

a. the total space E is contractible,

b. G acts freely on the total space, and

c. $B \cong E/G$.

Example 2.1.15. *Examples of EG and BG*

a. Let $G = \mathbb{Z}/2$, a finite group. Then, the space $EG(n)$ is S^n (up to homeomorphism). We think of G as the two homeomorphisms 1 and a of S^n where 1 is the identity map and a is the antipodal map. The base space $BG(n)$ is $\mathbb{R}P^n$, the real n-dimensional projective space.

b. Let $G = S^1$, an infinite group. Then, the space $EG(n)$ is S^{2n+1} (up to homeomorphism). For each $e^{i\theta} \in G$, the right action of G on $EG(n)$ defined by the relation $(z_0, z_1, \ldots, z_n)e^{i\theta} = (z_0 e^{i\theta}, z_1 e^{i\theta}, \ldots, z_n e^{i\theta})$. The base space $BG(n)$ is $\mathbb{C}P^n$, the complex n-dimensional projective space.

For any space Y, and commutative ring Λ, there exists the *cup product* pairing $H^n(Y, \Lambda) \times H^m(Y, \Lambda) \xrightarrow{\cup} H^{n+m}(Y, \Lambda)$ such that, for homogeneous elements $x \in H^n(Y, \Lambda)$ and $y \in H^m(Y, \Lambda)$, we have $x \cup y = (-1)^{nm} y \cup x$.

Lemma 2.1.16. *[Rot] If Y is a topological space and Λ is a commutative ring, then $H^*(Y, \Lambda) = \oplus H^n(Y, \Lambda)$ is a graded ring under cup product.*

Let G be a finite group with the discrete topology. Then, $H^*(BG, \Lambda)$, the singular cohomolgy of BG with coefficients in Λ, is isomorphic to $H^*(G, \Lambda) =$

$Ext^*_{\mathbb{Z}G}(\mathbb{Z}, \Lambda)$, using the algebraic definition of group cohomology [Bro82], [Ben04]. This is an important connection between group cohomology and topological cohomology. For more about the algebraic definition of group cohomology, see [Rot79].

Example 2.1.17. *Examples Computing Cohomology*

a. *Let $G = \mathbb{Z}/p$ for $p \neq 0$, $G = S^1$ for $p = 0$, and k be a field of characteristic p. Using cohomology with coefficients in k, we have*

$$H^*(BG) = \begin{cases} k[t] & p = 0, \\ k[s] & p = 2, \\ k[t] \otimes \wedge[s] & p \neq 0, 2. \end{cases} \tag{2.1.2.1}$$

where $t \in H^2(BG)$ and $s \in H^1(BG)$. [tD87, pg. 200].

b. *Let $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, with n factors. With coefficients in the field k,*

$$H^*_G = H^*(BG) = \begin{cases} k[x_1, \ldots, x_n] & p \text{ even}, \\ k[y_1, \ldots, y_n] \otimes_k \wedge[x_1, \ldots, x_n] & p \text{ odd}. \end{cases} \tag{2.1.2.2}$$

where $x_i \in H^1(BG)$ and $y_i \in H^2(BG)$ for all $i = 1, \ldots, n$, and where $\wedge[x_1, \ldots, x_n]$ is a graded vector space over $k = \mathbb{Z}/p$ with graded basis

$$x_{i_1} x_{i_2} \cdots x_{i_j}, 1 \leq i_1 < \cdots < i_j \leq n,$$

and the following properties hold:

(a) *$x_i x_j = -x_j x_i$ for all $i \neq j$,*

(b) *$x_i x_i = 0$ for all i,*

(c) *distributive laws*

(d) *$(\alpha_1 \hat{x}_1)(\alpha_2 \hat{x}_2) = \alpha_1 \alpha_2 (\hat{x}_1 \hat{x}_2)$ where $\hat{x}_1 = x_{i_1} \cdots x_{i_l}$, $\hat{x}_2 = x_{j_1} \cdots x_{j_m}$ and $\alpha_1, \alpha_2 \in k$.*

2.2 The Borel Construction

In this section, we let G be a compact Lie group, let X be a G-space, and choose a principal G-bundle $p : EG \to BG$ with EG contractible. $EG \times X$ has the diagonal action (i.e., $g(a, b) = (ga, gb)$ for all $g \in G$, for all $a \in EG$, and for all $b \in X$) since X is a G-space, and we can form the associated orbit space

$$EG \times_G X \doteq (EG \times X)/G \doteq X_G.$$

Example 2.2.1. *Suppose that $X = \{x_0\}$, a one-point G-space. Then one sees that*

$$EG \times_G \{x_0\} \cong BG.$$

Note that the universal bundle $p : EG \to BG$ induces a bundle map

$$p_X : EG \times_G X = (EG \times X)/G \to EG/G = BG$$

such that $[e, x] \mapsto [e]$. One can show that this is a fibration with fiber X. The *equivariant cohomology* with coefficients in Λ is defined by

$$H^*_G(X, \Lambda) \doteq H^*(EG \times_G X, \Lambda) = H^*(X_G, \Lambda)$$

where by $H^*(Y, \Lambda)$ we usually mean singular cohomology of the space Y with coefficients in the ring Λ. This cohomology was introduced by Borel in [Bor60].

A benefit to this construction is that after constructing X_G, we can work with a topological space rather than a group action. Since the action of G on $EG \times X$ is free, then passing to X_G preserves many of the properties of X. Another benefit of equivariant cohomology is that we have a tool for computing the cohomology of fibrations, namely, the Leray-Serre spectral sequence. (See [McC01] for more on Leray-Serre spectral sequences.) The Leray-Serre spectral sequence allows one to compute the cohomology of the total space from the cohomology of the base space and the fiber space.

Equivariant cohomology inherits the structure of a $H^*(BG)$-module as follows. Given the definitions of the maps p and p_X above, we see that with information about $H^*(X)$ and $H^*(BG)$, one can deduce information about $H^*_G(X)$. Since cohomology is a contravariant functor, p_X induces a map $p_X^* : H^*(BG) \to H^*_G(X)$. We use this map to define the following action of $H^*(BG)$ on $H^*_G(X)$: Given $x \in H^*(BG)$ and $y \in H^*_G(X)$, define $xy = p_X^*(x) \cup y \in H^*_G(X)$ where $\cup$ denotes the cohomology cup product. With this definition, $H^*_G(X)$ is a graded $H^*(BG)$-module.

When G is a fixed topological group, the category of G-spaces has morphisms called "equivariant maps" (or "G-maps").

Definition 2.2.2. *For X and Y G-spaces, an* equivariant map $\phi : X \to Y$ *is a map which commutes with the group actions; that is, $\phi(g(x)) = g(\phi(x))$ for all $g \in G$ and $x \in X$.*

More generally, a pair of continuous maps $\tilde{\phi} : G_1 \to G_2$, $\phi : X \to Y$, where G_1, G_2 are topological groups acting on the spaces X, Y respectively, is an equivariant pair *if $\tilde{\phi}$ is a homomorphism, and if for every $g \in G$ and $x \in X$, $\phi(gx) = \tilde{\phi}(g)\phi(x)$.*

With the definitions above, one can show that the Borel construction is functorial: given an equivariant pair $(\tilde{\phi}, \phi)$ as above, there is a continuous function $(\tilde{\phi}, \phi) : X_{G_1} \to Y_{G_2}$ that satisfies the usual functorial rules (see [Qui71a, pp. 550-551]).

2.3 Differentiable Actions

In this section, we will not define basic topological notions. The reader may refer to [Hus66] or [tD87] for these definitions.

In the following discussion, we follow the exposition of Duflot in [Duf83], although this material is standard in algebraic topology.

Definition 2.3.1. *A* Riemannian metric *on a differentiable manifold X is a differentiable inner product on the tangent bundle τ_X.*

Theorem 2.3.2. *[Bre72, p. 305] Let G be a compact Lie group. A differentiable G-manifold X has an invariant Riemannian metric.*

Let X be a differentiable manifold with a Riemannian metric, and let $Y \subseteq X$ be a submanifold with Y embedded in X. Y has a tangent space, $\tau_Y = \coprod_{y \in Y} \tau_Y(y)$ where $\tau_Y(y)$ is the tangent space to Y at the point y, that is a subset of the tangent space of X, $\tau_X = \coprod_{x \in X} \tau_X(x)$. The existence of the Riemannian metric allows one to construct the normal bundle to Y in X, ν_Y, a bundle over Y, whose fiber at $y \in Y$ is the orthogonal complement to $\tau_Y(y)$ in $\tau_X(y)$. If the differentiable manifold X has a smooth G-invariant Riemannian metric and if Y is a closed G-invariant submanifold of X, then the normal bundle ν_Y is a G-vector bundle since the metric defining ν_Y is G-invariant. This allows one to construct the bundle ν_{Y_G} to $Y_G \doteq EG \times_G Y$ in $X_G \doteq EG \times_G X$. Note that the dimension of ν_{Y_G} is the same as the dimension of ν_Y. We will assume that ν_Y, and therefore ν_{Y_G}, are orientable vector bundles. (See Husemoller [Hus66] or Atiyah [Ati67], for example, for more about normal bundles and orientability.)

We now briefly define tubular neighborhoods and the Thom isomorphism, for use later.

Definition 2.3.3. *[Bre72] Let G be a Lie group, and let X be a differentiable G-manifold. If $Y \subseteq X$ is a smooth, invariant, closed submanifold, then an* open invariant tubular neighborhood *of Y in X is a differentiable G-vector bundle ξ on Y with total space $E(\xi)$ and an equivariant diffeomorphism $\phi : E(\xi) \to X$ onto some open neighborhood of Y in X such that the restriction of ϕ to Y (the zero-section of ξ) is the inclusion of Y in X. Furthermore, if ξ is a G-bundle on Y whose fibers have a G-invariant inner product varying continuously over Y (i.e., ξ is a G-invariant Euclidean vector bundle over Y), then the restriction of the* *diffeomorphism ϕ to the unit disk bundle $D(\xi) \to X$ is called a* closed invariant tubular neighborhood *of Y.*

Theorem 2.3.4. *[Bre72, Thm. 2.2, Ch. VI] Let G be a compact Lie group, and let X be a differentiable G-manifold. If $Y \subseteq X$ is a closed, invariant submanifold, then Y has an open invariant tubular neighborhood in X. If X has a G-invariant Riemannian metric, this tubular neighborhood is obtained from the disk bundle of the normal bundle to Y in X.*

Applying Theorem 2.1 in Chapter VI from Bredon [Bre72], when G is compact, every open invariant tubular neighborhood "contains" a closed invariant tubular neighborhood.

For future use, we present a theorem from Spanier.

Theorem 2.3.5. *[Spa66, Thm. 10, pg. 259] (Thom Isomorphism Theorem). Let ξ be an oriented n-disk bundle (a fiber bundle whose fiber is a unit ball $\mathbf{B}^n$) over the base space B. There exist natural isomorphisms for any coefficient ring Λ*

$$\gamma : H^q(B, \Lambda) \overset{\cong}{\to} H^{q+n}(E, \dot{E}, \Lambda).$$

In fact, $\gamma(v) = p^*v \cup U_\xi$ where $p : E \to B$ is the bundle projection for ξ, and $U_\xi \in H^n(E, \dot{E}, \Lambda)$ is the orientation class of ξ.

Another version of this theorem is the following corollary.

Corollary 2.3.6. *Let G be a compact Lie group, let X be a differentiable G-manifold with a smooth G-invariant Riemannian metric, and let Y be a smooth invariant submanifold of X. Let ν_Y be the normal bundle to Y in X, and let $d = \dim \nu_Y$. Consider the embedding $D(\nu_Y) \to X$ of $D(\nu_Y)$ as a closed invariant tubular neighborhood of Y. Then, if ν_Y is orientable, there exist natural isomorphisms for all q, for any coefficient ring Λ,*

$$\tau : H^q(Y, \Lambda) \overset{\cong}{\to} H^{q+d}(D(\nu_Y), D(\nu_Y) - Y, \Lambda).$$

Proof. Let $\dot{D}$ be the boundary (or sphere part) of $D(\nu_Y) \doteq D$. Applying Theorem 2.3.5, there exists an isomorphism

$$\tilde{\gamma} : H^q(Y) \xrightarrow{\cong} H^{q+d}(D, \dot{D})$$

for all q. There also exists a G-invariant deformation retraction $D - Y \xrightarrow{r} \dot{D}$. This fact results in the following exact columns:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{q-1}(D) \arrow[r, "="] \arrow[d] & H^{q-1}(D) \arrow[d] \\
H^{q-1}(\dot{D}) \arrow[r, "\cong", "r^*"'] \arrow[d] & H^{q-1}(D-Y) \arrow[d] \\
H^q(D, \dot{D}) \arrow[r, "\theta"'] \arrow[d] & H^q(D, D-Y) \arrow[d] \\
H^q(D) \arrow[r, "="] \arrow[d] & H^q(D) \arrow[d] \\
H^q(\dot{D}) \arrow[r, "\cong", "r^*"'] \arrow[d] & H^q(D-Y) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Therefore, by the Five Lemma, $H^q(D, \dot{D}) \cong H^q(D, D-Y)$.

The composition map,

$$\tau : H^q(Y) \to H^{q+d}(D, D-Y),$$

such that $\tau = \theta \circ \tilde{\gamma}$, is an isomorphism. We will call this the Thom isomorphism, also. $\square$

Chapter 3

LINKING COMMUTATIVE ALGEBRA AND TOPOLOGY

This chapter summarizes some results from Quillen [Qui71a] and Maiorana [Mai76] and concludes with our own results, which use Quillen's results but do not rely on Maiorana's results. In the first section, we discuss ideas needed for the remainder of the paper. The second section, presents some of Quillen's results, followed by a section in which we make modifications to Maiorana's work to fit within our context, resulting in a topological sum formula for computing the C-multiplicity of a compact manifold using fixed point sets. We then apply these results to the special case when a compact Lie group, G, embeds in a unitary group, U. We prove that the C-multiplicity of H_G^* (as an H_G^{ev}-module) may be computed using the C-multiplicity of $H^*_{C_G(A)}$ where A is a maximal rank p-torus of G. We conclude our discussion with some remarks relating this last topological sum formula for computing the C-multiplicity to the sum formula from commutative algebra in Theorem 1.6.63.

3.1 H^* and H^{ev}

Since we will ultimately be applying results of Chapter 1 to cohomology rings, we switch indexing for our graded rings to upper indexing, and we use "H" instead of "R".

Consider the graded ring $H^* = \bigoplus\limits_{i \geq 0} H^i$, where $H^0 = k$, a field with characteristic $p > 0$ with p odd prime, and H^* is graded-commutative (i.e. for every $x, y \in H^*$, if $x \in H^i$ and $y \in H^j$ are homogeneous elements, then $xy = (-1)^{ij}yx$.) Define the subring $H^{ev} \doteq \bigoplus_{i \geq 0} H^{2i}$, the even part of H^*. Note that H^{ev} is a strictly commutative graded ring.

If $p = 2$, then H^* is already commutative, and we do not need to consider H^{ev}. In this paper, we address the p odd case, and we leave the corresponding adjustments to the reader for the $p = 2$ case.

Proposition 3.1.1. *Let the graded ring H^* be finitely generated as a graded-commutative algebra over $H^0 = k$, a field of characteristic $p > 0$ odd. Then*

a. H^{ev} is a finitely generated graded k-algebra.

b. H^* is finitely generated as a graded module over its subring H^{ev}.

c. If M is a finitely generated graded H^*-module, then M is also a finitely generated graded H^{ev}-module.

Proof. Note first that $H^* = H^{ev} \oplus H^{odd}$ where $H^{odd} \doteq \underset{i \geq 0}{H}{}^{2i+1}$. By hypothesis, there exist homogeneous elements $x_1, \dots, x_m, y_1, \dots, y_n$ that generate H^* as a k-algebra where the x_i are of odd degree and the y_j are of even degree.

Thus, everything in H^{ev} can be written as a linear combination of elements of the form $x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}$ with coefficients in k. Notice that since y_i has even degree for every i, $y_1^{b_1} \cdots y_n^{b_n}$ has even degree, and thereby $x_1^{a_1} \cdots x_m^{a_m}$ must also have even degree. For characteristic $p > 0$, p odd, we know that $x_i^{a_i} = 0$ for $a_i \geq 2$, so $a_i = 0$ or $a_i = 1$ for all i. In addition, we know that $x_i x_j = -x_j x_i$ for $i \neq j$. Applying these two properties implies that H^{ev} is spanned as a vector space over k by "monomials" of the form $x_{j_1} x_{j_2} \cdots x_{j_l} y_1^{b_1} \cdots y_n^{b_n}$ where $1 \leq j_1 < j_2 < \cdots < j_l \leq m$ and l is even. We can group the x_{j_i}'s in pairs as follows: let $w_{j_1,j_2} \doteq x_{j_1} x_{j_2}$, $j_1 < j_2$. Clearly w_{j_1,j_2} must have even degree. Now we see that H^{ev} is spanned by elements of the form $w_{j_1,j_2} w_{j_3,j_4} \cdots w_{j_{l-1},j_l} y_1^{b_1} \cdots y_n^{b_n}$. As there are no more than

$\binom{m}{2}$ of the $w_{i,j}$ where $1 \le i < j \le m$, we see that H^{ev} is finitely generated as an algebra over k.

Similarly, since $x_i^{a_i} = 0$ for $a_i \ge 2$ and $x_i x_j = -x_j x_i$, the spanning set for H^* over H^{ev} consists of elements of the form $x_{j_1} \cdots x_{j_l}$, where $1 \le j_1 < j_2 < \cdots < j_l \le m$ and l is odd. (If l were even, then $x_{j_1} \cdots x_{j_l} \in H^{ev}$.) There are at most $\binom{m}{l}$ of the $x_{j_1} \cdots x_{j_l}$, so we see that H^* is a finitely generated H^{ev}-module.

Finally, if M is finitely generated as a module over H^*, and H^* is finitely generated as a module over H^{ev}, then M is finitely generated as a module over H^{ev}. □

Combining results from previous sections, specifically Proposition 1.6.61, with Proposition 3.1.1, we have the following theorem.

Theorem 3.1.2. *Suppose that H^* is finitely generated as a graded-commutative algebra over $H^0 = k$. If M is a nonzero finitely generated graded H^*-module, then M is a finitely generated graded H^{ev}-module, and the order of the pole of the Poincaré series $PS(M,t)$ at $t = 1$ is*

$$D(M) = \mathrm{Dim}_{H^{ev}}(M),$$

the Krull dimension of M as a finitely generated graded H^{ev}-module. In particular, the order of the pole at $t = 1$ of $PS(H^*,t)$ is equal to the Krull dimension of the graded ring H^{ev}.

Adapting results from commutative algebra to the graded-commutative case, we obtain the following lemma.

Lemma 3.1.3. *Let H_1^* and H_2^* be finitely generated as graded-commutative algebras over $k = H_1^0 = H_2^0$, and let $\phi : H_1^* \to H_2^*$ be a map of graded-commutative k-algebras such that ϕ makes H_2^* into a finitely generated H_1^*-module. Then, $\phi : H_1^{ev} \to H_2^{ev}$, making H_2^{ev} a finitely generated H_1^{ev}-module, and*

$$\mathrm{Dim}_{H_1^{ev}} H_2^* = \mathrm{Dim}_{H_2^{ev}} H_2^* = \mathrm{Dim}\,(H_1^{ev}/\ker\phi) = \mathrm{Dim}\, H_2^{ev}.$$

Proof. Since H_2^* is finitely generated as an H_1^*-module and H_1^* is finitely generated as a H_1^{ev}-module, then H_2^* is finitely generated as an H_1^{ev}-module. Therefore, by Theorem 3.1.2,

$$\mathrm{Dim}_{H_1^{ev}} H_2^* = \text{ the order of the pole, at } t=1, \text{ of } \mathrm{PS}(H_2^*, t).$$

In addition, since H_2^* is finitely generated as an H_2^{ev}-module, we also have that

$$\mathrm{Dim}_{H_2^{ev}} H_2^* = \text{ the order of the pole, at } t=1, \text{ of } \mathrm{PS}(H_2^*, t),$$

resulting in

$$\mathrm{Dim}_{H_1^{ev}} H_2^* = \mathrm{Dim}_{H_2^{ev}} H_2^*.$$

On the other hand, H_2^* is also an H_1^{ev}-algebra, via ϕ, that is finitely generated as an H_1^{ev}-module. Since H_1^{ev} is Noetherian, the H_1^{ev}-submodule H_2^{ev} of H_2^* is finitely generated as a module over H_1^{ev}, via ϕ. In other words, H_2^* is a finitely generated module over its subring $H_1^{ev}/\ker\phi$. Thus, by Theorem 17 of Kaplansky [Kap74], H_2^{ev} is integral over $H_1^{ev}/\ker\phi$. Therefore, by Theorem 48 of Kaplansky [Kap74],

$$\text{the order of the pole, at } t=1, \text{ of } \mathrm{PS}(H_2^*, t) = \mathrm{Dim}\, H_2^{ev} = \mathrm{Dim}\left(H_1^{ev}/\ker\phi\right).$$

□

Now we can define Maiorana's C-multiplicity in this context.

Definition 3.1.4. *If M is a nonzero finitely generated graded H^*-module, the* Maiorana multiplicity $C(M)$ *is equal to the Maiorana multiplicity of M considered as an object in $\mathcal{C}(R)$ where $R = H^{ev}$,*

$$C(M) = \lim_{t\to 1}(1-t)^{D(M)} PS(M,t),$$

where $D(M) = \mathrm{Dim}_R(M)$. We define $C(0) \doteq 0$.

3.2 Quillen's Results

In this section, we state some of Quillen's results [Qui71a] without proof, and we include several additional useful results, many of which rely on Quillen's work. Let G be a compact Lie group. Let X be a compact (or paracompact of finite mod p cohomological dimension) manifold on which G acts continuously. We define

$$H_G^*(X) \doteq H_G^*(X, \mathbb{Z}/p\mathbb{Z})$$

where p is a prime number. We will assume that p is odd; the $p = 2$ case can be handled similarly. In addition, for the rest of this section we let $k = \mathbb{Z}/p\mathbb{Z}$.

Recall that to say A is a p-torus is the same as saying that A is elementary abelian; i.e.,

$$A \cong \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$$

as a group. We say that $\operatorname{rank} A$, the *rank* of A, is equal to the number of $\mathbb{Z}/p\mathbb{Z}$ factors.

Lemma 3.2.1. *Let K be a p-torus of rank n. If $H \leq K$, then H is a p-torus of rank $\leq n$.*

Proof. H is a finite abelian group, so there exist $i_1, i_2, \ldots, i_m$ with $i_j \geq 1$ such that $H \cong \mathbb{Z}/p^{i_1} \times \cdots \mathbb{Z}/p^{i_m}$. For all $h \in H$ where $x \neq e_H$, $|x| = p$. Hence, $i_j = 1$ for all $j = 1, \ldots, m$, implying that H is a p-torus of rank m. Since $H \leq K$, then $\operatorname{rank} H = m \leq n = \operatorname{rank} K$. $\square$

For G a compact Lie group which acts smoothly on the manifold X, let $\mathcal{A}(G, X)$ denote Quillen's category of pairs with objects (A, c), where A is a p-torus in G and c is a (nonempty) connected component of $X^A \neq \emptyset$, and morphisms $\theta : (A, c) \to (A', c')$, where θ is conjugation of A into (a subgroup of) A' by an element $g \in G$ such that $c' \subseteq gc$. The objects of $\mathcal{A}(G, X)$ are partially ordered using the definition $(A, c) \leq (A', c')$ if and only if $A \leq A'$ and $c' \subseteq c \subseteq X^A$. [Qui71a]

Recall the definitions of conjugate, subconjugate, conjugacy class, and inequality of conjugacy classes for $A \leq G$ from Definition 2.1.7. We extend these definitions to $\mathcal{A}(G, X)$.

Definition 3.2.2. *(A, c) is* subconjugate *to (A', c'), or $(A, c) \lesssim (A', c')$, in $\mathcal{A}(G, X)$ if and only if $\operatorname{Hom}_{\mathcal{A}(G,X)}((A, c), (A', c')) \neq \emptyset$. (A, c) is* conjugate *to (A', c'), or $(A, c) \sim (A', c')$, in $\mathcal{A}(G, X)$ if an only if (A, c) and (A', c') are isomorphic objects.*

Let $[(A, c)]$ be the conjugacy class *of (A, c) in $\mathcal{A}(G, X)$. Then, $[(A, c)] = [(A', c')]$ if and only if $(A, c) \sim (A', c')$. We say that $[(A, c)] \leq [(A', c')]$, if and only if $(A, c) \lesssim (A', c')$.*

Let $\mathcal{A}(G) = \{A \mid A$ is an elementary abelian p-subgroup of $G\}$.

Definition 3.2.3. *If X is a G-space:*

a. $\mathcal{A}_0(G, X) \doteq \{A \in \mathcal{A}(G) \mid X^A \neq \emptyset\}$.

b. $\mathcal{B}(G, X) \doteq \{[A] | A \in \mathcal{A}_0(G, X)$ *and A is of maximal rank in* $\mathcal{A}_0(G, X)\}$.

When $X = \{x_0\}$ is a one point space, note that $\mathcal{A}_0(G, \{x_0\})$ and $\mathcal{B}(G, \{x_0\})$ are

$$\mathcal{A}_0(G, \{x_0\}) = \mathcal{A}(G)$$

and

$$\mathcal{B}(G) \doteq = \mathcal{B}(G, \{x_0\}) = \{[A] \mid A \text{ a maximal rank } p\text{-torus in } G\},$$

respectively.

Note the following lemma.

Lemma 3.2.4. *[Qui71a, Lem. 6.3] Every compact Lie group has finitely many conjugacy classes of elementary abelian p-subgroups.*

Recall that for a map $X \to \{x_0\}$, there exists a ring homomorphism

$$H_G^*(\{x_0\}) \doteq H_G^* \to H_G^*(X)$$

making $H_G^*(X)$ a graded H_G^*-module.

Theorem 3.2.5. *Let G be a compact Lie group, acting on a topological space X. Suppose that $H^*(X)$ is finite dimensional as a graded vector space over k. Then the following hold:*

a. $H_G^*(X)$ is a finitely generated graded k-algebra. [Qui71a, Cor. 2.2]

b. $H_G^*(X)$ is a finitely generated graded module over H_G^*. [Qui71a, Cor. 2.3]

c. If $H \leq G$, then the natural homomorphism of rings $H_G^*(X) \stackrel{\text{res}}{\to} H_H^*(X)$ makes $H_H^*(X)$ into a finitely generated graded module over $H_G^*(X)$. [Qui71a, Cor. 2.3]

As a result of this theorem, we apply Proposition 3.1.1 to conclude the following corollary.

Corollary 3.2.6. *Let G be a compact Lie group, and let $H^*(X)$ be finite dimensional as a graded vector space over k. Then, $H_G^*(X)$ is a finitely generated graded module over H_G^{ev}, and H_G^{ev} is a finitely generated k-algebra.*

Therefore, applying the theory for graded rings developed in Chapter 1, the following invariants exist as in Corollary 1.6.40 and in Definitions 1.5.5, 1.6.42, and 1.5.6:

- $D(H_G^*(X))$, defined by considering $H_G^*(X)$ as an $R = H_G^{ev}$-module, is equal to the order of the pole of the Poincaré series for $H_G^*(X)$, $\ell(H_G^*(X))$,

- for $\mathcal{I} \subseteq R$ an ideal of definition for $H_G^*(X)$ considered as an object in $\mathcal{C}(R)$, the integer $e(H_G^*(X), \mathcal{I})$ is the Samuel multiplicity of $H_G^*(X)$ with respect to the ideal $\mathcal{I}$, and

- the rational number $C(H_G^*(X)) = \lim_{t \to 1}(1-t)^{D(H_G^*(X))} \operatorname{PS}(H_G^*(X), t)$ is Maiorana's C-multiplicity. Note that C is defined in terms of the Poincaré series, so C appears to depend only on the vector space structure of $H_G^*(X)$.

We also have the following theorem from Quillen.

Theorem 3.2.7. *[Qui71a, Thm. 7.7] (Quillen's Main Theorem) Let G be a compact Lie group. For p prime, let X be a compact (or paracompact with finite mod-p cohomological dimension) G-space. If $H^*(X)$ is finite dimensional, then $\ell(H^*_G(X))$ equals the maximum rank of a p-torus A in G such that $X^A \neq \emptyset$, i.e.,*

$$\ell(H^*_G(X)) = \max\{\operatorname{rank} A \mid A \textit{ a p-torus in } G, X^A \neq \emptyset\}$$

$$= \max\{\operatorname{rank} A \mid A \in \mathcal{A}_0(G, X)\}.$$

We do not define paracompact nor mod-p cohomological dimension here. See [Mun00] and [Qui71a] for definitions.

For H any compact Lie group, we define the p-rank of H as follows:

$$p\text{-}\operatorname{rank} H \doteq \max\{\operatorname{rank} B \mid B \text{ is a } p\text{-torus in } H\}.$$

Corollary 3.2.8. *Let X be a one-point space, and let G be a compact Lie group. Then, for p prime, since $H^*_G(X) \doteq H^*(BG) \doteq H^*_G$,*

$$\ell(H^*_G) = p\text{-}\operatorname{rank} G.$$

The following corollary follows from Lemma 2.1.8 and the fact that every subgroup of a p-torus is a p-torus, as shown in Lemma 3.2.1.

Corollary 3.2.9. *Suppose that G is a compact Lie group acting on the topological space X. If G_x is a p-torus for $x \in X^A$, then A is a p-torus also.*

Moreover, if $A \leq G$, then, for every $z \in GX^A$, $p\text{-rank}\, G_z \geq p\text{-rank}\, A$.

Corollary 3.2.10. *With the same hypotheses on G and X as Theorem 3.2.7, we have*

$$\begin{aligned} \ell(H^*_G(X)) &= \max\{p\text{-}\operatorname{rank} G_x \mid x \in X\} \\ &= \max\{\ell(H^*_{G_x}) \mid x \in X\}. \end{aligned}$$

Proof. By Quillen's Main Theorem and the definitions of a p-torus, we have that

$$\begin{aligned}\ell\,(H_G^*(X)) &= \max\{\operatorname{rank} A \mid A \text{ a } p\text{-torus in } G, X^A \neq \emptyset\} \\ &= \max\{\operatorname{rank} A \mid A \in \mathcal{A}_0(G,X)\}.\end{aligned}$$

For G_x the isotropy group for $x \in X$, let

$$\ell(H_{G_x}^*) = \max\{\operatorname{rank} B \mid B \in \mathcal{A}(G_x)\}.$$

Consider $A \in \mathcal{A}_0(G,X)$. Since $X^A \neq \emptyset$, there exists $x \in X$ such that $A \leq G_x$. Thus, $A \in \mathcal{A}(G_x)$. On the other hand, for some $x \in X$, consider $B \in \mathcal{A}(G_x)$. Then, $B \leq G_x$ and $x \in X^B$. Thus, $X^B \neq \emptyset$, and $B \in \mathcal{A}_0(G,X)$. Therefore,

$$\{A \mid A \in \mathcal{A}_0(G,X)\} = \bigcup_{x\in X}\{B \mid B \in \mathcal{A}(G_x)\},$$

and

$$\ell(H_G^*(X)) = \max\{\ell(H_{G_x}^*) \mid x \in X\}.$$

□

Corollary 3.2.11. *Let G and X be as defined in Theorem 3.2.7. If Y is a G-invariant subspace of X such that Y also satisfies the hypotheses of Theorem 3.2.7, then*

$$\ell(H_G^*(Y)) \leq \ell(H_G^*(X)).$$

Proof. By Corollary 3.2.10, $\ell(H_G^*(Y)) = \max\{\ell(H_{G_y}^*) \mid y \in Y\}$ and $\ell(H_G^*(X)) = \max\{\ell(H_{G_x}^*) \mid x \in X\}$. Then, since $Y \subseteq X$, we have that

$$\max\{\ell(H_{G_y}^*) \mid y \in Y\} \leq \max\{\ell(H_{G_x}^*) \mid x \in X\}.$$

□

Using Quillen's Main Theorem and our previous results that the Krull dimension equals the order of the pole of the Poincaré series at $t = 1$, we have the following corollary.

Corollary 3.2.12. *With the same hypothesis as Theorem 3.2.7, $D(H_G^*(X))$, the Krull dimension of $H_G^*(X)$ as an H_G^{ev}-module, equals the maximum rank of an elementary abelian p-subgroup A of G such that $X^A \neq \emptyset$.*

For future use, we note the following lemma.

Lemma 3.2.13. *Let G be a compact Lie group, and let $H^*(X)$ be finite dimensional as a graded vector space over k. Then, the Krull dimension of $H_G^*(X)$ as an H_G^{ev}-module is equal to the Krull dimension of $H_G^{ev}(X)$ as a commutative ring.*

Proof. We know that $H_G^*(X)$ is finitely generated as an H_G^{ev}-module. Therefore, since $H_G^{ev}(X)$ is an H_G^{ev}-submodule of $H_G^*(X)$ and H_G^{ev} is a commutative Noetherian ring, we have that $H_G^{ev}(X)$ is finitely generated as an H_G^{ev}-module. Therefore, using Lemma 3.1.3, for the ring homomorphisms $\phi : H_G^{ev} \to H_G^{ev}(X)$ and $\phi : H_G^{ev} \to H_G^*(X)$, we have

$$D_{H_G^{ev}} H_G^{ev}(X) = D(H_G^{ev}(X)) = D_{H_G^{ev}}(H_G^*(X)) = D\left(H_G^{ev} / \ker \phi\right).$$

$\square$

We take some time to discuss relationships between the various subspaces GX^A, as A varies.

Lemma 3.2.14. *Let $A, B \leq G$, a compact Lie group, and let X be a compact topological space on which G acts. Let $z \in X$. Then, $z \in GX^A \cap GX^B$ if and only if $[A] \leq [G_z]$ and $[B] \leq [G_z]$.*

Proof. Suppose there exists $z \in GX^A \cap GX^B$. By Corollary 3.2.9, G_z contains a conjugate of A and a conjugate of B. This implies that $A \lesssim G_z$ and $B \lesssim G_z$. Therefore, $[A] \leq [G_z]$ and $[B] \leq [G_z]$.

Suppose that $[A] \leq [G_z]$ and $[B] \leq [G_z]$ for some $z \in X$. Then there exists $g, \tilde{g} \in G$ such that $gAg^{-1} \subseteq \tilde{g}G_z\tilde{g}^{-1}$, which implies that $\tilde{g}^{-1}gAg^{-1}\tilde{g} \subseteq G_z$. For all $a \in A$, we have that $\tilde{g}^{-1}gag^{-1}\tilde{g}z = z$, so $g^{-1}\tilde{g}z \in X^A$, and $z \in GX^A$. Similarly, we can show that $z \in GX^B$. Therefore, $z \in GX^A \cap GX^B$. $\square$

Lemma 3.2.15. *Let A and B be maximal p-tori in a group G. Then*

$$[A] \leq [B] \Leftrightarrow A \sim B.$$

Proof. Since A, B maximal in $\mathcal{A}(G)$ and $[A]$ maximal with respect to $\leq$,

$$[A] \leq [B] \Leftrightarrow [A] = [B] \Leftrightarrow A \sim B.$$

□

Proposition 3.2.16. *Let G be a compact Lie group acting on a compact topological space X such that each isotropy group G_x for all $x \in X$ has a unique maximal p-torus, and let A and B be maximal in $\mathcal{A}_0(G, X)$. Then, $[A] = [B]$ in $\mathcal{A}_0(G, X)$ if and only if $GX^A = GX^B$.*

Proof. Suppose that $[A] = [B]$. Then $A \sim B$, so there exists $g \in G$ such that $gAg^{-1} = B$, and we see that

$$GX^B = GX^{gAg^{-1}}.$$

Hence, for all $y \in X^B$ and for all $a \in A$,

$$\begin{aligned} gag^{-1}y &= y \\ ag^{-1}y &= g^{-1}y. \end{aligned}$$

However, $g^{-1}y \in X$, so

$$g^{-1}y \in X^A.$$

If $hy \in GX^B$, where $h \in G$, we have

$$hy = h(gag^{-1}y) = (hga)(g^{-1}y) \in GX^A$$

since $hga \in G$ and $g^{-1}y \in X^A$, and we see that

$$GX^B \subseteq GX^A.$$

Similarly, we can show the reverse inclusion, and we conclude that $GX^B = GX^A$.

On the other hand, suppose that $GX^B = GX^A$. Then for all $gx \in GX^A$ (where $g \in G$ and $x \in X^A$), there exists $hy \in GX^B$ (where $h \in G$ and $y \in X^B$) such that $gx = hy$. Note that since B is maximal in $\mathcal{A}_0(G, X)$, then B is maximal in $\mathcal{A}(G_y)$. Similarly, since A is maximal in $\mathcal{A}_0(G, X)$, then A is maximal in $\mathcal{A}(G_x)$. Then, by the hypothesis, B is the unique maximal p-torus in G_y, and A is the unique maximal p-torus in G_x.

Let $\hat{g} = h^{-1}g$. For all $a \in A$,

$$\begin{aligned}
\hat{g}a\hat{g}^{-1}y &= h^{-1}gag^{-1}hy \\
&= h^{-1}gx \\
&= y.
\end{aligned}$$

This implies that $\hat{g}A\hat{g}^{-1} \subseteq G_y$.

Consider the p-torus $\hat{g}A\hat{g}^{-1}$. Suppose there exists $\tilde{A} \subseteq G_y$ such that $\hat{g}A\hat{g}^{-1} \subseteq \tilde{A} \subseteq G_y$. Then, for $\tilde{a} \in \tilde{A}$,

$$\begin{aligned}
\hat{g}^{-1}\tilde{a}\hat{g}x &= g^{-1}h\tilde{a}h^{-1}gx \\
&= g^{-1}h\tilde{a}y \\
&= g^{-1}hy \\
&= g^{-1}gx \\
&= x
\end{aligned}$$

Hence, we see that $A \subseteq \hat{g}^{-1}\tilde{A}\hat{g} \subseteq G_x$. However, since A is the unique maximal p-torus in G_x, then $A = \hat{g}^{-1}\tilde{A}\hat{g}$. Thus, $\hat{g}A\hat{g}^{-1} = \tilde{A}$, so $\hat{g}A\hat{g}^{-1}$ is a maximal p-torus in G_y. Therefore, $\hat{g}A\hat{g}^{-1} = B$, and $A \sim B$. □

Lemma 3.2.17. *Let G be a compact Lie group acting on a compact topological space X such that each isotropy group G_x for all $x \in X$ has a unique maximal p-torus. Then, $GX^A \cap GX^B = \emptyset$ for all $[A] \neq [B]$, where A, B maximal in $\mathcal{A}_0(G, X)$.*

Proof. Suppose that for some A, B maximal in $\mathcal{A}_0(G, X)$, $[A] \neq [B]$ and that $GX^A \cap GX^B \neq \emptyset$. Then there exists $z \in GX^A \cap GX^B$. By Lemma 3.2.14, we have that $[A] \leq [G_z]$ and $[B] \leq [G_z]$. Since A and B are maximal, and G_z has a unique maximal p-torus, $[A] = [B]$. Contradiction. □

Lemma 3.2.18. *Let G be a compact Lie group and let X be a compact topological space on which G acts. Then $\ell(H_G^*(X)) = \ell(H_G^*(GX^A))$ for all $[A] \in \mathcal{B}(G, X)$.*

Proof. By Theorem 3.2.7, for all $A \in \mathcal{A}_0(G, X)$ of maximal rank,

$$\ell(H_G^*(X)) = \operatorname{rank} A.$$

Using Corollary 3.2.10,

$$\ell(H_G^*(GX^A)) = \max\{p\text{-}\operatorname{rank} G_x | x \in GX^A\}.$$

If $x \in GX^A$, then G_x contains a conjugate of A by Lemma 2.1.8. Since A is isomorphic to its conjugates,

$$\operatorname{rank} A \leq p\text{-}\operatorname{rank} G_x.$$

Since A is a maximal rank p-torus in $\mathcal{A}_0(G, X)$, $\operatorname{rank} A = p\text{-}\operatorname{rank} G_x$. Therefore, $\operatorname{rank} A = p\text{-}\operatorname{rank} G_x$ for all $x \in GX^A$, and

$$\ell(H_G^*(X)) = \operatorname{rank} A = \ell(H_G^*(GX^A)) \text{ for all } [A] \in \mathcal{B}(G, X).$$

$\square$

Lemma 3.2.19. *Let G be a compact Lie group and let X be a compact topological space on which G acts. Suppose that each isotropy group G_x, for $x \in X$, has a unique maximal p-torus. Then:*

a. *The union $\bigcup_{[A] \in \mathcal{B}(G,X)} GX^A$ is a disjoint union: $\coprod_{[A] \in \mathcal{B}(G,X)} GX^A$.*

b. *If $z \notin \coprod_{[A] \in \mathcal{B}(G,X)} GX^A$, then $\ell(H_{G_z}^*) < \ell(H_G^*(GX^A))$ for all $[A] \in \mathcal{B}(G, X)$.*

Proof. For a., the indicated union is disjoint using 3.2.17.

For b., since $z \notin \coprod_{[A] \in \mathcal{B}(G,X)} GX^A$, then $z \notin GX^A$ for all $[A] \in \mathcal{B}(G, X)$. This implies that G_z does not contain a conjugate of A for all $[A] \in \mathcal{B}(G, X)$, and

$$\ell(H_{G_z}^*) = p\text{-}\operatorname{rank} G_z < \operatorname{rank} A = \ell(H_G^*(GX^A)) \text{ for all } [A] \in \mathcal{B}(G, X).$$

$\square$

3.3 Results Regarding Maiorana's C-Multiplicity

From this point forward, unless otherwise stated, let X be a differentiable compact manifold with a Riemannian metric, on which a compact Lie group G acts differentiably. In this section, we state Maiorana's main result in [Mai76] without proof. Then we prove variations of Maiorana's theorem suitable for our purposes. We do not use any of Maiorana's results in our proofs. With our definition of equivariant cohomology $H^*_G(X)$ for an action of G on X, we will obtain a formula for the invariant $D(H^*_G(X))$ in terms of the isotropy groups, and ultimately we will obtain a formula for $C(H^*_G(X))$ involving the collection of fixed point sets of subgroups. (Note that all cohomology has $\mathbb{Z}/p\mathbb{Z}$ coefficients, where p is a prime number.)

Definition 3.3.1. *Suppose a group G acts on a smooth compact manifold X. Let F be an invariant submanifold of X. F is* isolated *if there is some neighborhood U_x of x for each $x \in F$, such that $\ell(H^*_{G_y}) < \ell(H^*_G(F))$ for $y \in U_x - F$.*

Using the above definition for an isolated submanifold, Maiorana [Mai76] proves the following proposition, without using Quillen's Main Theorem (Theorem 3.2.7).

Proposition 3.3.2. *[Mai76, Prop. 4.2] Let G be a p-group which acts on a compact manifold X with $F_1, F_2, \ldots, F_m$ closed, invariant, disjoint, and isolated submanifolds. If p is odd, then suppose that each normal bundle ν_{F_i} is oriented. If $\ell(H^*_G(F_i)) \geq \ell(H^*_G(X))$ for each i, and if $\ell(H^*_{G_x}) < \ell(H^*_G(X))$ for all $x \in X - \bigcup_{i=1}^{m} F_i$, then*

$$\sum_{i=1}^{m} C(H^*_G(F_i)) = C(H^*_G(X)).$$

Using Quillen's Main Theorem (Theorem 3.2.7), we will prove a generalization of Maiorana's Proposition 3.3.2 that suits our purposes. We do not use Maiorana's results.

Theorem 3.3.3. *Let a compact Lie group G act smoothly on a compact manifold X with $Z = \bigcup_{i=1}^{n} Z_i$ where the Z_i's are closed, G-invariant, disjoint submanifolds of X such that ν_{Z_i} is orientable for all i (but ν_{Z_i} may have different dimensions over different components of Z_i). Assume that*

(i.) $\ell\left(H_G^*(X)\right) = \ell\left(H_G^*(Z_i)\right)$ *for all* i,

(ii.) if $z \notin Z$, *then* $\ell\left(H_{G_z}^*\right) < \ell\left(H_G^*(Z_i)\right)$ *for all* i, *and*

(iii.) $\pi_0(Z_i)$, *the set of connected components of* Z_i, *has a finite number,* q_i, *of orbits under the* G *action.*

Then,

$$C\left(H_G^*(X)\right) = \sum_{i=1}^{n} C\left(H_G^*(Z_i)\right).$$

The proof of this theorem, finally accomplished on page 93, requires some notation and several lemmas.

3.3.1 Notation

We begin the proof of Theorem 3.3.3 by setting notation and stating some basic facts.

Let $Y = \bigcup_{i=1}^{m} Y_i$ where the Y_i's are closed, connected, G-invariant disjoint submanifolds of X such that ν_{Y_i} is orientable and $\dim \nu_{Y_i} = d_i \geq 0$ for all i. There is an exact equivariant Gysin sequence for the embedding $Y \to X$,

$$\cdots \to H_G^q(Y_i)(-d_i) \xrightarrow{\phi_i} H_G^q(X) \xrightarrow{g_i^*} H_G^q(X - Y_i) \xrightarrow{\sigma_i} H_G^{q+1}(Y_i)(-d_i) \to \cdots$$

for every i.

To see this, we begin by letting D_i be a closed, G-invariant, tubular neighborhood of Y_i in X obtained by equivariantly embedding the total space of the disk bundle associated to ν_{Y_i} in X such that $D_i \cap D_j = \emptyset$ for all $i \neq j$.Define $D = \bigcup_{i=1}^{m} D_i$. Note also that $\bigcup_{i=1}^{m}(D_i - Y_i) = \bigcup_{i=1}^{m} D_i - \bigcup_{i=1}^{m} Y_i$.

Define the following canonical maps, all of which are restriction maps:

$$\begin{array}{rcl} f:(X,\emptyset)\to(X,X-Y) & \text{induces} & f^*:H^q_G(X,X-Y)\to H^q_G(X), \\ g:X-Y\to X & \text{induces} & g^*:H^q_G(X)\to H^q_G(X-Y). \end{array}$$

For each $i=1,...,m$, define the following canonical maps, all of which are restriction maps also:

$$\begin{array}{rcl} f_i:(X,\emptyset)\to(X,X-Y_i) & \text{induces} & f_i^*:H^q_G(X,X-Y_i)\to H^q_G(X), \\ g_i:X-Y_i\to X & \text{induces} & g_i^*:H^q_G(X)\to H^q_G(X-Y_i), \\ h_i:(X,X-Y)\to(X,X-Y_i) & \text{induces} & h_i^*:H^q_G(X,X-Y_i)\to H^q_G(X,X-Y), \\ \tilde{h}_i:X-Y\to X-Y_i & \text{induces} & \tilde{h}_i^*:H^q_G(X-Y_i)\to H^q_G(X-Y). \end{array}$$

Diagram 3.3.4. *These degree-preserving maps, along with the connecting homomorphisms Δ_Y and Δ_i, yield the following commutative diagram with exact rows of the pairs $(X_G,(X-Y_i)_G)$ and $(X_G,(X-Y)_G)$:*

$$\begin{tikzcd} \cdots \arrow[r] & H^q_G(X,X-Y_i) \arrow[r, "f_i^*"] \arrow[d, "h_i^*"] & H^q_G(X) \arrow[r, "g_i^*"] \arrow[d, "="] & H^q_G(X-Y_i) \arrow[r, "\Delta_i"] \arrow[d, "\tilde{h}_i^*"] & H^{q+1}_G(X,X-Y_i) \arrow[r] \arrow[d, "h_i^*"] & \cdots \\ \cdots \arrow[r] & H^q_G(X,X-Y) \arrow[r, "f^*"] & H^q_G(X) \arrow[r, "g^*"] & H^q_G(X-Y) \arrow[r, "\Delta_Y"] & H^{q+1}_G(X,X-Y) \arrow[r] & \cdots \end{tikzcd}$$

Diagram 3.3.5. *Using excision, we have the isomorphisms*

$$e_D:H^q_G(X,X-Y)\to H^q_G\left(\bigcup_{i=1}^m D_i,\bigcup_{i=1}^m(D_i-Y_i)\right)$$

obtained by excising $X-D$ and

$$e_i:H^q_G(X,X-Y_i)\to H^q_G(D_i,D_i-Y_i)$$

obtained by excising $X-D_i$ for all i, resulting in the following commutative diagram for all q:

$$\begin{tikzcd} H^q_G(X,X-Y_i) \arrow[r, "e_i"] \arrow[d, "h_i^*"] & H^q_G(D_i,D_i-Y_i) \arrow[d, hook, "\iota_i^*"] \\ H^q_G(X,X-Y) \arrow[r, "e_D"] & H^q_G\left(\bigcup_{i=1}^m D_i,\bigcup_{i=1}^m(D_i-Y_i)\right)\cong\bigoplus_{i=1}^m H^q_G(D_i,D_i-Y_i) \end{tikzcd}$$

where $\iota_i^*:H^q_G(D_i,D_i-Y_i)\to H^q_G\left(\bigcup_{i=1}^m D_i,\bigcup_{i=1}^m(D_i-Y_i)\right)$ is inclusion onto the appropriate direct summand.

Diagram 3.3.6. *Since the space $(Y_i)_G$ is equivariantly embedded in $(D_i)_G$ as the zero section of the disk bundle associated to $\nu_G : N_G \to Y_G$, then by Proposition 2.3.6, there is a Thom isomorphism $\tau_i : H^q_G(Y_i)(-d_i) \to H^q_G(D_i, D_i - Y_i)$. The Gysin sequence for this normal bundle,*

$$\begin{tikzcd}
\cdots \arrow[r] & H^q_G(Y_i)(-d_i) \arrow[rr, "\phi_i"] \arrow[d, "\tau_i"] & & H^q_G(X) \arrow[r, "g_i^*"] & \cdots \\
& H^q_G(D_i, D_i - Y_i) \arrow[r, "e_i^{-1}"] & H^q_G(X, X - Y_i) \arrow[ur, "f_i^*"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "g_i^*"] & H^q_G(X - Y_i) \arrow[rr, "\sigma_i"] \arrow[d, "\Delta_i"] & & H^{q+1}_G(Y_i)(-d_i) \arrow[r] & \cdots \\
& H^{q+1}_G(X, X - Y_i) \arrow[r, "e_i"] & H^{q+1}_G(D_i, D_i - Y_i), \arrow[ur, "\tau_i^{-1}"] & &
\end{tikzcd}$$

is obtained using $\phi_i \doteq f_i^* e_i^{-1} \tau_i$ and $\sigma_i \doteq \tau_i^{-1} e_i \Delta_i$ for all i.

3.3.2 Basic Lemmas

At this point, we will prove a series of lemmas which will aid in the proof of Theorem 3.3.3.

Definition 3.3.7. *Define*

$$\tilde{H}^q_G(Y) \doteq \bigoplus_{i=1}^{m} H^q_G(Y_i)(-d_i) \doteq \bigoplus_{i=1}^{m} H^{q-d_i}_G(Y_i)$$

as a graded object.

Lemma 3.3.8. *Let G be a compact Lie group and let $Y = \bigcup_{i=1}^{m} Y_i$, where the Y_i's are closed, connected, G-invariant, disjoint submanifolds of a compact manifold X such that ν_{Y_i} is orientable for all i and $\dim \nu_{Y_i} = d_i \geq 0$ for all i. Then,*

a. $\tilde{H}^*_G(Y)$ is a graded module over H^*_G.

b. $\tilde{H}^*_G(Y)$ is a finitely generated H^*_G-module.

c. $\ell(\tilde{H}^*_G(Y)) = \ell(H^*_G(Y))$.

d. $C(\tilde{H}_G^*(Y)) = C(H_G^*(Y))$.

Proof. Consider $\xi \in H_G^p$ and $\tilde{\xi} \in \tilde{H}_G^q(Y) = \bigoplus_{i=1}^{m} H_G^q(Y_i)(-d_i)$. We write

$$\tilde{\xi} = \tilde{\xi}_1 + \cdots + \tilde{\xi}_m,$$

where $\tilde{\xi}_i \in H^{q-d_i}(Y_i)$. Then,

$$\xi\tilde{\xi} = \xi\tilde{\xi}_1 + \cdots + \xi\tilde{\xi}_m,$$

where $\xi\tilde{\xi}_i \in H_G^{p+q-d_i}(Y_i)$. Hence, $\xi\tilde{\xi} \in \tilde{H}_G^{p+q}(Y)$, confirming that $\tilde{H}_G^*(Y)$ is a graded module over H_G^*.

Since $H_G^*(Y_i)$ is finitely generated as an H_G^*-module for all i by Theorem 3.2.5, then $H_G^*(Y_i)(-d_i)$ is also a finitely generated H_G^*-module for all i. Therefore, $\tilde{H}_G^*(Y) = \bigoplus_{i=1}^{m} H_G^*(Y_i)(-d_i)$ is a finitely generated H_G^*-module.

Recall that $\tilde{H}_G^q(Y) = \bigoplus_{i=1}^{m} H_G^{q-d_i}(Y_i)$. Then we have that

$$\begin{aligned}
\mathrm{PS}(\tilde{H}_G^*(Y), t) &= \sum_{q=0}^{\infty} \dim\left(\bigoplus_{i=1}^{m} H_G^{q-d_i}(Y_i)\right) t^q = \sum_{q=0}^{\infty}\sum_{i=1}^{m} \dim\left(H_G^{q-d_i}(Y_i)\right) t^q \\
&= \sum_{i=1}^{m}\sum_{q=0}^{\infty} \dim\left(H_G^{q-d_i}(Y_i)\right) t^q = \sum_{i=1}^{m} t^{d_i}\left(\sum_{q=0}^{\infty} \dim\left(H_G^{q-d_i}(Y_i)\right) t^{q-d_i}\right) \\
&= \sum_{i=1}^{m} t^{d_i}\left(\sum_{q=0}^{\infty} \dim\left(H_G^{q}(Y_i)\right) t^{q}\right) = \sum_{i=1}^{m} t^{d_i}\left(\mathrm{PS}\left(H_G^*(Y_i), t\right)\right) \\
&= \sum_{i=1}^{m} \left(\mathrm{PS}\left(H_G^*(Y_i)(-d_i), t\right)\right)
\end{aligned}$$

Now, we see that $\ell(\tilde{H}_G^*(Y)) = \max\{\ell\left(H_G^*(Y_i)(-d_i)\right) \mid i = 1, \ldots, m\}$. Recall that $\ell(M(-r)) = \ell(M)$. Then, $\ell\left(H_G^*(Y_i)(-d_i)\right) = \ell\left(H_G^*(Y_i)\right)$ for all $i = 1, \ldots, m$. Therefore, $\ell(\tilde{H}_G^*(Y)) = \max\{\ell\left(H_G^*(Y_i)\right) | i = 1, \ldots, m\}$.

Since $\mathrm{PS}(\tilde{H}_G^*(Y), t) = \sum_{i=1}^{m}(\mathrm{PS}(H_G^*(Y_i)(-d_i), t)) = \mathrm{PS}(H_G^*(Y), t)$, as shown above, and since $\ell(\tilde{H}_G^*(Y)) = \ell(H_G^*(Y))$, multiplying by $(1-t)^{\ell(\tilde{H}_G^*(Y))}$ and taking the limit as $t \to 1$ results in

$$C(\tilde{H}_G^*(Y)) = C(H_G^*(Y)).$$

□

Define the homomorphism of graded H_G^*-modules

$$\phi : \tilde{H}_G^*(Y) \to H_G^*(X)$$

such that if $\xi \in \tilde{H}_G^q(Y)$, write $\xi =! \sum_{i=1}^{m} \xi_i$ where $\xi_i \in H_G^q(Y_i)(-d_i)$, then

$$\phi(\xi) \doteq \sum_{i=1}^{m} \phi_i(\xi_i) \in H_G^q(X).$$

Notice that with this definition, ϕ preserves grading.

Define the homomorphism of graded H_G^*-modules

$$\sigma : H_G^*(X - Y) \to \tilde{H}_G^*(Y)$$

as follows. Let $\zeta \in H_G^q(X - Y)$. Consider the connecting homomorphism Δ_Y for the long exact sequence of the pair $X, X - Y$,

$$\Delta_Y : H_G^q(X - Y) \to H_G^{q+1}(X, X - Y),$$

and the excision isomorphism e_D obtained by excising $M - D$,

$$e_D : H_G^{q+1}(X, X - Y) \to H_G^{q+1}\left(\bigcup_{i=1}^{m} D_i, \bigcup_{i=1}^{m} (D_i - Y_i)\right) = \bigoplus_{i=1}^{m} H_G^{q+1}(D_i, D_i - Y_i),$$

for all q. There exists a unique

$$\left[e_D\left(\Delta_Y(\zeta)\right)\right]_i \in H_G^{q+1}(D_i, D_i - Y_i) \text{ such that } e_D\left(\Delta_Y(\zeta)\right) = \sum_{i=1}^{m} \left[e_D\left(\Delta_Y(\zeta)\right)\right]_i.$$

Then

$$\tau_i^{-1}\left[e_D\left(\Delta_Y(\zeta)\right)\right]_i \in H_G^{q+1}(Y_i)(-d_i) \text{ for all } i.$$

Now define

$$\sigma(\zeta) = \sum_{i=1}^{m} \tau_i^{-1}\left[e_D\left(\Delta_Y(\zeta)\right)\right]_i \in \bigoplus_{i=1}^{m} H_G^{q+1}(Y_i)(-d_i) = \tilde{H}_G^{q+1}(Y).$$

Notice that with this definition, σ raises degree by 1.

The following lemma and proof are an elaboration of [Duf83].

Lemma 3.3.9. *(The Gysin Triangle). Let a compact Lie group G act smoothly on a compact manifold X with $Y = \bigcup\limits_{i=1}^{m} Y_i$, where the Y_i's are closed, connected, G-invariant, disjoint submanifolds of X such that the normal bundle ν_{Y_i} is orientable and $\dim \nu_{Y_i} = d_i \geq 0$ for all i. Using the definitions of the maps above, we have that*

$$\begin{tikzcd} H_G^*(X-Y) \arrow[dr, "\sigma"'] & & H_G^*(X) \arrow[ll, "g^*"'] \\ & \tilde{H}_G^*(Y) \arrow[ur, "\phi"'] & \end{tikzcd}$$

is an exact triangle, where ϕ and g^* preserve degree and σ raises degree by 1.

Proof. We will approach this proof in three parts.

a. $\ker \phi \subseteq \operatorname{im} \sigma$.

Take $\xi \in \tilde{H}_G^q(Y)$ such that $\xi =! \sum\limits_{i=1}^{m} \xi_i$ where $\xi_i \in H_G^q(Y_i)(-d_i)$ for all i and $\phi(\xi) = 0$ in $H_G^q(X)$. Then we can write

$$\phi(\xi) = \sum_{i=1}^{m} \phi_i(\xi_i) = 0$$

and

$$\phi_i(\xi_i) = 0 \text{ in } H_G^q(X) \text{ for all } i.$$

Since the Gysin sequence is exact, we have that

$$\xi_i \in \operatorname{im} \sigma_i \text{ for all } i.$$

Thus, there exists $\omega_i \in H_G^{q-1}(X - Y_i)$ such that $\sigma_i(\omega_i) = \xi_i$ for all i. This implies that, for all i, we have $\tilde{h}_i^*(\omega_i) \in H_G^{q-1}(X - Y)$, so $\sum\limits_{i=1}^{m} \tilde{h}_i^*(\omega_i) \in H_G^{q-1}(X - Y)$. Using the diagrams above and the facts that Δ_Y is a homomorphism and ι_i^* is inclusion, we see that

$$\begin{aligned} \sigma\left(\sum_{i=1}^{m} \tilde{h}_i^*(\omega_i)\right) &= \sum_{i=1}^{m} \tau_i^{-1} \left[e_D \left(\Delta_Y \left(\sum_{i=1}^{m} \tilde{h}_i^*(\omega_i) \right) \right) \right]_i \\ &= \sum_{i=1}^{m} \tau_i^{-1} \left[e_D \left(\sum_{i=1}^{m} \Delta_Y \tilde{h}_i^*(\omega_i) \right) \right]_i = \sum_{i=1}^{m} \tau_i^{-1} \left[\sum_{i=1}^{m} e_D h_i^* \Delta_i(\omega_i) \right]_i \\ &= \sum_{i=1}^{m} \tau_i^{-1} \left[\sum_{i=1}^{m} \iota_i^* e_i \Delta_i(\omega_i) \right]_i = \sum_{i=1}^{m} \tau_i^{-1} e_i \Delta_i(\omega_i) \\ &= \sum_{i=1}^{m} \sigma_i(\omega_i) = \sum_{i=1}^{m} \xi_i = \xi \end{aligned}$$

We leave the reverse inclusion to the reader.

b. $\ker g^* \subseteq \operatorname{im} \phi$.

Consider $\eta \in \ker g^*$. Since g^* preserves degree, without loss of generality we assume that $\eta \in H_G^q(X)$ and $g^*\eta = 0$ in $H_G^q(X - Y)$. There exists $\hat{\eta} \in H_G^q(X, X - Y)$ such that $\eta = f^*(\hat{\eta})$ since the rows in Commutative Diagram 3.3.4 are exact. In addition, we see that

$$e_D(\hat{\eta}) = \sum_{i=1}^{m} \hat{\eta}_i \in H_G^q\left(\bigcup_{i=1}^{m} D_i, \bigcup_{i=1}^{m} (D_i - Y_i)\right),$$

where $\hat{\eta}_i \in H_G^q(D_i, D_i - Y_i)$. This implies that $(\iota_i^*)^{-1}\left[e_D(\hat{\eta})\right]_i = \hat{\eta}_i \in H_G^q(D_i, D_i - Y_i)$. Since the Gysin sequence is exact,

$$\tau_i^{-1}(\iota_i^*)^{-1}\left[e_D(\hat{\eta})\right]_i = \tau_i^{-1}\hat{\eta}_i \doteq \xi_i \in H_G^q(Y_i)(-d_i).$$

Let $\xi \in \tilde{H}_G^q(Y)$ such that $\xi = \sum_{i=1}^{m} \xi_i$ where $\xi_i \in H_G^q(Y_i)(-d_i)$ for all i. Using our various diagrams, we have

$$\begin{aligned} \phi\xi &= \sum_{i=1}^{m} \phi_i \xi_i = \sum_{i=1}^{m} f_i^* e_i^{-1} \tau_i \xi_i \\ &= \sum_{i=1}^{m} f^* h_i^* e_i^{-1} \tau_i \xi_i = f^* \sum_{i=1}^{m} e_D^{-1} \iota_i^* \hat{\eta}_i \\ &= f^* e_D^{-1} \sum_{i=1}^{m} \hat{\eta}_i = f^* \hat{\eta} = \eta \end{aligned}$$

Therefore, $\ker g^* \subseteq \operatorname{im} \phi$. We leave the reverse inclusion to the reader.

c. $\ker \sigma \subseteq \operatorname{im} g^*$.

Suppose that $\rho \in \ker \sigma$. Then $\sigma(\rho) = 0$ and $\rho \in H_G^q(X - Y)$. Notice that

$$\sigma(\rho) = \sum_{i=1}^{m} \tau_i^{-1}\left[e_D\left(\Delta_Y \rho\right)\right]_i = 0 \text{ in } \bigoplus_{i=1}^{m} H_G^{q+1}(Y_i)(-d_i).$$

$\Rightarrow \tau_i^{-1}\left[e_D\left(\Delta_Y \rho\right)\right]_i = 0$ in $H_G^{q+1}(Y_i)(-d_i)$ for all i

$\Rightarrow \left[e_D\left(\Delta_Y \rho\right)\right]_i = 0$ in $H_G^{q+1}(D_i, D_i - Y_i)$ since τ_i an isomorphism for all i

$\Rightarrow e_D\left(\Delta_Y \rho\right) = 0$ in $H_G^{q+1}\left(\bigcup_{i=1}^{m} D_i, \bigcup_{i=1}^{m} (D_i - Y_i)\right)$

$\Rightarrow \Delta_Y \rho = 0$ in $H_G^{q+1}(X, X-Y)$ since e_D an isomorphism

$\Rightarrow \rho \in \ker \Delta_Y = \operatorname{im} g^*$ by exactness in Commutative Diagram 3.3.4

Therefore, $\ker \sigma \subseteq \operatorname{im} g^*$. We leave the reverse inclusion to the reader. This completes our proof.

$\square$

Corollary 3.3.10. *Suppose that $G = \{e\}$. Then, if any two corners of the Gysin Triangle in Lemma 3.3.9 are finite dimensional graded vector spaces over k, then so is the third.*

In order to compare Poincaré series in the exact triangle in Lemma 3.3.9, we prove a lemma about Poincaré series for general exact triangles.

Lemma 3.3.11. *Consider three graded vector spaces, A, B, and C, over k such that their Poincaré series are defined. Let $A = \bigoplus_{q=0}^{\infty} A^q$, $B = \bigoplus_{q=0}^{\infty} B^q$, and $C = \bigoplus_{q=0}^{\infty} C^q$ such that there is a long exact sequence of finite dimensional spaces*

$$\cdots \to C^{q-1} \xrightarrow{\sigma} A^q \xrightarrow{\phi} B^q \xrightarrow{g^*} C^q \to \cdots$$

where $A^q = B^q = C^q = 0$ for all $q < 0$. Then,

(i.) $\mathrm{PS}(B,t) \leq \mathrm{PS}(A,t) + \mathrm{PS}(C,t)$.

(ii.) $\mathrm{PS}(A,t) \leq t\,\mathrm{PS}(C,t) + \mathrm{PS}(B,t)$.

(iii.) $t\,\mathrm{PS}(C,t) \leq \mathrm{PS}(A,t) + t\,\mathrm{PS}(B,t)$.

Proof. (i.) We know that the sequence

$$0 \longrightarrow \phi(A^q) \hookrightarrow B^q \xrightarrow{g^*} g^*(B^q) \longrightarrow 0$$

is exact for all q. This results in

$$\dim B^q = \dim \phi(A^q) + \dim g^*(B^q)$$

for all q. We have that $\dim \phi(A^q) \leq \dim A^q$ for all q, and since $g^*(B^q) \subseteq C^q$, we also have that $\dim g^*(B^q) \leq \dim C^q$ for all q. These facts imply that

$$\dim B^q \leq \dim A^q + \dim C^q$$

for all q. Therefore,

$$\mathrm{PS}(B,t) \leq \mathrm{PS}(A,t) + \mathrm{PS}(C,t).$$

(ii.) For all q, we know that the sequence

$$0 \longrightarrow \sigma(C^{q-1}) \hookrightarrow A^q \overset{\phi}{\twoheadrightarrow} \phi(A^q) \longrightarrow 0$$

is exact. This sequence results in

$$\dim A^q = \dim \sigma(C^{q-1}) + \dim \phi(A^q)$$

for all q. We have that $\dim \sigma(C^{q-1}) \leq \dim C^{q-1}$ for all q, and since $\phi(A^q) \subseteq B^q$, we also have that $\dim \phi(A^q) \leq \dim B^q$ for all q. These facts imply that

$$\dim A^q \leq \dim C^{q-1} + \dim B^q$$

for all q. Certainly, then, we have

$$\begin{aligned}
\sum_{q=0}^{\infty} \dim A^q t^q &\leq \sum_{q=0}^{\infty} \dim C^{q-1} t^q + \sum_{q=0}^{\infty} \dim B^q t^q \\
&= t \sum_{q=1}^{\infty} \dim C^{q-1} t^{q-1} + \sum_{q=0}^{\infty} \dim B^q t^q \\
&= t \sum_{q=0}^{\infty} \dim C^q t^q + \sum_{q=0}^{\infty} \dim B^q t^q,
\end{aligned}$$

resulting in

$$\mathrm{PS}(A,t) \leq t\,\mathrm{PS}(C,t) + \mathrm{PS}(B,t).$$

(iii.) For $q = -1$, $B^{-1} = C^{-1} = 0$, so $\dim B^{-1} = \dim C^{-1} = 0$. Certainly, $\dim A^0 \geq 0$. These facts result in the inequality

$$\dim C^q \leq \dim A^{q+1} + \dim B^q$$

for $q = -1$. In addition, for $q \neq -1$, we have the exact sequence

$$0 \longrightarrow g^*(B^q) \hookrightarrow C^q \xrightarrow{\sigma} \sigma(C^q) \longrightarrow 0$$

resulting in the dimension relationship

$$\dim C^q = \dim g^*(B^q) + \dim \sigma(C^q).$$

We have that $\dim g^*(B^q) \leq \dim B^q$, and since $\sigma(C^q) \subseteq A^{q+1}$, we also have that $\dim \sigma(C^q) \leq \dim A^{q+1}$ for $q \neq -1$. Thus,

$$\dim C^q \leq \dim A^{q+1} + \dim B^q$$

for all q. Certainly, then, we have

$$\begin{aligned} \sum_{q=0}^{\infty} \dim C^q t^{q+1} &\leq \sum_{q=-1}^{\infty} \dim A^{q+1} t^{q+1} + \sum_{q=0}^{\infty} \dim B^q t^{q+1} \\ t\sum_{q=0}^{\infty} \dim C^q t^q &\leq \sum_{q=0}^{\infty} \dim A^q t^q + t\sum_{q=0}^{\infty} \dim B^q t^q \\ t\,\mathrm{PS}(C,t) &\leq \mathrm{PS}(A,t) + t\,\mathrm{PS}(B,t). \end{aligned}$$

□

3.4 Two Main Theorems

The following two theorems are two of the main theorems of this paper.

Theorem 3.4.1. *Let a compact Lie group G act smoothly on a compact manifold X with $Y = \bigcup_{i=1}^{m} Y_i$ where the Y_i's are closed, connected, G-invariant, disjoint submanifolds of X such that each ν_{Y_i} is orientable and $\dim \nu_{Y_i} = d_i \geq 0$ for all i. Assume that*

(i.) $\ell\left(H_G^*(X)\right) = \ell\left(H_G^*(Y_i)\right)$ for all i and

(ii.) if $z \notin Y$, then $\ell\left(H_{G_z}^*\right) < \ell\left(H_G^*(Y_i)\right)$ for all i.

Then

$$C\left(H_G^*(X)\right) = \sum_{i=1}^{m} C\left(H_G^*(Y_i)\right).$$

Proof. Since X is compact and Y is closed, therefore compact also, we have that $H^*(X)$ and $H^*(Y)$ are finite dimensional graded vector spaces. By Corollary 3.3.10, $H^*(X-Y)$ is also finite dimensional.

For D a closed tubular neighborhood of Y as defined on page 82, let $\mathring{D}$ be the interior of D. Notice that since X is compact, $X-\mathring{D}$ is also compact. Of course, $X-\mathring{D} \subset X-Y$.

By Quillen's results, as stated in Corollary 3.2.10, we have that

$$\ell(H_G^*(X-\mathring{D})) = \max\left\{\ell\left(H_{G_z}^*\right) | z \in X-\mathring{D}\right\},$$

so $\ell\left(H_G^*(X-\mathring{D})\right) < \ell\left(H_G^*(Y_i)\right) = \ell(H_G^*(X))$ for all i, using hypotheses (i.) and (ii.). In addition, since there exists a G-deformation retraction $X-Y \to X-\mathring{D}$, resulting in

$$H_G^*(X-Y) \cong H_G^*(X-\mathring{D}),$$

we have that $\ell(H_G^*(X-Y)) < \ell(H_G^*(X))$.

Recall from Lemma 3.3.8 that $\ell\left(\tilde{H}_G^*(Y)\right) = \ell\left(H_G^*(Y)\right)$. Also,

$$\mathrm{PS}\left(\bigoplus_{i=1}^{m} H_G^*(Y_i), t\right) = \mathrm{PS}\left(H_G^*(Y_1), t\right) + \cdots \mathrm{PS}\left(H_G^*(Y_m), t\right),$$

and $\ell\left(\bigoplus_{i=1}^{m} H_G^*(Y_i)\right) = \ell\left(H_G^*(Y_i)\right) = \ell(H_G^*(X))$ for all $i = 1, \ldots, m$. Therefore,

$$\ell\left(H_G^*(X-Y)\right) < \ell\left(\tilde{H}_G^*(Y)\right) = \ell(H_G^*(X)).$$

From Lemma 3.3.11 applied to the Gysin triangle of Lemma 3.3.9,

$$\begin{aligned} \mathrm{PS}\left(H_G^*(Y), t\right) \;\leq\; & t\,\mathrm{PS}\left(H_G^*(X-Y), t\right) \\ & + \mathrm{PS}\left(H_G^*(X), t\right). \end{aligned}$$

Hence, if $\ell \doteq \ell(H_G^*(X)) = \ell(H_G^*(Y)) = \ell(H_G^*(Y_i))$ for all i, since all Poincaré series have nonnegative integer coefficients,

$$\begin{aligned} \lim_{t\to 1}(1-t)^\ell\,\mathrm{PS}\left(\tilde{H}_G^*(Y), t\right) \;\leq\; & \lim_{t\to 1}(1-t)^\ell t\,\mathrm{PS}\left(H_G^*(X-Y), t\right) \\ & + \lim_{t\to 1}(1-t)^\ell\,\mathrm{PS}\left(H_G^*(X), t\right), \end{aligned}$$

which results in

$$C\left(H_G^*(Y)\right) = C(\tilde{H}_G^*(Y)) \leq C\left(H_G^*(X)\right)$$

since $\ell\left(H_G^*(X-Y)\right) < \ell\left(H_G^*(X)\right) = \ell$, using Lemma 3.3.8, part (d.).

On the other hand, from Lemma 3.3.11 applied to the Gysin triangle of Lemma 3.3.9,

$$\mathrm{PS}\left(H_G^*(X), t\right) \leq \mathrm{PS}\left(\tilde{H}_G^*(Y), t\right) + \mathrm{PS}\left(H_G^*(X-Y), t\right).$$

Hence,

$$\begin{aligned} \lim_{t\to 1}(1-t)^\ell\, \mathrm{PS}\left(H_G^*(X), t\right) &\leq \lim_{t\to 1}(1-t)^\ell\, \mathrm{PS}\left(\tilde{H}_G^*(Y), t\right) \\ &\quad + \lim_{t\to 1}(1-t)^\ell\, \mathrm{PS}\left(H_G^*(X-Y), t\right), \end{aligned}$$

which results in

$$C\left(H_G^*(X)\right) \leq C\left(\tilde{H}_G^*(Y)\right) = C(H_G^*(Y))$$

since $\ell\left(H_G^*(X-Y)\right) < \ell\left(H_G^*(X)\right)$. Therefore,

$$C\left(H_G^*(X)\right) = C\left(\bigoplus_{i=1}^{m} H_G^*(Y_i)\right) = \sum_{i=1}^{m} C(H_G^*(Y_i)).$$

□

Our last step is to prove finally Theorem 3.3.3, which was stated on page 82.

Proof. We can write

$$Z_i = \coprod_{j=1}^{q_i} Gc_{i,j}$$

where $c_{i,j} \in \pi_0(Z_i)$, $Gc_{i,j} = \bigcup_{g\in G} gc_{i,j}$, and $Gc_{i,j} \cap Gc_{i,k} = \emptyset$ for all $j \neq k$. In other words, $\{c_{i,j} \mid 1 \leq j \leq q_i\}$ is a set of representatives for the orbits of the G-action on $\pi_0(Z_i)$. Notice that $Gc_{i,j}$ is closed and G-invariant and that $\nu_{Gc_{i,j}}$ is orientable and of constant dimension for each i, j. Without loss of generality, we can order the $Gc_{i,j}$ such that

$$Z_i = Gc_{i,1} \coprod \cdots \coprod Gc_{i,r_i} \coprod Gc_{i,r_i+1} \coprod \cdots \coprod Gc_{i,q_i}$$

where

$$\ell\left(H_G^*(Gc_{i,j})\right) = \ell\left(H_G^*(X)\right) \text{ for } 1 \leq j \leq r_i$$

and

$$\ell\left(H_G^*(Gc_{i,j})\right) < \ell\left(H_G^*(X)\right) \text{ for } r_i < j \leq q_i.$$

Here we use the hypothesis that $\ell(H_G^*(X)) = \ell(H_G^*(Z_i))$ for all i.

Let $\tilde{Z}_i = \coprod_{j=1}^{r_j} Gc_{i,j} \subseteq Z_i$ for each i, and let $\tilde{Z} = \coprod_{i=1}^{n} \tilde{Z}_i \subseteq Z$. Note that

$$\ell(H_G^*(\tilde{Z}_i)) = \ell(H_G^*(X)) \text{ for all } i,$$

by definition. Suppose that $z \in X - \tilde{Z}$. On the one hand, suppose $z \notin Z$. Then $\ell(H_{G_z}^*) < \ell(H_G^*(X)) = \ell(H_G^*(\tilde{Z}_i))$ for all i by the hypothesis. On the other hand, suppose $z \in Z$. Then $z \in Z_i$ but $z \notin \tilde{Z}_i$ for some unique i, and so $z \in Gc_{i,j}$ for some j, $r_i < j \leq q_i$. By definition,

$$\ell(H_G^*(Gc_{i,j})) < \ell(H_G^*(X)) \text{ since } r_i < j \leq q_i.$$

Now, since $z \in Gc_{i,j}$, Lemma 3.2.19 shows that

$$\ell(H_{G_z}^*) \leq \ell(H_G^*(Gc_{i,j})),$$

and we know that, by definition of $\tilde{Z}_i$,

$$\ell(H_G^*(X)) = \ell(H_G^*(\tilde{Z}_i)) \text{ for all } i.$$

So, we see that, if $z \in X - \tilde{Z}$,

$$\ell(H_{G_z}^*) < \ell(H_G^*(\tilde{Z}_i)) \text{ for all } i.$$

By Theorem 3.4.1 applied to $\tilde{Z} \subseteq X$, decomposed as $\coprod_{i=1}^{n}\coprod_{j=1}^{r_j} Gc_{i,j}$, we see that

$$C(H_G^*(X)) = \sum_{i=1}^{n}\sum_{j=1}^{r_i} C\left(H_G^*(Gc_{i,j})\right).$$

Since

$$Z = \coprod_{i=1}^{n} Z_i = \coprod_{i=1}^{n}\coprod_{j=1}^{q_i} Gc_{i,j},$$

we have

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{PS}(H_G^*(Z_i),t) &= \sum_{i=1}^{n}\sum_{j=1}^{q_i} \mathrm{PS}(H_G^*(Gc_{i,j}),t)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{r_i} \mathrm{PS}(H_G^*(Gc_{i,j}),t) + \sum_{i=1}^{n}\sum_{j=r_i+1}^{q_i} \mathrm{PS}(H_G^*(Gc_{i,j}),t).\end{aligned}$$

Multiplying by $(1-t)^{\ell(H_G^*(X))}$, and taking the limit as $t \to 1$, we have

$$\sum_{i=1}^{n} C(H_G^*(Z_i)) = \sum_{i=1}^{n}\sum_{j=1}^{r_i} C(H_G^*(Gc_{i,j}))$$

since $\ell(H_G^*(Gc_{i,j})) = \ell(H_G^*(X))$ for $1 \le j \le r_i$ and $\ell(H_G^*(Gc_{i,j})) < \ell(H_G^*(X))$ for $r_i < j \le q_i$ for all i.

Therefore,

$$C(H_G^*(X)) = \sum_{i=1}^{n} C(H_G^*(Z_i)).$$

□

3.5 Properties of U/S

In this section, we apply our results from Section 3.2 to a special manifold in order to apply Theorem 3.3.3. Recall from Chapter 2, that the unitary group is defined as

$$U(n,k) = \{A \in GL_n(k) \mid AA^* = I\},$$

where $k = \mathbb{R}$ or $k = \mathbb{C}$. For convenience, we let $U(n) \doteq U(n,\mathbb{C})$. We have seen in Chapter 2 that $U(n)$ is a compact smooth manifold of real dimension n^2.

Let

$$T(n) = \{A \in GL_n(\mathbb{C}) \mid a_{ij} = 0 \text{ for all } i \neq j, |a_{ii}| = 1 \text{ for all } i\} \cong \underbrace{S^1 \times \cdots \times S^1}_{n}$$

and let

$$S_p(n) = \{A \in T(n) \mid a_{ii}^p = 1 \text{ for all } i = 1,\ldots,n\} \cong \underbrace{\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}}_{n},$$

the "diagonal" p-torus of rank n in $U(n)$. Since $U(n)$ is a manifold and $S_p(n)$ acts freely and differentiably on $U(n)$ via left multiplicaiton, the orbit space

$$F(n) \doteq U(n)/S_p(n)$$

is a manifold, too [tD87, Ch. 1, Prop. 5.2].

From this point forward, we will use $U = U(n)$, $T = T(n)$, $S = S_p(n)$, and $F = F(n)$ for some n.

Fix an embedding of a compact Lie group G in the unitary group U. This makes U a differentiable G-manifold with G acting on U by left multiplication. Of course, S also acts on U by left multiplication. Let $F = U/S$ be the compact smooth G-manifold of orbits. We now consider several properties involving F.

Lemma 3.5.1. *Let $F = U/S$, and let G be a compact Lie group embedded in U. For all $z \in F$, G_z is a p-torus in G. Furthermore, for every p-torus A of G, there exists a $z \in F$ such that $A \leq G_z$. Thus, $\mathcal{A}_0(G, F) = \mathcal{A}(G)$, and $\mathcal{B}(G, F) = \mathcal{B}(G)$.*

Proof. Recall that $S \cong \underbrace{\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p}_{n}$, a p-torus.

Since $z \in F = U/S$, there exists $u \in U$ such that $z = uS$. Then,

$$\begin{aligned} G_z &= \{g \in G \mid gz = guS = uS\} = \{g \in G \mid u^{-1}gu \in S\} \\ &= \{g \in G \mid g \in uSu^{-1}\} = uSu^{-1} \cap G. \end{aligned}$$

Notice that $G_z \leq uSu^{-1} \cong S$, a finite abelian group and a p-torus of rank n. Therefore, by Lemma 3.2.1, G_z is a p-torus also.

Now, suppose that A is a p-torus in G. Then A is a set of commuting matrices over $\mathbb{C}$, all of which are diagonalizable. Therefore the set A is "simultaneously diagonalizable"; i.e., there exists an element $u \in U$ such that $uAu^{-1} \subset T$. Since uAu^{-1} is a p-torus, we must have $uAu^{-1} \subseteq S$. So, $G_{u^{-1}S} = u^{-1}Su \cap G \geq A$. $\square$

A direct application of Lemmas 2.1.8, 3.5.1, 3.2.18, and 3.2.19 results in the following corollary. Recall that $\mathcal{B}(G)$ is the set of conjugacy classes of p-tori in G of maximal rank.

Corollary 3.5.2. *For G a compact Lie subgroup of U and $F = U/S$, the following properties hold.*

a. If A is a maximal (with respect to $\leq$) p-torus in G, then $F^A \neq \emptyset$, and for all $x \in F^A$, $A = G_x$. Furthermore, for all $x \in GF^A$, $A \sim G_x$.

b. $\ell(H_G^*(F)) = \ell(H_G^*(GF^A))$ for all $[A] \in \mathcal{B}(G)$.

c. $GF^A \cap GF^B = \emptyset$ for all $[A] \neq [B]$ with $[A], [B] \in \mathcal{B}(G)$.

d. If $z \notin \coprod_{[A] \in \mathcal{B}(G)} GF^A$, then $\ell(H_{G_z}^*) < \ell(H_G^*(GF^A))$ for all $[A] \in \mathcal{B}(G)$.

From Duflot [Duf83, Cor. 1 and 2], we can conclude the following lemma.

Lemma 3.5.3. *[Duf83] Let G be a compact Lie group, and let A be a p-torus acting smoothly on a differentiable manifold X. Assume that X has a smooth G-invariant Riemannian metric. Then, the normal bundle ν_{X^A} has a complex structure and, therefore, is orientable.*

Note that Duflot's paper [Duf83] requires p odd. For the $p = 2$ case, all bundles are orientable mod-2, so we do not need Lemma 3.5.3 if $p = 2$.

Applying this lemma to the differentiable manifold $F = U/S$ and to the compact Lie group G embedded in U, we have the following corollary.

Corollary 3.5.4. *For G a compact Lie group embedded in U and $F = U/S$, the normal bundle for F^A, ν_{F^A}, is orientable for every p-torus A of G.*

3.6 Application of Theorem 3.3.3 to $F = U/S$

Given the many facts concerning the space $F = U/S$ where G is a compact Lie group embedded in U as shown in Lemmas 2.1.10, 2.1.11, 3.2.17, 3.2.4, and 3.5.3 and Corollaries 3.2.10 and 3.2.11, we now apply Theorem 3.3.3 to this special case, resulting in a significant formula for calculating the C-multiplicity of $H_G^*(F)$.

Theorem 3.6.1. *Let G be a compact Lie group which embeds in U, let $F = U/S$, and let $\mathcal{B}(G)$ be the set of conjugacy classes of maximal rank p-tori of G. Then*

$$C\left(H_G^*(F)\right) = \sum_{[A]\in\mathcal{B}(G)} C\left(H_G^*(GF^A)\right).$$

Using further results from [Duf84], we will use the above theorem to conclude with a nice formula for finding the C-multiplicity of H_G^*. Further embellishments will be considered in future research.

3.6.1 Review of Duflot's Results

Definition 3.6.2. *The* normalizer *of A in G is defined by*

$$N_G(A) = \{g \in G | gA = Ag\},$$

the largest subgroup of G having A as a normal subgroup. The centralizer *of A in G is defined by*

$$C_G(A) = \{g \in G | ga = ag \text{ for all } a \in A\}.$$

It is important to note that $N_G(A)$ and $C_G(A)$ are both subgroups of G and that $C_G(A)$ is a normal subgroup of $N_G(A)$. The Weyl group *of A in G is defined by*

$$W_G(A) \doteq N_G(A)/C_G(A).$$

Note that $W_G(A)$ is a finite group. More generally, in order to apply the above definitions to Quillen's category of pairs, we make the following definiton.

Definition 3.6.3. *For G a group which acts smoothly on a manifold X and $(A, c) \in \mathcal{A}(G, X)$, we define*

$$\begin{aligned} N_G(A,c) &= \{g \in G \mid gA = Ag \text{ and } gc = c\}, \\ C_G(A,c) &= \{g \in G \mid ga = ag \text{ for all } a \in A \text{ and } gc = c\}, \\ W_G(A,c) &= N_G(A,c)/C_G(A,c). \end{aligned}$$

Lemma 3.6.4. *[Duf84, Lem. 3.4] Let G be a compact Lie group, and fix an embedding of G in a unitary group U. Let $F = U/S$, and let (A, c) be maximal in $\mathcal{A}(G, X)$, where X is either compact and every orbit of X is a G-deformation retract of one of its neighborhoods, or X is paracompact with finite mod-p cohomological dimension. Then, there are isomorphisms for $i \geq 1$,*

$$H^*_G\left(G \cdot \left(c \times (F^A)^i\right)\right) \xrightarrow{\cong} H^*_{N_G(A,c)}\left(c \times (F^A)^i\right).$$

Recall that if Y is a topological space, then $\pi_0(Y)$ is the set of connected components of Y.

The following theorem was proved on pages 98–99 of Duflot [Duf84].

Theorem 3.6.5. *[Duf84] Let G be a compact Lie group, and let A be a p-torus of G. Fix an embedding of G in a unitary group U, and let $F = U/S$ as a G-space. Then, for every $A \in \mathcal{A}(G, F) = \mathcal{A}(G)$, and for every component c of F^A,*

a. $C_G(A, c) = C_G(A)$, and $C_G(A)$ acts trivially on $\pi_0(F^A)$. Therefore the group $W_G(A) = N_G(A)/C_G(A)$ acts on $\pi_0(F^A)$.

b. $W_G(A) \doteq N_G(A)/C_G(A)$ acts freely on $\pi_0(F^A)$.

Applying the theorem above, Duflot concludes the following lemma.

Lemma 3.6.6. *[Duf84, Lem. 3.5 and 3.6] Let G be a compact Lie group, and fix an embedding of G in a unitary group U. If $(A, c) \in \mathcal{A}(G, X)$, where X is either compact and every orbit of X is a G-deformation retract of one of its neighborhoods, or X is paracompact with finite mod-p cohomological dimension, then*

a. for $F = U/S$, $H^*_{C_G(A,c)}\left(c \times (F^A)^i\right)$ is a free $\mathbb{Z}/p\mathbb{Z}[W_G(A, c)]$-module for every $i \geq 1$, where $W_G(A, c) = N_G(A, c)/C_G(A, c)$.

b. there are isomorphisms for $i \geq 1$,

$$H^*_{N_G(A,c)}\left(c \times (F^A)^i\right) \xrightarrow{\cong} H^*_{C_G(A,c)}\left(c \times (F^A)^i\right)^{W_G(A,c)}.$$

Applying Lemmas 3.6.4 and 3.6.6 to $X = \{x_0\}$, a one-point space, we have the following corollary.

Corollary 3.6.7. *Let G be a compact Lie group, and fix an embedding of G in a unitary group U. Let $F = U/S$, and let A be maximal in $\mathcal{A}(G)$. Then, there are isomorphisms*

$$H_G^*\left(GF^A\right) \cong H_{C_G(A)}^*\left(F^A\right)^{W_G(A)}.$$

3.6.2 Application of Duflot's Results

We first consider the following well-known lemma from representation theory, which we will apply to the lemmas stated earlier in this section.

Lemma 3.6.8. *Let W be a finite group and let V be a free $k[W]$-module, for k a field, of rank d. Then*

$$\dim_k V^W = \frac{\dim_k V}{|W|}.$$

Proof. Since V is a free $k[W]$-module, then

$$V \cong \underbrace{k[W] \oplus \cdots \oplus k[W]}_{d}$$

with d copies of $k[W]$. Consider an element $\xi \in k[W]$. We may write the element $\xi = \sum_{g \in W} \alpha_g g$ where $\alpha_g \in k$. Then we see that $\dim_k k[W] = |W|$ because W is a basis for $k[W]$ over k, and we determine that $\dim_k V = d|W|$.

Now consider

$$V^W \cong \left(\underbrace{k[W] \oplus \cdots \oplus k[W]}_{d}\right)^W \cong \underbrace{(k[W])^W \oplus \cdots \oplus (k[W])^W}_{d}.$$

For an element $\xi \in (k[W])^W$, we know that $\xi \in k[W]$ and $w \cdot \xi = \xi$ for all $w \in W$. Therefore, for all $w \in W$,

$$\begin{aligned} w \cdot \xi &= w \cdot \sum_{g \in W} \alpha_g g \\ &= \sum_{g \in W} \alpha_g (wg) \\ &= \sum_{g \in W} \alpha_g g. \end{aligned}$$

Hence, $\alpha_g = \alpha_{wg}$ for all $w \in W$, implying that $\alpha_g = \alpha$ for all $g \in W$. Define $N \doteq \sum_{g \in W} g$. Then for all $\xi \in (k[W])^W$,

$$\xi = \sum_{g \in W} \alpha g = \alpha \sum_{g \in W} g = \alpha N.$$

Then we see that $(k[W])^W$ is one-dimensional, generated over k by N. Therefore, $\dim_k V^W = d \cdot 1 = d$, and we can conclude that $\dim_k V^W = d = \frac{\dim_k V}{|W|}$. $\square$

As a direct result of Lemmas 3.6.6 and 3.6.8, we get the following corollary.

Corollary 3.6.9. *Let G be a compact Lie group, and fix an embedding of G in U. Let $F = U/S$ and let A be a maximal rank p-torus in G. Then,*

$$\dim_{\mathbb{Z}/p\mathbb{Z}} \left(H^q_{C_G(A)}(F^A)^{W_G(A)} \right) = \frac{1}{|W_G(A)|} \dim_{\mathbb{Z}/p\mathbb{Z}} H^q_{C_G(A)}(F^A)$$

for all $q \geq 0$.

Applying Corollaries 3.6.7 and 3.6.9 to the definition of Poincaré series, we have the following result.

Corollary 3.6.10. *Let G be a compact Lie group, and fix an embedding of G in U. Let $F = U/S$, and let A be a maximal rank p-torus in G. Then,*

$$PS\left(H^*_G(GF^A)\right) = \frac{1}{|W_G(A)|} PS\left(H^*_{C_G(A)}(F^A)\right).$$

3.7 Main Results Regarding Maiorana's C-Multiplicity

In this portion of this chapter, we conclude with a series of significant theorems. The first of these theorems results in a formula for computing $C(H^*_G(F))$, which is more simple than that found in Theorem 3.6.1. The final theorem states a nice formula for computing the C-multiplicity of H^*_G using the C-multiplicities of $H^*_{C_G(A)}$ for A a maximal rank p-torus in G.

Theorem 3.7.1. *Let G be a compact Lie group embedded in a unitary group U, $F = U/S$, and $\mathcal{B}(G)$ be the set of conjugacy classes of maximal rank p-tori in G. Then*

$$C\left(H_G^*(F)\right) = \sum_{[A]\in\mathcal{B}(G)} \frac{1}{|W_G(A)|} C\left(H_{C_G(A)}^*(F^A)\right).$$

Proof. From Corollary 3.6.10, we can conclude that

$$\ell \doteq \ell\left(H_G^*(GF^A)\right) = \ell\left(H_{C_G(A)}^*(F^A)\right).$$

Using the definition of the C-multiplicity and applying it to the result in Corollary 3.6.10, we see that

$$\begin{aligned} C\left(H_G^*(GF^A)\right) &= \lim_{t\to 1}(1-t)^\ell PS\left(H_G^*(GF^A)\right) \\ &= \lim_{t\to 1}(1-t)^\ell \tfrac{1}{|W_G(A)|} PS\left(H_{C_G(A)}^*(F^A)\right) \\ &= \tfrac{1}{|W_G(A)|}\lim_{t\to 1}(1-t)^\ell PS\left(H_{C_G(A)}^*(F^A)\right) \\ &= \tfrac{1}{|W_G(A)|} C\left(H_{C_G(A)}^*(F^A)\right) \end{aligned}$$

From Theorem 3.6.1, we have that

$$\begin{aligned} C\left(H_G^*(F)\right) &= \sum_{[A]\in\mathcal{B}(G)} C\left(H_G^*(GF^A)\right) \\ &= \sum_{[A]\in\mathcal{B}(G)} \tfrac{1}{|W_G(A)|} C\left(H_{C_G(A)}^*(F^A)\right). \end{aligned}$$

□

Now we want to connect $H_G^* = H^*(BG)$ to $\bigoplus_{[A]\in\mathcal{B}(G)} H_G^*(GF^A)$.

Corollary 3.7.2. *Let $F = U/S$ and G be a compact Lie group embedded in U. Then*

$$H_G^*(F) \cong H_G^* \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F).$$

Proof. From Quillen's Lemma 6.5 in [Qui71a], we can conclude that

$$H_G^*(F) \cong H_G^* \otimes_{H_U^*} H_S^*.$$

The proof of Quillen's lemma cited above also gives us the following fact:

$$H^*\left(BU, H^*(F)\right) \cong H^*(BS).$$

Here, the coefficients $H^*(F)$ are simple coefficients, since, considering the long exact sequence in homotopy for a fibration and using the facts that U is connected and EU is contractible, we know that $\pi_1(BU)$ is trivial. Since the cohomology of BU is free over the integers, applying the Universal Coefficient Theorem (as found in [Mun84]) then gives us

$$H^*\left(BU, H^*(F)\right) \cong H^*(BU) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F).$$

Since $H^*_S = H^*(BS)$ and $H^*_U = H^*(BU)$, we can conclude

$$\begin{aligned} H^*_G(F) &\cong H^*_G \otimes_{H^*_U} H^*_S \\ &\cong H^*_G \otimes_{H^*_U} \left(H^*_U \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F)\right) \\ &\cong \left(H^*_G \otimes_{H^*_U} H^*_U\right) \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F) \\ &\cong H^*_G \otimes_{\mathbb{Z}/p\mathbb{Z}} H^*(F). \end{aligned}$$

□

Theorem 3.7.3. *If a compact Lie group G embeds in a unitary group U and $F = U/S$, then*

$$C\left(H^*_G(F)\right) = C(H^*_G)f(1)$$

where $f(t) = PS\left(H^*(F), t\right)$.

Proof. Applying our facts about Poincaré series in Proposition 1.5.2 to Corollary 3.7.2, we have

$$PS\left(H^*_G(F), t\right) = \left[PS\left(H^*_G, t\right)\right] \cdot \left[PS\left(H^*(F), t\right)\right].$$

Recall that F is a compact manifold with finite dimension, so $H^q(F) = 0$ for all $q > \dim(F)$, implying that $PS\left(H^*(F), t\right)$ is a polynomial: $f(t)$. In addition, $f(1) \neq 0$ since $f(1) = \sum_{i=0}^{dim(F)} \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^i(F)\right)$, the total dimension of the cohomology of F, which is not 0. Hence,

$$\ell \doteq \ell\left(H^*_G(F)\right) = \ell\left(H^*_G\right).$$

Using the definition of the C-multiplicity, we have that

$$\begin{aligned} C(H^*_G(F)) &= \lim_{t\to 1}(1-t)^{\ell} PS\left(H^*_G(F),t\right) \\ &= \lim_{t\to 1}(1-t)^{\ell} PS\left(H^*_G,t\right) PS\left(H^*(F),t\right) \\ &= \lim_{t\to 1}(1-t)^{\ell} PS\left(H^*_G,t\right) f(t) \\ &= C\left(H^*_G\right)\cdot f(1). \end{aligned}$$

□

Corollary 3.7.4. *Let G be a compact Lie group which embeds in U, let $F = U/S$, and let A be a maximal rank p-torus in G. Then,*

$$C\left(H^*_{C_G(A)}(F^A)\right) = C\left(H^*_{C_G(A)}\right) f(1),$$

where $f(t) = PS\left(H^*(F),t\right)$.

Proof. Notice that $C_{C_G(A)}(A) = C_G(A)$ and that $N_{C_G(A)}(A) = C_{C_G(A)}(A)$, so $W_{C_G(A)}(A)$ contains only the identity. Applying the results of Theorem 3.7.1 to $C_G(A)$ instead of G, since A is of maximal rank in $C_G(A)$ and there is only one conjugacy class of A in $C_G(A)$, we have that

$$\begin{aligned} C\left(H^*_{C_G(A)}(F)\right) &= \sum_{[A]\in\mathcal{B}(C_G(A))} \frac{1}{|W_{C_G(A)}(A)|} C\left(H^*_{C_G(A)}(F^A)\right) \\ &= C\left(H^*_{C_G(A)}(F^A)\right). \end{aligned}$$

Applying Theorem 3.7.3 with $C_G(A)$ instead of G, $C_G(A) \subseteq G \subseteq U$, we also have that

$$C\left(H^*_{C_G(A)}(F)\right) = C\left(H^*_{C_G(A)}\right)\cdot f(1),$$

where $f(t) = PS\left(H^*(F),t\right)$. Therefore,

$$C\left(H^*_{C_G(A)}(F^A)\right) = C\left(H^*_{C_G(A)}\right)\cdot f(1).$$

□

Finally, we have our "centerpiece" Theorem:

Theorem 3.7.5. *Let G be a compact Lie group, and let $\mathcal{B}(G)$ be the set of conjugacy classes of maximal rank p-tori of G. Then*

$$C\left(H_G^*\right) = \sum_{[A]\in\mathcal{B}(G)} \frac{1}{|W_G(A)|} C\left(H^*_{C_G(A)}\right).$$

Proof. Representation theory tells us that there exists a unitary group U and an embedding $G \hookrightarrow U$ of G as a closed subgroup of U (see [Pri77], Theorem 6.1.1). Fix such an embedding, and let $F = U/S$, as usual.

By Theorem 3.7.3, we have that

$$C\left(H_G^*(F)\right) = C\left(H_G^*\right) \cdot f(1),$$

where $f(t) = PS\left(H^*(F), t\right)$. By Theorem 3.7.1, we have that

$$C\left(H_G^*(F)\right) = \sum_{[A]\in\mathcal{B}(G)} \frac{1}{|W_G(A)|} C\left(H^*_{C_G(A)}(F^A)\right).$$

Combining these two theorems and using the fact from Corollary 3.7.4, we have that

$$C(H_G^*) \cdot f(1) = \sum_{[A]\in\mathcal{B}(G)} \frac{1}{|W_G(A)|} C\left(H^*_{C_G(A)}\right) f(1).$$

Therefore, dividing both sides by $f(1)$ results in the nice formula

$$C(H_G^*) = \sum_{[A]\in\mathcal{B}(G)} \frac{1}{|W_G(A)|} C\left(H^*_{C_G(A)}\right).$$

□

3.8 Final Remarks

Let G be a compact Lie group. Let the graded ring H^* be finitely generated as a graded-commutative algebra over $H^0 = k$, a field of characteristic $p > 0$ odd prime.

We have seen that there is a filtration of H_G^* by graded H_G^{ev}-submodules

$$0 = \mathcal{F}^{n+1}(H_G^*) \subseteq \cdots \subseteq \mathcal{F}^0(H_G^*) = H_G^*,$$

homogeneous prime ideals $\mathfrak{p}_i$ in H_G^{ev}, and positive integers $\tilde{d}_i$, for $0 \le i \le n$, with

$$\mathcal{F}^i(H_G^*)/\mathcal{F}^{i+1}(H_G^*) \cong H_G^{ev}/\mathfrak{p}_i(-\tilde{d}_i)$$

as H_G^{ev}-modules, for every i. We also know that every minimal prime ideal for H_G^* as an H_G^{ev}-module occurs as at least one of the $\mathfrak{p}_i$.

Recall from Chapter 1, we have

$$C(H_G^*) = \sum_{\mathfrak{p} \in \mathcal{D}(H_G^*)} N_{\mathfrak{p}} C(H_G^{ev}/\mathfrak{p})$$

where $N_{\mathfrak{p}}$ is the number of times that $H_G^{ev}/\mathfrak{p}$, twisted possibly, occurs as a factor in the filtration, and $\mathcal{D}(H_G^*)$ is the subset of minimal primes defined by: $\mathfrak{p} \in \mathcal{D}(H_G^*)$ if and only if $\mathrm{Dim}(H_G^*) = \mathrm{Dim}(H_G^{ev}/\mathfrak{p})$.

By Quillen [Qui71b, Prop. 11.2], as applied to the context of X a one-point set, there is a one-to-one correspondence between conjugacy classes of maximal p-tori in $\mathcal{A}(G)$ and minimal prime ideals of H_G^{ev} given by associating to A the prime ideal $\mathfrak{p}_A$. For $\mathrm{rank}\, A = n$, this correspondence is defined by

$$H_G^{ev} \overset{\mathrm{res}}{\to} H_A^{ev} \overset{\pi}{\to} H_A^{ev}/\sqrt{0} \cong k[y_1, \ldots, y_n].$$

(Recall Example 2.1.17.) $\pi \circ \mathrm{res} : H_G^{ev} \to H_A^{ev}/\sqrt{0}$ has kernel

$$\mathfrak{p}_A \doteq \ker\left(\pi \circ \mathrm{res}\right).$$

(Note: $\sqrt{0}$ is the nilradical of H_A^{ev}.)

Using Lemma 3.1.3 and 3.2.5, we see that $H_A^{ev}/\sqrt{0}$ is a finitely generated H_G^{ev}-module, and using theorems from Kaplansky, as in the proof of Lemma 3.1.3,

$$\mathrm{Dim}\left(H_G^{ev}/\mathfrak{p}_A\right) = \mathrm{Dim}\left(H_A^{ev}/\sqrt{0}\right) = n = \mathrm{rank}\, A,$$

since $H_A^{ev}/\sqrt{0}$ is a polynomial ring in n-variables. Therefore, the $\mathfrak{p}_A \in H_G^{ev}$ such that $\mathrm{Dim}\, H_G^{ev} = \mathrm{Dim}\left(H_G^{ev}/\mathfrak{p}_A\right)$ are exactly those corresponding to $A \in \mathcal{A}(G)$ with maximal rank; that is, $[A] \in \mathcal{B}(G)$.

Therefore, we also have

$$C(H_G^*) = \sum_{[A]\in\mathcal{B}(G)} N_{\mathfrak{p}_A} C(H_G^{ev}/\mathfrak{p}_A).$$

Notice that this sum formula has the same index set as the sum formula in Theorem 3.7.5. A comparison of the summands of each formula is reserved for future research.

Chapter 4

FUTURE RESEARCH

We first note that straightforward modifications may be applied to extend our results to $X \times F$, rather than just F, to obtain the following conjecture:

Conjecture 4.0.1. *Let G be a compact Lie group, and let either X be compact with every orbit of X a G-deformation retract of one of its neighborhoods, or X be paracompact with finite mod-p cohomological dimension. Let $\tilde{\mathcal{B}}(G,X)$ be the set of conjugacy classes of $(A,c) \in \mathcal{A}(G,X)$ of maximal rank in $\mathcal{A}_0(G,X)$. Then,*

$$C(H_G^*(X)) = \sum_{[(A,c)] \in \tilde{\mathcal{B}}(G,X)} \frac{1}{|W_G(A,c)|} C(H^*_{C_G(A,c)}(c)).$$

Some form of this conjecture is certainly true, but we chose not to explicitly develop the modifications necessary in this paper in order to simplify the exposition. An initial future project will be to explicitly develop and write down these modifications.

Recall that the ultimate goal of this paper was to determine how to relate the algebraic definitions and computations of multiplicity from commutative algebra to computations done in the cohomology theory of group actions on manifolds. In this paper, we have accomplished major steps toward this goal:

a. We have set definitions and constructed explicit proofs extending the theory of the Samuel multiplicity to the context of graded rings suitable for our purposes. This work resulted in an associativity (or linearity) formula for $e(M, \mathcal{I})$ and $C(M)$, the Samuel multiplicity and Maiorana's C-multiplicity, respectively.

b. We related the Samuel multiplicity, $e(M, \mathcal{I})$, in general to Maiorana's C-multiplicity, $C(M)$.

c. Using Quillen's results in [Qui71a] and [Qui71b], we have proven a generalization of Maiorana's work in [Mai76]. We did not use any of Maiorana's results in our proofs. This theorem resulted in a topological formula for computing the C-multiplicity, expressing $C(M)$ in terms of fixed point sets.

d. Using our own results and Quillen's work, we studied the topological sum formula for the situation where, for U the unitary group and S the p-torus in U, $F = U/S$ is a compact differentiable manifold, with G a compact Lie group embedded in U, as constructed by Duflot in [Duf84]. This work resulted in a topological sum formula, not involving F, for $C(H_G^*)$.

e. Using Quillen, we explored the connection between the algebraic and topological sum formulas for $C(H_G^*)$.

In the future, we plan to extend the work done in this thesis in the following ways:

a. We will continue to investigate the commutative algebra sum formula and the topological sum formula for the C-multiplicity of H_G^* to determine connections between the summands of these formulas.

b. We will consider the lists of computations of cohomology of groups which have been done in the past ten years and compare these computations with our theorems.

c. We will study the Steenrod algebra and investigate the work we framed in terms of modules over the Steenrod algebra. We will study how the invariants presented in this paper can be related to the invariants Kuhn presents in [Kuh07].

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