UNSTEADY FLOW OF GROUND WATER INTO EQUALLY SPACED DRAINS[1]

by

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SYNOPSIS

A solution to the nonlinear differential equation describing unsteady flow toward equally spaced drains above a horizontal impermeable boundary is presented.

The solution is compared with field data and a numerical solution. The solution was found to agree with the field data and the numerical solution when the drain spacing was large relative to the depth of drains.

INTRODUCTION

On agricultural lands underlain by impermeable boundaries of zero slope where there is little natural subsurface drainage, equally spaced

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drains may be installed to adequately lower the water table for crop production. The depth and spacing of the drains are two important design factors which control the lowering of the water table.

In recent years, some approximations (1, 2, 4, 11, 13) have been obtained for the solution of drainage problems involving a falling water table. Many of the approximations are objectionable (12) because the assumptions upon which they are based are not realistic and hence the solutions may not be sufficiently accurate for design purposes. This discussion deals with improving an approximation developed by Glover (2) by eliminating an assumption in his development. Such an approximation will provide information on depth and spacing of drains as well as water table recession rates.

As reported by Dumm (2), Glover developed a solution based upon the heat flow equation for the problem of the falling water table. Using the Dupuit assumption, he considered equally spaced tile drains in a homogeneous soil overlying a horizontal impermeable layer. In the development of Glover's solution an approximation was made which restricts the use of his equation to cases in which the distance from the tile to the impermeable boundary is large in relation to the drawdown of the water table. Experience has indicated (2) that when the assumptions are satisfied, satisfactory results may be obtained. In summarizing some of the recent studies on the falling water table van Schilfgaarde (12) stated that "Glover's equation, based on the assumption of horizontal flow, appears to be more nearly correct than any other, but it is not sufficiently accurate to be used for design purposes."

The approximate solution presented herein takes into account the drawdown of the water table which implies that the distance from the tile to the impermeable boundary may be small compared to the drawdown of the water table. However, the approximate solution still involves the Dupuit assumption.

THEORY

Consider a system of equally spaced drains in a homogeneous soil overlying an impermeable boundary of zero slope as shown in figure 1.

Figure 1. Model of equally spaced drains above an impermeable boundary with accompanying boundary and initial conditions.

The equation for flow based upon the Dupuit assumption and Darcy's law may be written as

$$Q = K(D + h)\frac{\partial h}{\partial x}, \tag{1}$$

in which Q is the volume rate of flow in the x direction per unit length of tile, K is the hydraulic conductivity, and the other symbols are as defined in figure 1.

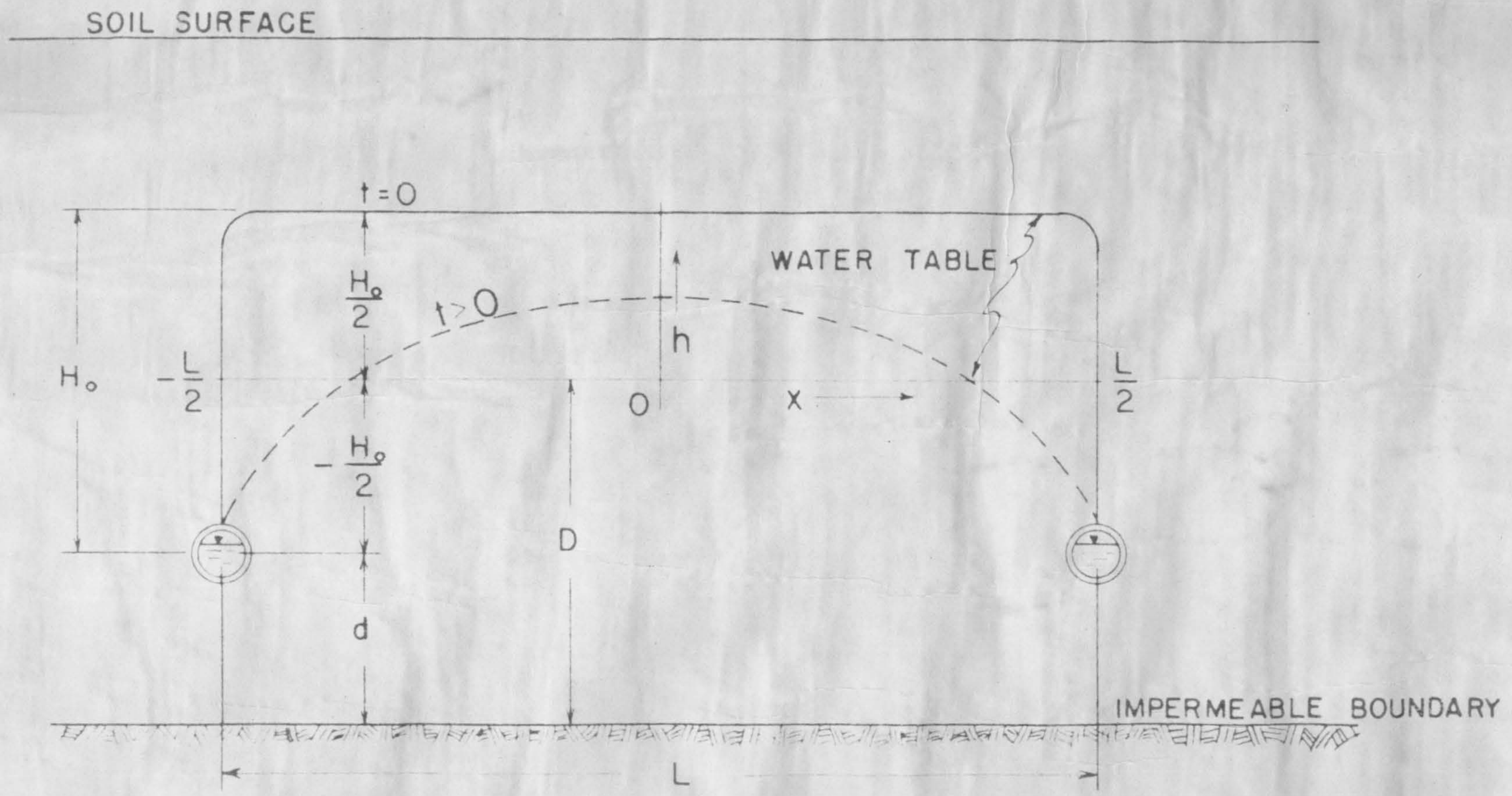

Fig. 1. Model of equally spaced drains above an impermeable boundary with accompanying boundary and initial conditions.

If equation (1) is substituted into equation of continuity, the differential equation describing flow toward the drain becomes

$$\alpha \frac{\partial^2 h}{\partial x^2} - \frac{\partial h}{\partial t} = - \frac{\alpha}{D} \left(\frac{\partial h}{\partial x}\right)^2 - \frac{\alpha}{D} h \frac{\partial^2 h}{\partial x^2} \tag{2}$$

in which α is DK/f, f is the specific yield, and t is time.

The boundary and initial conditions imposed upon the system are given as

$$\begin{aligned} h(\pm \tfrac{L}{2}, t) &= - \frac{H_o}{2}, \ (t \geq 0) \\ h(x, 0) &= \frac{H_o}{2}, \ -\frac{L}{2} \leq x \leq \frac{L}{2} \end{aligned} \tag{3}$$

Picard's method of successive approximations (10) was suggested by Glover[3] for solving the non-linear differential equation (2) with its accompanying conditions given by equations (3). The method of Picard is described below as it applies to this problem.

Equation (2) becomes

$$\alpha \frac{\partial^2 h}{\partial x^2} - \frac{\partial h}{\partial t} = 0. \tag{4}$$

when the non-linear terms are discarded. A first approximation, h_1, is obtained for equation (2) by solving equation (4). A second approximation, h_2, is obtained such that

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[3] Glover, R. E. Consultant, Agricultural Experiment Station, Colorado State University, Fort Collins, Colorado; Personal Communication 1958.

$$\alpha \frac{\partial^2 h_2}{\partial x^2} - \frac{\partial h_2}{\partial t} = - \frac{\alpha}{D} \left(\frac{\partial h_1}{\partial x} \right)^2 - \frac{\alpha}{D} h_1 \frac{\partial^2 h_1}{\partial x^2}. \qquad (5)$$

Similarly, successive approximations, $h_3 \ldots h_n$, are formed such that

$$\alpha \frac{\partial^2 h_3}{\partial x^2} - \frac{\partial h_3}{\partial t} = - \frac{\alpha}{D} \left(\frac{\partial h_2}{\partial x} \right)^2 - \frac{\alpha}{D} h_2 \frac{\partial^2 h_2}{\partial x^2}$$

$$\vdots$$

$$\alpha \frac{\partial^2 h_n}{\partial x^2} - \frac{\partial h_n}{\partial t} = - \frac{\alpha}{D} \left(\frac{\partial h_{(n-1)}}{\partial x} \right)^2 - \frac{\alpha}{D} h_{(n-1)} \frac{\partial^2 h_{(n-1)}}{\partial x^2}$$

In the process of obtaining each approximation, the boundary and initial conditions must be satisfied. Glover (2) solved equation (4) using the boundary and initial conditions given by equations (3), with the exception that he considered his origin of the coordinate axes at the tile instead of as shown in figure 1. For the origin as shown in figure 1, the solution to equation (4) is

$$h_1 = \frac{4H_o}{\pi} \sum_{n = 1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{1}{n} \exp\left(- \frac{\alpha n^2 \pi^2 t}{L^2} \right) \cos \frac{n\pi}{L} x - \frac{H_o}{2} \qquad (6)$$

The solution presented below is a result of carrying Picard's process to a second approximation. The expression is made up of terms consisting of the first approximation, particular integrals, and terms necessary to permit restoration of the initial and boundary conditions. It may be expressed in terms of the first approximation as

$$h_2 = \left(1 + \frac{H_o}{2D}\right)\left(h_1 + \frac{H_o}{2}\right) - \frac{\alpha}{2D} H_o \quad t\frac{\partial^2 h_1}{\partial x^2} + \frac{1}{2D}\frac{\partial h_1}{\partial x}\int\left(h_1 + \frac{H_o}{2}\right) dx$$

$$- \frac{1}{2D}\left(h_1 + \frac{H_o}{2}\right)^2 - \int_o^t \frac{\partial F}{\partial T}\left(H_o - G\right) dT \doteq \frac{H_o}{2}, \tag{7}$$

in which h_1 is as given above, and

$$F = \frac{8 H_o^2}{\pi^2 D}\sum_{n=1,3,5,\ldots}^{\infty}\quad\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m^2}\exp\left[-\frac{(m^2+n^2)\alpha\pi^2 T}{L^2}\right] \tag{8}$$

The term G, under the time integral is equal to $(h_1 + \frac{H_o}{2})$ which in turn is a function of $\left[(t - T), x\right]$ where T is a dummy variable. The particular integral

$$\frac{1}{2D}\frac{\partial h_1}{\partial x}\int\left(h_1 + \frac{H_o}{2}\right)dx$$

did not satisfy the boundary condition, hence, the time integral in equation (7) was introduced to permit restoration of the boundary condition which was evaluated numerically.[4]

Another and simpler approach was considered which is described below. A reasonable approximation for flow toward the drain in figure 1 is given by

$$Q = KD\frac{\partial h_1}{\partial x}, \tag{9}$$

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[4] The method used to restore the boundary condition and the manner in which the time integral was evaluated is postponed for discussion at a later date.

in which $h \ll D$. A more accurate expression for flow toward the drain is given by equation (1),

$$Q = K(D + h_2) \frac{\partial h_2}{\partial x} \tag{10}$$

An approximate solution, h_2, is obtained by substituting equation (9) into equation (10) which when integrated yields

$$Dh_2 + \frac{h_2^2}{2} = Dh_1 + c \tag{11}$$

where c is the constant of integration. Using the boundary conditions given by equation (3), the constant of integration is found to be $(\frac{H_o}{2})^2$. After substituting the value of c into equation (11) and rearranging, the new approximation, h_2, is expressed as

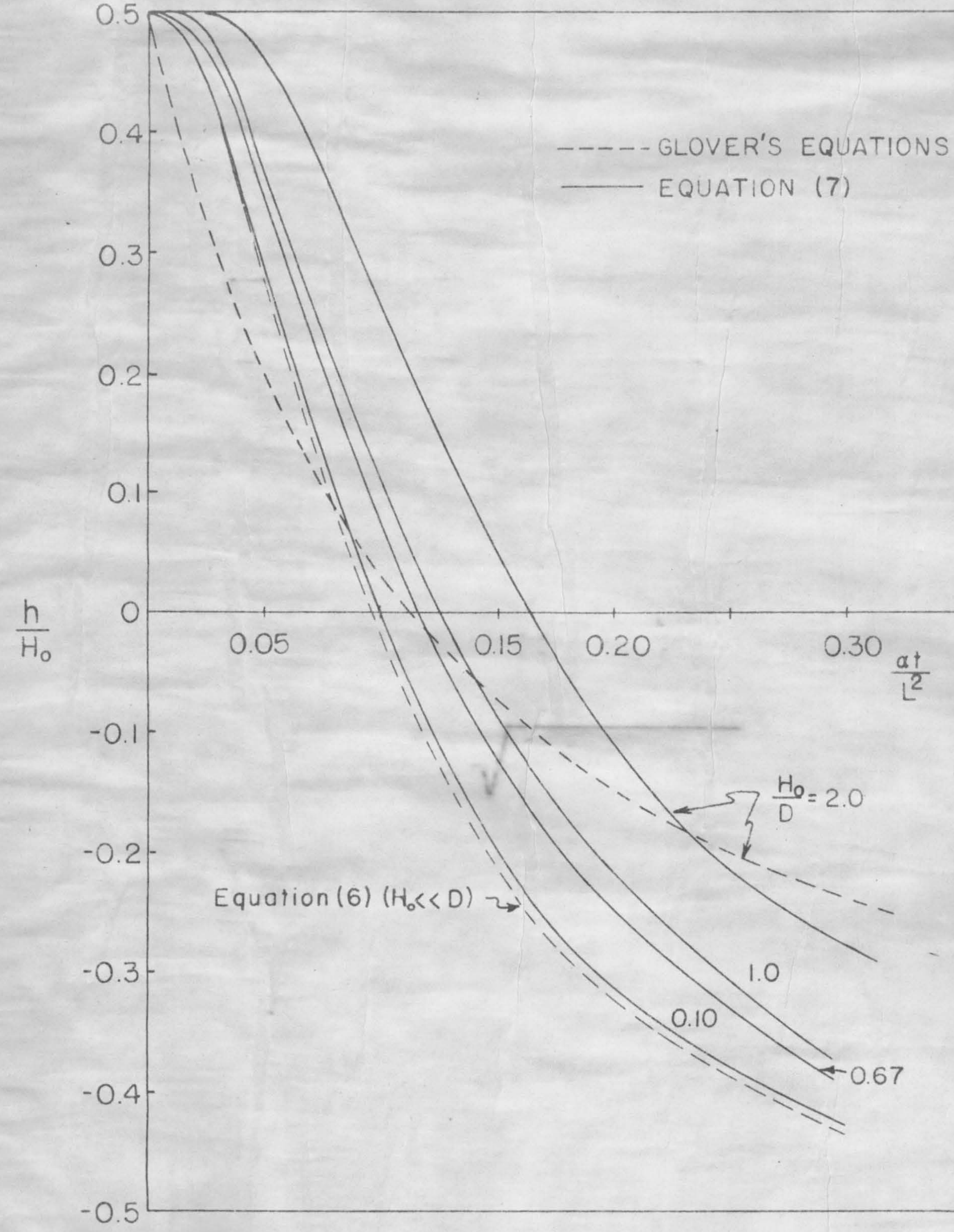
$$h_2 = - D + \sqrt{D^2 + 2Dh_1 + (\frac{H_o}{2})^2} . \tag{12}$$

This expression also satisfies the initial condition.

THEORETICAL RESULTS AND DISCUSSION

The solution of equation (7) is presented graphically in figure 2

Figure 2. Theoretical curves of relative water table height, h/H_o, as a function of the time parameter $\alpha t/L^2$ for tile at various relative distances $\frac{H_o}{D}$ above the impermeable boundary and for $x = 0$.



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for $x = 0$ and for various values of H_o/D. Glover's solutions are also shown in figure 2 for comparison with this theory.

The reason for the apparent inconsistency between Glover's special equation for $H_o/D = 2.0$, which places the drain on the impermeable boundary, and equation (7) is that Glover's solution (2) does not satisfy the same initial conditions assumed herein.

The curves in figure 2 obtained from equation (7), show that the water table midway between drains does not begin to recede until a finite period of time has elapsed depending upon the distance from the drain to the impermeable boundary. The range of relative distances to the impermeable boundary, $0 < \frac{H_o}{D} \leq 2.0$, constitutes 100 percent of the total drainable depth. Kirkham (6) showed that flow into tile drains is considerably reduced when the drain is placed on the impermeable boundary. A comparison of the curves in figure 2 for the drain on the impermeable boundary, $H_o/D = 2.0$, and at two-thirds of the drainable depth, $H_o/D = 1.0$, also indicates that when the boundary is near the drain the flow is considerably reduced.

A comparison of the two approximations, equations (7) and (12), is shown in table 1. The agreement of the two approximations is excellent for $0.1 \leq \frac{H_o}{D} \leq 1.0$. However, for $H_o/D = 2.0$, the agreement is not entirely satisfactory.

Table 1. A comparison of theoretical solutions obtained from equations (7) and (12) for $x = 0$ and for various values of H_o/D.

Table 1. A comparison of theoretical solutions obtained from equations (7) and (12) for $x = 0$ and for various values of H_o/D.

Time Parameter	Relative Position of the Water Table, h/H_o, from Equation (7)				Relative Position of the Water Table, h/H_o, from Equation (12)			
$\alpha t/L^2$ \ H_o/D	0.1	0.67	1.0	2.0	0.1	0.67	1.0	2.0
0.02	0.48	0.49	0.49	0.50	0.46	0.48	0.48	0.49
0.03	0.43	0.46	0.48	0.49	0.41	0.44	0.45	0.46
0.05	0.28	0.33	0.37	0.46	0.27	0.32	0.34	0.38
0.07	0.15	0.22	0.26	0.38	0.14	0.21	0.23	0.30
0.10	-0.01	0.07	0.11	0.25	-0.02	0.06	0.10	0.19
0.16	-0.23	-0.16	-0.13	-0.01	-0.23	-0.17	0.12	0.01
0.20	-0.31	-0.25	-0.21	-0.11	-0.31	-0.26	0.22	-0.08
0.25	-0.38	-0.33	-0.30	-0.22	-0.39	-0.34	0.31	-0.16
0.30	-0.43	-0.40	-0.38	-0.28	-0.43	-0.41	0.38	-0.24

In a recent paper by Isherwood (5), the general method of Kirkham and Gaskell (7) was employed with the use of a high speed computer to solve the boundary value problem presented herein. Isherwood solved the problem as a two-dimensional system, hence his solution should be more accurate than the one-dimensional system which is based upon the Dupuit assumption. By comparing Isherwood's solution with the approximate theoretical solution presented here, it is possible to determine the validity of equation (7) to a reasonable degree.

Isherwood's data for the case of an impermeable boundary existing 10 feet below the soil surface is reproduced in figures 3 and 4, using

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Figure 3. A comparison of the theoretical curve and the solution of Isherwood (5) for $\frac{H_o}{D} = \frac{6}{7}$ and for various values of $\frac{L}{d}$.

Figure 4. A comparison of the theoretical curve and the solution of Isherwood (5) for $\frac{H_o}{D} = \frac{8}{6}$ and for various values of $\frac{L}{d}$.

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dimensionless parameters. From this comparison it seems necessary to place a restriction upon equation (6) and equations (7) and (12). This restriction may be made by use of the ratio of the tile spacing, L , to the distance from the tile to the impermeable boundary, d , i.e., L/d . It

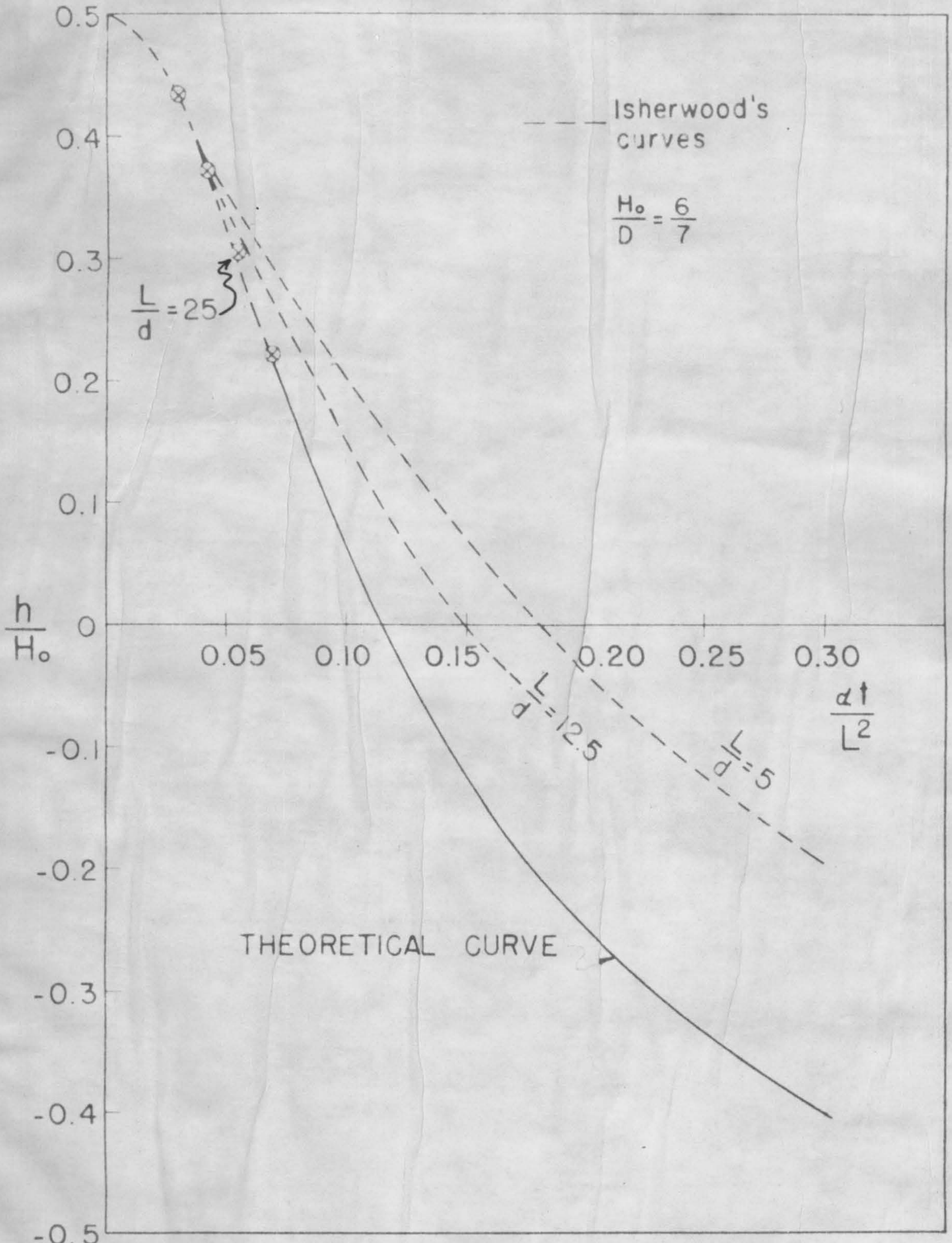

Fig. 3. A comparison of the theoretical curve and the solution of Isherwood (6) for $\frac{H_o}{D} = \frac{6}{7}$ and for various values of $\frac{L}{d}$.

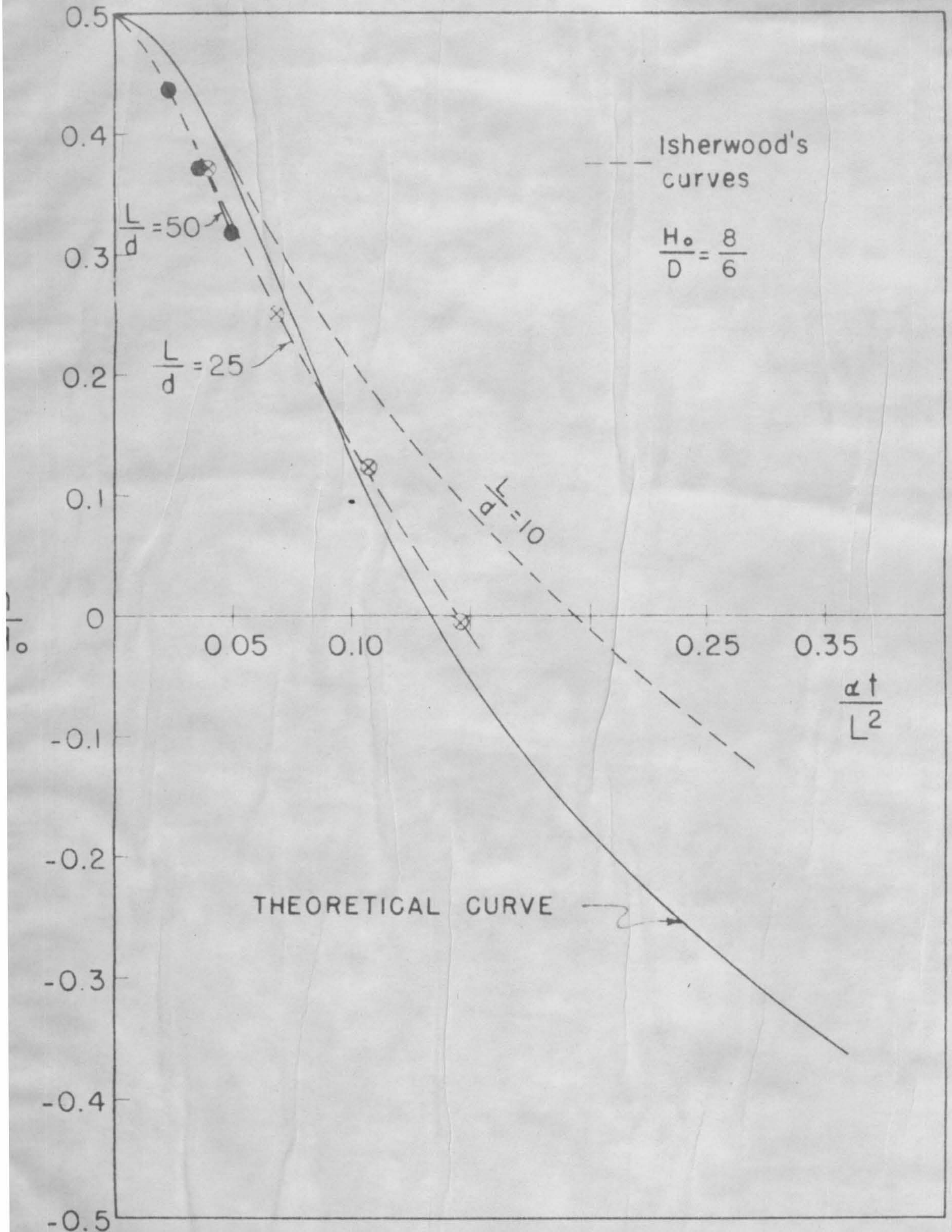

Fig. 4. A comparison of the theoretical curve and the solution of Isherwood (6) for $\frac{H_o}{D} = \frac{8}{6}$ and for various values of $\frac{L}{d}$.

is reasonable to expect that for large ratios of $\frac{L}{d}$ that a solution to this problem based upon the Dupuit assumption would closely approximate the two dimensional solution, and conversely for small ratios of $\frac{L}{d}$ such a solution might be in considerable error. This fact is clearly shown in figures 3 and 4 in which Isherwood's solution agrees very closely with the theoretical curve for $\frac{L}{d} \geq 25$ while for $\frac{L}{d} < 25$ there is considerable lack of agreement. It appears from the data available from Isherwood's paper that the theory of this paper is independent of $\frac{L}{d}$ for ratios greater than 25.

A comparison of Klinge's (8) unpublished field data, and equation (7) is shown in figure 5. The theoretical curve appears as a solid line while the data appear as points. These data of Klinge's agree remarkable well with

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Figure 5. A comparison of the theoretical curve and the data of Klinge (8) for $\frac{H_o}{D} = 1.08$.

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the theoretical curve. These data were reported for shallow drainage systems in which the impermeable boundary was 3-4 feet below the surface. A number of these comparisons were made and similar agreement was found.

An example of the use of the theoretical curves follows. Assume a system of drains are to be installed to a depth of 6 feet below the soil surface and an impermeable boundary exists at 9 feet below the soil surface.

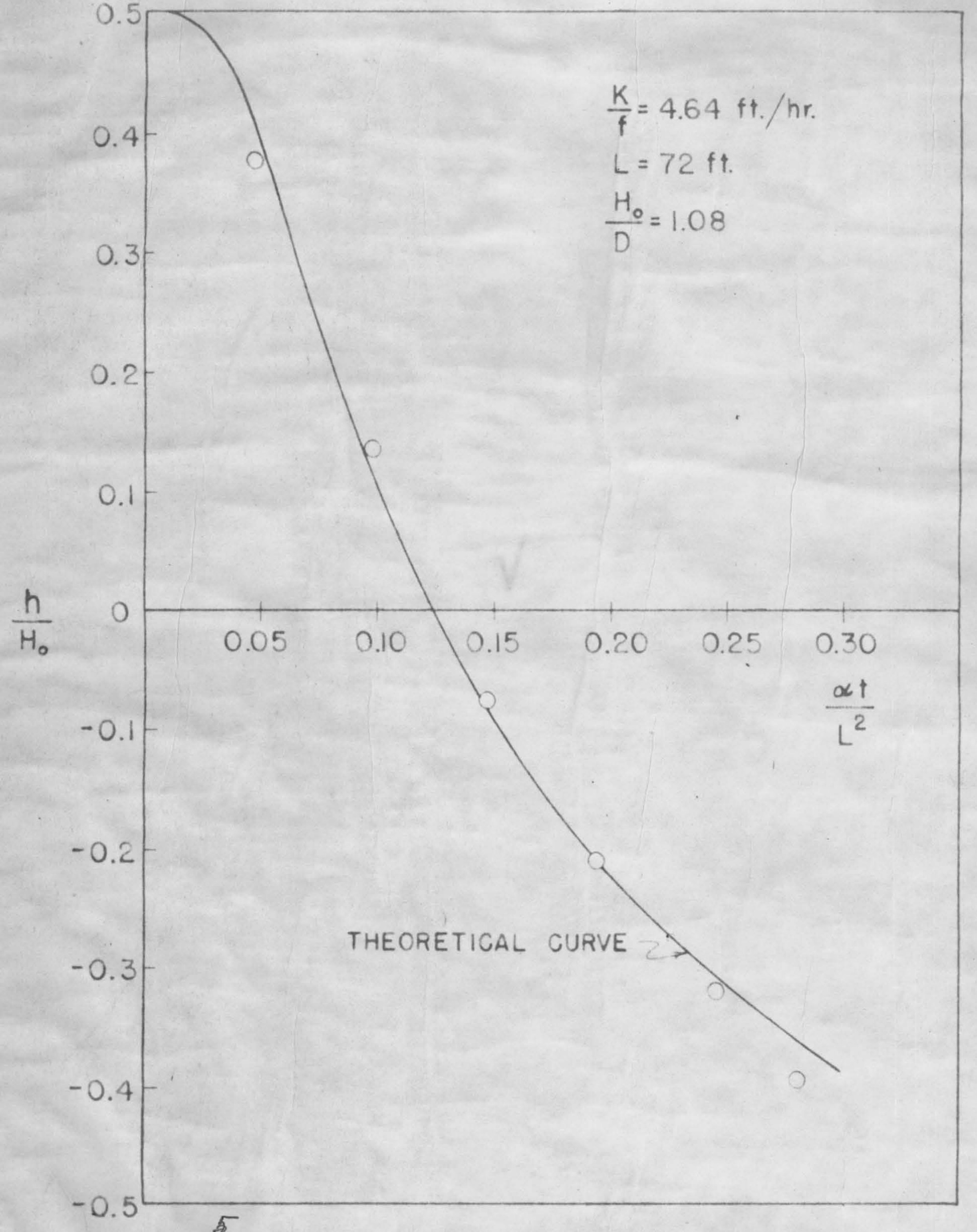

Fig. 5. A comparison of the theoretical curve and the data of Klinge (9) for $\frac{H_o}{D}$ = 1.08. *at $x=0$*

If the position of the water table initially exists at the soil surface, compute the tile spacing which will cause the water table to recede to a depth of 3.6 feet in 3 days when the hydraulic conductivity-specific yield ratio, $\frac{K}{f}$, is 2.0 ft/hr. The scaled time variable, $\frac{\alpha t}{L^2}$, is read from the theoretical curve, $\frac{H_o}{D} = 1.0$ for $\frac{h}{H_o} = -0.1$, which is found to be 0.150. Since t and α are known, the spacing, L, is computed from the relation

$$\frac{\alpha t}{L^2} = 0.150$$

or

$$L = \sqrt{\frac{\alpha t}{0.150}}$$

in which α is 12.0 ft^2/hr. The spacing is found to be 76 ft (approximately) and $\frac{L}{d} = \frac{76}{3} = 25.3$ which is within the limits set forth using the data of Isherwood. The theoretical predictions of the tile spacings which can be obtained from figure 2 will be no better than the methods used to obtain characteristic hydraulic conductivity - specific yield data for the drainage system. Methods of measuring hydraulic conductivity on a large scale are needed. The recent work of Nelson (9) seems to be a step forward in this direction.

The assumption that there is a constant quantity called specific yield is indeed naive as discussed by Childs (3). The true solution of this problem as indicated by Childs (3) "Will demand the study of the soil as a whole, both above and below the water table, as an essay in the field of water movement in a medium whose hydraulic conductivity is a function of moisture content." However, it is hoped that the approximate solution presented herein will enable the drainage engineer to more accurately design drainage installations and predict drainage costs.

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