DISSERTATION

TECHNIQUES IN INTERPOLATION PROBLEMS

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY OLIVIA DUMITRESCU ENTITLED TECHNIQUES IN INTERPOLATION PROBLEMS BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

TECHNIQUES IN INTERPOLATION PROBLEMS

This dissertation studies degeneration techniques in interpolation problems, that can be phrased as computing the dimension of the space of plane curves of degree d having general multiple points. The general interpolation problem goes back to the origin of algebraic geometry and is still far from being solved. We approach it using algebraic geometry techniques, by systematically exploiting degenerations of the projective plane. Degenerating the plane into a union of planes we prove the planar case of the interpolation problem for double points, and we present results obtained for higher multiplicities. We will generalize this technique and using toric geometry methods, we prove the interpolation problems for triple points. Using non-toric degenerations we prove the emptiness of a linear system with ten multiple points for different ratios, a result that approximates from below Nagata's bound by rational numbers. In the introduction we also state other results obtained and we mention different directions for further research.

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To my father, Marin

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First I would like to express my gratitude to my advisor, Professor Rick Miranda for guiding me in all these years with patience, support, enthusiasm and knowledge, for the english (and italian) lessons, for making Fort Collins feel like home and for making me feel always surrounded by family.

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I want to dedicate this thesis to my family, to my mother and sister, for always making me feel loved and protected in both happy and difficult moments, and especially to my father. I am grateful to him for guiding my first steps in math, for making me love life and people, for keeping me honest with myself. There is not enough space to express all my gratitude and love to them. Finally, I would like to thank Cristi, for his unconditional help and support and for being so close (although so far away).

I could have not done this without all their help.

Contents

CHAPTER 1

Introduction

1. Introduction to Interpolation Problems.

We will briefly introduce the general interpolation problem, which goes back to the origin of algebraic geometry. Given r general points $P_1, ..., P_r$ in the projective space of n dimensions and $d, m_1, ..., m_r$ positive integers, one could ask to find a polynomial of a degree d for which all its higher order derivatives up to order m_i at the points P_i match an assigned set of values. This problem, far from being solved, is called polynomial interpolation. To lower the difficulty of the problem, people reduce it to asking that at each point all the polynomial derivatives up to order m_i vanish. In this form, the problem can be rephrased geometrically to describe $\mathcal{L}_{n,d}(m_1, ..., m_r)$ the linear system of hypersurfaces in $\mathbb{P}^n$ of degree d, that pass through r points $P_1, ..., P_r$, with multiplicity at least m_i. A natural question would be to compute the projective dimension of the linear system $\mathcal{L}$. If all the conditions imposed are linearly independent we can easily compute it as the difference between the dimension of the space of divisors of degree d (i.e. polynomials of degree d in $n+1$ variables) and the number of conditions imposed by asking a polynomial f to vanish to order m_i at each point, (i.e. the number of terms in the Taylor expansion of f at each point up to order $m_i - 1$). We will define this difference to be the virtual dimension of $\mathcal{L}$

$$v(\mathcal{L}_{n,d}) := \binom{d+n}{n} - \sum_{i=1}^{r} \binom{m_i + n - 1}{n} - 1$$

and the expected dimension to be $e(\mathcal{L}) := max\{v(\mathcal{L}), -1\}$. A naive conjecture is $dim(\mathcal{L}) = e(\mathcal{L})$.

To restrict the generality of this problem, we will consider the homogeneous case when all multiplicities are equal $m_1 = ... = m_r = m$ so the linear system becomes $\mathcal{L}_{n,d}(m^r)$. There are some elementary cases for which $dim(\mathcal{L}) \neq e(\mathcal{L})$.

Consider $n = d = r = m_1 = m_2 = 2$, for example. One can notice that $dim(\mathcal{L}_{2,2})(2^2) = 0$ since the plane system of conics with two general double points consists of a fixed divisor: the unique double line through the two points, so its projective dimension is 0. This is different than the expected dimension $e(\mathcal{L}_{2,2})(2^2) = -1$. A linear system for which $dim\mathcal{L} > e(\mathcal{L})$ is called a special linear system.

The interpolation problem for double points is closely related to the defectivity of the secant varieties. Indeed, we denote $V_{n,d}$ to be the Veronese variety in $\mathbb{P}^{\binom{d+n}{n}-1}$ and $Sec_k(X)$ to be the closure of the union of all $k+1$–secant $\mathbb{P}^k$'s of X. The Veronese embedding transforms hypersurfaces in $\mathbb{P}^n$ with r double points into hyperplanes tangent to $V_{n,d}$ at r general points. Using Terracini's lemma we remark that the $r-1$ secant variety to $V_{n,d}$, $Sec_{r-1}(V_{n,d})$ has the expected dimension if and only if $\mathcal{L}_{n,d}(m^r)$ is non-special. In this case $V_{n,d}$ is said to be not $r-1$-defective. Alexander and Hirschowitz proved the double points interpolation theorem ($m = 2$) for any n dimensional space, by classifying all the cases for which the secant variety $Sec_k(V_{n,d})$ is defective and proving that it is not k defective otherwise.

For the planar case progress has been made, however for $n \geq 3$ few things are known, so the problem is still open.

Consider the blow–up X of the plane at the points $p_1, \ldots, p_n$. We denote by $\mathcal{L}$ also the proper transform of $\mathcal{L}$ to X. Suppose $\mathcal{L}$ not empty, and assume that there is a (-1)–curve C on X such that $C \cdot \mathcal{L} \leq -2$. This forces C to be a multiple fixed curve in the system and it is easy to see that $\mathcal{L}$ is special in this case. We will then say that $\mathcal{L}$ is (-1)–*special*. The Harbourne-Hirshowitz Conjecture says that a system is special if and only if it is (-1)–special Related to this conjecture, but weaker, is Nagata's Conjecture: if $n > 9$ and $d^2 < nm^2$ then $\mathcal{L}_d(m^n)$ is empty.

There has been a partial progress on these conjectures; let us recall some of the results. The Harbourne-Hirschowitz Conjecture is true for $n \leq 9$, or for $m_i \leq 7$ (S. Yang, 2004); Nagata proved that his conjecture is true for $n = k^2$ points or if $n < 10$.

One of the main techniques used to work on polynomial interpolation problems consists of performing a degeneration of the ambient space $\mathbb{P}^k$. By doing this we degenerate the bundle $\mathcal{L}$ and the points, moving them to particularly special positions and we end by a semicontinuity argument. Finding a suitable position for this kind of argument is delicate, since it should be special enough to be treatable and at the same time general enough that the dimension of the degenerated linear system is still the expected one.

2. About the Degeneration Technique.

Degeneration theory has been used for solving many different problems in algebraic geometry. Degenerations of curves are used for analyzing the moduli space of curves since one of the roles of the moduli space is to deduce facts about certain smooth curves by studying a limit curve of the family. They were used to give a proof to the classical Brill Noether problem, saying that a general curve of genus g carries a g^r_d if and only if the Brill Noether number is non-negative, and if so then it equals the dimension of the locus $W^r_d(C)$ of linear series in $Pic^d(C)$. A linear series is a pair consisting of the line bundle of degree d and its global sections, a vector space of dimension r. Since examples for general curves of higher genus are not known, the problem has been analyzed by studying the degenerations of line bundles and of linear systems. However, if a linear series is replaced by a pair of rank r bundle together with its global section, then the smoothness, emptiness or dimension of $W^r_d(C)$ are open problems. Degenerations have also been used to solve the problem of the irreducibility of the family of curves $V_{d,g}$ of a given degree and genus (fixing the genus is the same as giving the number of nodes). Degenerations of curves parametrized by a given component of the Severi variety, were used for analyzing the irreducibility of a family of curves, while in my thesis I use it for analyzing different interpolation problems. Degeneration techniques were also used to answer questions concerning the corank of a Gaussian map. For hyperplane sections of smooth K3 surfaces of degree $2g-2$, the corank of the Gaussian map is 1 for $g \geq 13$. The proof uses was a degeneration of the $K3$ surface to a configuration of planes whose hyperplane sections are graph curves with a corank one Gaussian map (see [**7**]).

My research has had two directions corresponding to two methods of degeneration of the ambient space: toric degenerations and a non-toric degenerations consisting of a successive sequence of blow ups as in [**4**] and [**5**]. I will briefly describe the main results presented in this thesis.

3. Degenerations of the Veronese in dimension two.

Consider a trivial family of planes, $\mathcal{X}$ and blow up a point of the fiber over zero in the total space of the family. We get a new family $\mathcal{X}'$ having as a central fiber the union of two surfaces $\mathbb{F}$ which is a plane blown up at a point and $\mathbb{P}$ the exceptional divisor. Denote by $\mathcal{O}(d)$ the line bundle which is the pull-back to $\mathcal{X}'$ of $\mathcal{O}_{\mathbb{P}^2}(d)$. The bundle $\mathcal{O}(d) \otimes \mathcal{O}(-\mathbb{F})$ embeds the general fiber as the Veronese V_d and the central fiber as the union of V_{d-1} and a scroll $S(d-1,d)$. Similarly the scroll $S(d,d-1)$ degenerates to $d-1$ quadrics and a plane. Iterating this process one obtains a toric degeneration of the d fold Veronese to a union of d planes and $d(d-1)/2$ quadrics (see also [**17**] and [**4**]).

As a consequence we obtain results regarding the interpolation problem for k^2 number of points of arbitrary multiplicity m. Our technique uses the degeneration of V_d mentioned above together with a degeneration of the bundle $\mathcal{O}(d)$ which we will choose to have convenient degrees on the planes and convenient bidegrees on the quadrics. The next two results are proved in Section 2 by a degeneration argument (see [**4**]).

Theorem 1.3.1. *The linear systems $\mathcal{L}_{km}(m^{k^2})$ and $\mathcal{L}_{km+1}(m^{k^2})$ have the expected dimension.*

One could use our approach to generalize this result for all values d using induction.

For the double points interpolation problem we will further degenerate the quadrics into a union of two planes. We obtain a planar degeneration of V_d into a union of d^2 planes each plane containing exactly 3 coordinate points of the projective space, and each double curve being a line in the ambient space. Using a semicontinuity argument we reduce the problem to the case when $v(\mathcal{L}) = 0$ and we use induction to give a new proof of the planar case of the famous Alexander-Hirschowits conjecture (see also [**4**]).

Theorem 1.3.2. *For $d \geq 5$, $\mathcal{L}_d(2^r)$ has the expected dimension (in particular is empty if the expected dimension is negative).*

In fact, Alexander-Hirschowitz's Theorem for double points in dimension 3 has recently been proved by Silvia Brannetti using a similar technique (see [**3**]). Moreover as part of her doctoral thesis, Elisa Postinghel uses degenerations of $\mathbb{P}^r$ to give a new proof of the classical Alexander-Hirschowitz's Theorem (see [**21**]).

4. Triple Points in $\mathbb{P}^2$.

Consider now $\mathcal{L}_d(3^r)$ of plane curves having r triple points. We extend the double points interpolation problem to the triple points problem using toric degenerations of V_d to conclude that $\mathcal{L}_d(3^r)$ has the expected dimension. Indeed, curves in $\mathcal{L}_d(3^r)$ correspond by the Veronese embedding to hyperplanes meeting the Veronese variety at r points with multiplicity three. We will use toric degenerations of V_d into a disjoint union of surfaces, and place each triple point in one of the surfaces. We choose the surfaces such that a general hyperplane's restriction to each of them is a linear system that becomes empty when imposing a triple point. We will then conclude that the hyperplane needs to contain the surface, and therefore all of its coordinate points. If the union of these surfaces is the whole ambient space, then such a hyperplane can not exist since it should contain all the coordinate points of the Veronese surface V_d, and this leads to a contradiction. This proves that $\mathcal{L}_d(3^r)$ is empty. We also give a full classification of convex polytopes, for which the corresponding toric surfaces are used in the degeneration of V_d. The following theorem presented in Section 3 was obtained using the toric degeneration described above.

Theorem 1.4.1. *$\mathcal{L}_d(3^r)$ has the expected dimension if $d \neq 4$.*

An algebraic approach for triple points in $\mathbb{P}^1 \times \mathbb{P}^1$ have been also considered by T. Lenarcik in [**16**]. Later on we will focus on Nagata's conjecture and for this we will use degenerations of $\mathbb{P}^2$ that are not toric.

5. Ten Points of Arbitrary Multiplicity.

The virtual dimension of a homogeneous linear system in $\mathbb{P}^2$ of plane curves of degree d passing through n points is $e(\mathcal{L}_d(m^n)) = max\{-1, \frac{d(d+3)}{2} - \frac{nm(m+1)}{2}\}$. Nagata conjectured that if the leading term of the virtual dimension is negative for large values of m and d, then $\mathcal{L}$ is empty and he proved his conjecture when the number of points is a perfect square. The result is also known if $n \leq 9$, so the case when $n = 10$ appears to be a boundary case for this conjecture; this problem was analyzed by Harbourne-Roé [**13**] and by Dumnicki [**9**].

This is an old conjecture that is also connected to the symplectic packing problem in dimension four. A symplectic packing of a $2n$ dimensional symplectic manifold (M, Ω) is a symplectic embedding by n equal balls $B(\lambda)$ endowed with the standard symplectic structure. (M, Ω) admits a full symplectic packing if M can be symplectically packed by n equal balls. This problem is also equivalent to determining how much it is possible to blow up symplectically the manifold. In particular, Nagata's Conjecture's would imply that complex projective plane admits a full symplectic structure (see [**2**]).

Another approach to Nagata's conjecture involves the blow up of $\mathbb{P}^2$ at a finite set of general points, X. A weaker version of Segre's conjecture states that every integral curve with negative self-intersection on X is a (-1) curve. An equivalent statement states that the extremal rays contained in the positive side of the Mori cone (the closure of $NE(X)$) are contained in the closure of the cone of $1-cycles$ with non-negative intersection with an ample divisor and non-negative self-intersection. Segre's conjecture implies Nagata's conjecture ([**19**]). Therefore the positive side of the Mori cone on the blow up of $\mathbb{P}^2$ gives us information about a full symplectic structure of the complex projective plane. Our approach to Nagata's conjecture involves degeneration methods.

By constructing a degeneration of a family of planes X_t to a union of nine surfaces in the central fiber X_0, we calculate all possible limits $\mathcal{L}_0$ of the line bundle $\mathcal{L}_d(m^{10})$ on the central fiber of the family X. Such a limit line bundle is a line bundle on each surface, which agree on all of the double curves of the degeneration. If for a fixed ratio of d/m, one proves that for any limit line bundle $\mathcal{L}_0$, at least one of the restrictions of $\mathcal{L}_0$ to the surfaces from the central fiber is empty, then one concludes that there cannot be a limit curve in $\mathcal{L}_0$, and therefore $\mathcal{L}_d(m^{10})$ is empty.

We consider the degeneration of the plane $\mathbb{P}^2$ into a ruled surface isomorphic to $\mathbb{F}_1$, denoted by $\mathbb{F}$ and a plane $\mathbb{P}$, meeting along a double curve R, and choose four general points on $\mathbb{P}$ and six general points on $\mathbb{F}$. Whenever a -1 curve intersects the double curve twice, we perform a 2–throw by blowing it up twice and contracting it the other way. We construct the degeneration by performing a series of 2–throws (see [**5**] and [**8**])

(1) The cubic $\mathcal{L}_3(2, 1^6)$ on $\mathbb{F}$
(2) Six disjoint curves, two conics $\mathcal{L}_2(1^4, [1, 0], [0, 0])$ and four quadrics $\mathcal{L}_4(2^3, 1, [1, 1]^2)$ on $\mathbb{P}$.

where the notation $[a, b]$ stands for infinitely near multiplicities. In Section 4, using the 'centrally effective' argument described above to this degeneration we prove the best result known for 10 points (see also [**5**])

Theorem 1.5.1. *If* $\frac{d}{m} < \frac{117}{37} \approx 3.162162$ *then* $\mathcal{L}_d(m^{10})$ *is empty.*

We note that $\frac{117}{37}$ is a good approximation of $\sqrt{10} \cong 3.1622...$ We also remark that our emptiness result implies that the corresponding Seshadri constant for ten points in the plane is at least $117/370$; see [**13**].

Consider more general planar degeneration obtained by performing n–throws. They occur when blowing up curves that intersect the union of the double curves n times. The curve needs to be blown up n times in a row so it creates n pairs of infinitely near multiplicities, since it introduces n exceptional surfaces. The matching conditions for the bundles on the degenerated plane are not known for $n > 2$. This problem was also analyzed by Michele Nesci [**20**] in his doctoral thesis and it still remains an open question.

We remark that in general, by performing degenerations we get results regarding rational ratios of d/m, while Nagata's conjecture involves irrational numbers when the number of base points is not a perfect square. With every degeneration we get better irrational limits for d/m so we speculate that the sequence of rational bounds approaches from below the irrational Nagata's bound.

To prove Nagata's conjecture for a general number of points using the degeneration method described above, we speculate that we will need to use an inductive argument and to consider a sequence of successive n-throws to obtain a degeneration of $\mathbb{P}^2$ into a union of different surfaces. All surfaces must agree on the double curves and we speculate that we must systematically vary the number of multiple points on each surface and to use matching conditions and induction in order to prove the emptiness of corresponding the linear system (a method introduced in [**4**]).

6. Other Results.

We only mention here other results obtained without giving the details. In dimension three, few things are known except Alexander-Hirschowitz's theorem for double points. Using the theory developed in Section 2 and Section 3 we can give a description of the convex polytopes for which the corresponding linear system becomes empty when imposing a triple point in $\mathbb{P}^3$. Using toric degenerations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into a union of disjoint toric subvarieties corresponding to three dimensional polytopes enclosing 10 points and we also proved by induction the following result:

Theorem 1.6.1. *Linear systems in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with an arbitrary number of triple points and arbitrary tridegree (a, b, c) have the expected dimension if $a, b, c > 2$, $a \neq 5$, and $b \neq 5$.*

Using a similar technique, we have also proved the analogue of Theorem 2.1.3 for K_3 surfaces. Moreover, we were able to generalize the results of the lemma 3.4.2 for any multiplicity m in the projective plane. We were able to find general polytopes for which the associated linear system with an general m multiple point becomes empty, so using this and following the arguments used in Sections 2 and 3 one can reduce the interpolation problem to a combinatorial result.
Furthermore, since we easily generalized our triple point analysis from the projective plane to the projective space, one could study the generalization of the m multiplicity point from the projective plane to higher dimensional spaces. This is a challenging problem since ,as we mentioned before, for higher dimensional spaces very little is known.

Remark 1.6.2. Everywhere in the thesis we call a Cremona transformation the birational transformation on the projective plane obtained by blowing up three points and blowing down the three (-1) curves connecting the three points. For the projective plane we have that every birational transformation is a composition of Cremona transformations specifying the points that are being blown up and the order of such composition. For example, if we fix six points $P_1, ..., P_6$ in the plane, a Cremona transformation $123 - 456$ will represent the plane obtained by performing two Cremona transformations: to the first three points and then to the last three points. We notice that this changes the geometry of the plane by affecting the curves passing through the points and therefore it changes the linear systems with these base points.

CHAPTER 2

Degenerations of the Veronese

We denote by $S(a,b)$ and call it a *rational normal scroll* , with $0 < a \leq b$, to be a smooth scroll surface of degree $d = a + b$ in $\mathbb{P}^{d+1}$, which is described by the lines joining corresponding points of two rational normal curves of degrees a and b lying in two linearly independent subspaces of dimensions a and b respectively. As an abstract surface, $S(a,b)$ is isomorphic to the *Hirzebruch surface* $\mathbb{F}_{b-a}$.

First of all, the Veronese V_d degenerates to the union of V_{d-1} and a scroll $S(d-1,d)$:

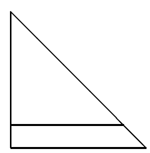


FIGURE 2.1.

In a trivial family $\mathcal{X}$ of $\mathbb{P}^2$'s parametrized by a disk Δ, blow up the central fibre along a line R, and get a new family $\mathcal{X}'$. The general fibre of $\mathcal{X}'$ is still a plane, while the central fibre consists of the union of two surfaces: the old plane $\mathbb{P}$ and the exceptional divisor $\mathbb{F}$, which is a $\mathbb{F}_1$, meeting $\mathbb{P}$ along the line R. We note that R is also the (-1)–curve on $\mathbb{F}$, meeting the ruling F in one point.

We consider the line bundle which is the pull–back to $\mathcal{X}'$ of $\mathcal{O}_{\mathbb{P}^2}(d)$. We call $\mathcal{O}(d)$ the pull–back of this line bundle to $\mathcal{X}'$ and we twist it by $-\mathbb{F}$, $L = \mathcal{O}(d) \otimes \mathcal{O}(-\mathbb{F})$. Since $\mathbb{F}$ is a surface disjoint from any other plane of the family, the restriction of L to the general fibre is still $\mathcal{O}_{\mathbb{P}^2}(d)$, whereas its restriction to $\mathbb{P}$ is $\mathcal{O}_{\mathbb{P}^2}(d-1)$ and to $\mathbb{F}$ is $\mathcal{O}_{\mathbb{F}}(dF+R)$. $dF+R$ is a very ample divisor and embeds $\mathbb{F}$ as a scroll of degree $2d-1$ in $\mathbb{P}^{d+1}$, $S(d-1,d)$.

This construction can be iterated, and we thus see that V_d degenerates to a union of a plane V_1 and a sequence of scrolls $S(1,2), S(2,3), \ldots, S(d-1,d)$ (see Figure 4.2).

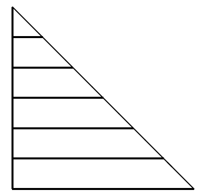


FIGURE 2.2.

Similarly, $S(d-1,d)$ degenerates to a quadric and a scroll $S(d-2,d-1)$:

FIGURE 2.3.

In the central fibre of a trivial family of $\mathbb{F}_1$'s one has to blow up a ruling, thus creating an exceptional divisor, G which is a $\mathbb{F}_0$. We twist by $-G$, i.e. we consider the bundle $\mathcal{O}(R+dF)\otimes\mathcal{O}(-G)$ that embeds the general fiber $\mathbb{F}_1$ as a scroll $S(d-1,d)$ and the fiber over zero as a union of a quadric $S(1,1)$ and a scroll $S(d-2,d-1)$. When $d=1$ the bundle $\mathcal{O}(R+F)$ contracts the negative section so it embeds $\mathbb{F}_1$ as a plane. Iterating this construction we have that $S(a-1,a)$ degenerates to $a-1$ quadrics and a plane (see Figure 2.4).

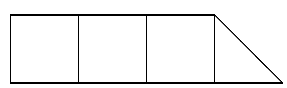


FIGURE 2.4.

We notice that these degenerations can be combined in order to give rise to a degeneration of V_d to a union of d planes and $\binom{d}{2}$ quadrics, which we illustrate below for $d=6$:

$d=6$:

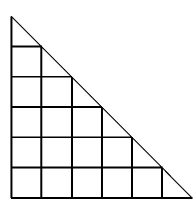


FIGURE 2.5.

Summing up, the *vertices* of the last configuration of planes and quadrics are independent and therefore can be taken as the coordinate points of the ambient $\mathbb{P}^{d(d+3)/2}$. We will call this degeneration the *quadrics degeneration* of the Veronese.

Moreover, each quadric can independently degenerate in its own space $\mathbb{P}^3$, to a union of two planes and it can be performed by moving the quadric in a pencil in its embedding $\mathbb{P}^3$ and leaving the corresponding quadrilateral of double lines fixed.

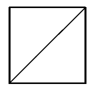
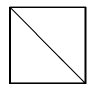
 or  for each quadric.

FIGURE 2.6.

If we degenerate each quadric to two planes, we obtain degenerations of V_d to d^2 planes; we will refer to these as to *planar degenerations* of the Veronese. Again, each double curve is a line in the ambient projective space, and the union of the planes spans this space, each plane contains exactly three of the coordinate points of the projective space. Since the vertices of these configurations are independent in the ambient projective space, any subset of n of the planes which are pairwise disjoint will span a maximal dimensional space, of dimension $3n-1$. Any such subset of a given planar degeneration D of V_d will be called a *skew n-set* of planes of D.

1. Double point interpolation problems.

We consider $\mathcal{L}_d(2^n)$, i.e. the linear system of plane curves of degree d with n general double points. We recall the *Veronese embedding*

$$v_d : \mathbb{P}^2 \longrightarrow \mathbb{P}^{d(d+3)/2}$$

with image V_d, the *Veronese surface* of degree d^2. A plane curve of degree d corresponds via v_d to a hyperplane section of V_d; and such a plane curve has a double point at p if and only if the corresponding hyperplane is tangent to V_d at $v_d(p)$. Therefore the linear system $\mathcal{L}_d(2^n)$ corresponds to the linear system $\mathcal{H}$ of hyperplanes in $\mathbb{P}^{d(d+3)/2}$ which are tangent to V_d at n fixed (but general) points. The Terracini's Lemma relates this linear system to the tangent space to the a secant variety of V_d: the base locus of $\mathcal{H}$ is the general tangent space to $\mathrm{Sec}_{n-1}(V_d)$, the $(n-1)$–*secant variety* to V_d, i.e. the variety described by all linear spaces of dimension $n-1$ which are n–secant to V_d. One thus concludes that $\mathcal{L}_d(2^n)$ is special if and only if $\mathrm{Sec}_{n-1}(V_d)$ has smaller dimension than expected, namely V_d is $(n-1)$–*defective*. Employing a planar degeneration of the Veronese, we are able to reduce this result to a purely combinatorial property of the resulting configuration of planes.

Lemma 2.1.1. *Suppose that there exists a planar degeneration D of V_d, and a skew n–set of planes of D. Then the linear system $\mathcal{L}_d(2^n)$ has the expected dimension $d(d+3)/2 - 3n$. In particular, if there is skew n–set of planes of D whose planes contain all of the $(d+1)(d+2)/2$ coordinate points of the configuration, then $3n = (d+1)(d+2)/2$ and the linear system $\mathcal{L}_d(2^n)$ is empty.*

Proof. We consider the degeneration D, and we let the general points $p_1, \ldots, p_n$ on V_d degenerate in such a way that each point goes to a general point of the planes in the subset S. The limit of the system of hyperplanes tangent to V_d at the points $p_1, \ldots, p_n$ is the system of hyperplanes tangent to the configuration D at each limit point; but a hyperplane which is tangent to a plane at a point must contain that plane. We conclude the limiting system of hyperplanes is the system that contains the subset S of n planes in the configuration, which is the system of hyperplanes containing the span of S. Since S consists of pairwise disjoint planes, it has maximal dimensional span, of dimension $3n-1$; and therefore this limiting system of hyperplanes has codimension equal to $3n$.

By semicontinuity, we conclude that the system $\mathcal{L}_d(2^n)$ has codimension at least $3n$ in $\mathcal{L}_d$; but this is also the maximum possible codimension, since we are imposing $3n$ linear conditions on the plane curves.

In particular, if one can find a skew n–set S of planes in D that contain all of the coordinate points of the configuration, then S will span the ambient space. Hence there can be no hyperplane that contains all of the planes of S, and we conclude, using the same argument as above, that the corresponding linear system must be empty. □

Lemma 2.1.2. $\mathcal{L}_5(2^7)$ *is empty, and* $\dim(\mathcal{L}_6(2^9)) = 0$*, i.e. these systems have the expected dimensions.*

Proof. We illustrate below a skew 7-subset (respectively 9-subset) for a planar degeneration D_5 of V_5 (respectively D_6 of V_6):

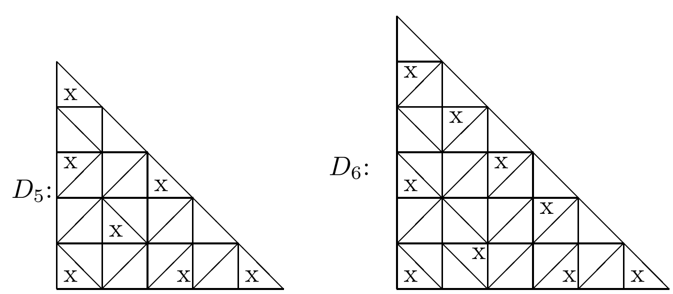


Figure 2.7.

Note that in the $d = 5$ case, the planes indicated by an 'x' form both a spanning skew 7-subset of the indicated total planar degeneration so the 7-subset of planes spans a $\mathbb{P}^{20}$. In the $d = 6$ example, the only vertex not covered by the 9-subset is the one at the upper left; the 9-subset spanning a $\mathbb{P}^{26}$ □

As announced, the lemmas above enable us to reduce the problem of determining the dimension of $\mathcal{L}_d(2^n)$ to a combinatorial one.

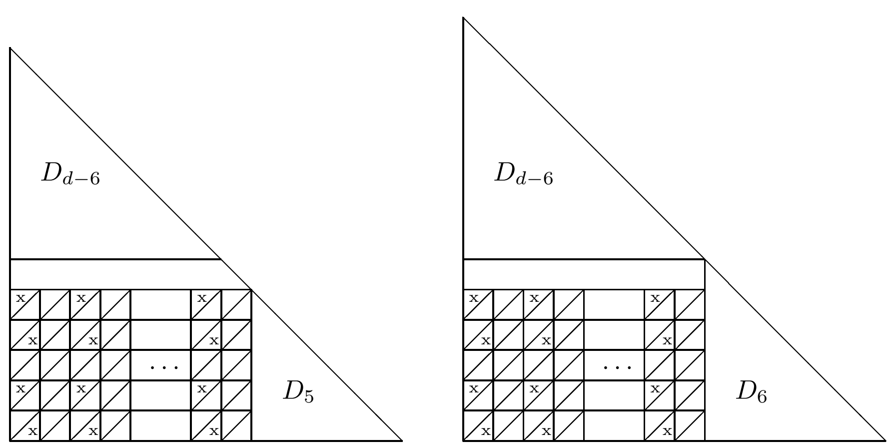
Theorem 2.1.3. *The linear system $\mathcal{L}_d(2^n)$ has the expected dimension whenever $d \geq 5$.*

PROOF. The proof will be by induction on the degree d. Fix $n_0 = \lfloor (d+1)(d+2)/6 \rfloor$; with this number of points, we see that the virtual dimension of $\mathcal{L}_d(2^{n_0})$ is

$$v = d(d+3)/2 - 3n_0 = \begin{cases} -1 & \text{if } d \equiv 1, 2 \mod 3 \\ 0 & \text{if } d \equiv 0 \mod 3. \end{cases}$$

Suppose that the theorem is true for this $n = n_0$. Since the virtual dimension of $\mathcal{L}_d(2^{n_0})$ is at least -1, we conclude that the $3n_0$ conditions imposed by the n_0 double points are independent. Hence any fewer number of points will also impose independent conditions, and so $\mathcal{L}_d(2^k)$ will have the expected dimension for any $k < n_0$.

We will show is that there is a skew n_0–subset S of planes for a certain planar degeneration D of V_d, if there is one for V_{d-6}. To start the induction, we must illustrate such degenerations and subsets for $5 \leq d \leq 10$; like in Lemma (2.1.2); this is not difficult and we leave it to the reader. By induction, we assume that a total planar degeneration D_{d-6} and a maximal skew subset S' of it are available. To be specific, we have two configurations (depending on the parity of d) which show that



d odd $\qquad$ d even

FIGURE 2.8.

if we have a solution for degree $d-6$, then we can construct one for d; this finishes the proof. $\square$

Remark 2.1.4. *When $d = 2$ or 4 it is easy to see that we can't find a 2 or 5 skew set of planes; however this does not prove the non-emptiness of the linear system. The argument here is the speciality of the systems. However for $d = 3$ there exists a 3-skew set, therefore the theorem also holds for $d = 3$*

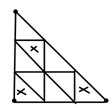


FIGURE 2.9.

2. Points of higher multiplicity.

In this section we go back to interpolation and we want to use the planar degenerations of the Veronese surfaces in order to study multiple points interpolation problems. We recall a basic fact.

Lemma 2.2.1. *Let X be a variety, M a line bundle on X, and D an effective Cartier divisor on X. Set $M(D) = M \otimes \mathcal{O}_X(D)$. Suppose that $H^0(X, M) = H^1(X, M) = 0$. Then the restriction map from $H^0(X, M(D))$ to $H^0(X, M(D)|_D)$ is an isomorphism.*

Lemma 2.2.2.

(a) *Let X be the blow up of $\mathbb{P}^2$ at a point, with exceptional divisor E and line class H. Let $M = \mathcal{O}_X((m-1)H - mE)$. Then $H^0(M) = H^1(M) = 0$.*

(b) *Let X be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two general points, with exceptional divisors E_1 and E_2, and denote by V the vertical fiber class and by H the horizontal fiber class. Let $M = \mathcal{O}_X((m-1)H + mV - mE_1 - mE_2)$ (or, symmetrically, $M = \mathcal{O}_X(mH + (m-1)V - mE_1 - mE_2)$). Then $H^0(M) = H^1(M) = 0$.*

PROOF. In both cases, we have that $H^0 = 0$ since the systems are empty.
Indeed, for (a) there are no curves of degree $m-1$ with one m-multiple point.
We apply Riemann-Roch for $D = (m-1)H - mE$

$$\chi(\mathcal{O}_X(D)) = \frac{D(D-K_X)}{2} + \chi(\mathcal{O}_X).$$

By Serre Duality we get $H^2(\mathcal{O}_X((m-1)H - mE)) = 0$; and since $H^0(\mathcal{O}_X((m-1)H - mE)) = 0$ and $\chi(\mathcal{O}_X) = 1$ we obtain

$$h^1(\mathcal{O}_X((m-1)H - mE) = 1 - \frac{D(D-K_X)}{2}$$

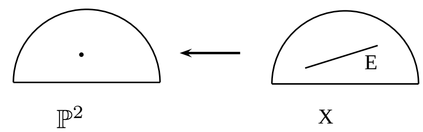


FIGURE 2.10.

We have $K_X = K_{\mathbb{P}^2} + E$ and $K_{\mathbb{P}^2} = -3H$, so

$D - K_X = (m+2)H - (m+1)E$ where $E^2 = -1$ and $H^2 = 1$

$$h^1(\mathcal{O}_X((m-1)H - mE)) = 1 - [(m-1)H - mE][(m+2)H - (m+1)E]/2$$

$$= 1 - [(m-1)(m+2) + m(m+1)]/2 = 0$$

For part (b) $D = mH + (m-1)V - mE_1 - mE_2$ and we get that

$H^0(\mathcal{O}_X(mH + (m-1)V - mE_1 - mE_2)) = 0$ since applying a Cremona transformation we have $\mathcal{L}_{m-1,m}(m^2) \cong \mathcal{L}_{m-1}(m) = \oslash$ In general, $K_{\mathbb{P}^1 \times C} = -2s + (2g(C) - 2)F$ so for $C = \mathbb{P}^1$ we obtain

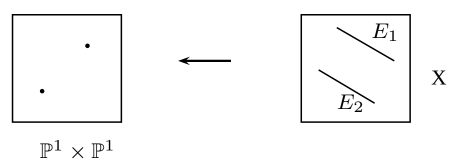


FIGURE 2.11.

$K_{\mathbb{P}^1 \times \mathbb{P}^1} = -2B - 2F$. Using $K_X = K_{\mathbb{P}^1 \times \mathbb{P}^1} + (E_1 + E_2)$ we obtain $D - K_X = (m+1)B + (m+2)F - (m+1)(E_1 + E_2)$ where $E_1^2 = E_2^2 = -1$ and $F^2 = B^2 = 0$. By Riemann-Roch

$\chi = D(D - K_X)/2 + 1 = 0$ it implies that $h^1(\mathcal{O}_X(mH + (m-1)V - mE_1 - mE_2)) = 0$. □

We can apply the previous Lemmas to different divisors D. It is useful for our applications, given the degeneration constructions we have introduced, that the divisors D be subdivisors of the double curves of the planes or quadrics in the degeneration.

In the planar case, this means that we will be applying the lemmas for a divisor D consisting of a subdivisor of a triangle of lines $L_1 + L_2 + L_3$. We have the following list of $M(D)$'s in this case.

Lemma 2.2.3. *Let X be the blow up of $\mathbb{P}^2$ at a point, with exceptional divisor E and line class H. Let L_1, L_2, and L_3 be a triangle T of lines not passing through the point. Let $M = \mathcal{O}_X((m-1)H - mE)$.*

Then the restriction maps

$$(1):\ H^0(X, \mathcal{O}_X(mH - mE)) \to H^0(L_i, \mathcal{O}_X(mH)|_{L_i})$$

$$(2):\ H^0(X, \mathcal{O}_X((m+1)H - mE)) \to H^0(L_i + L_j, \mathcal{O}_X((m+1)H - mE)|_{L_i+L_j})$$

$$(3):\ H^0(X, \mathcal{O}_X((m+2)H - mE)) \to H^0(T, \mathcal{O}_X((m+2)H - mE)|_T)$$

are isomorphisms.

Proof. We note that these three spaces have dimensions $m+1$, $2m+3$, and $3m+6$, respectively.

(1) This result also follows from Lemmas 2.2.1 and 2.2.2 but we will give a different proof. Indeed, we notice that the linear system $\mathcal{L}_m(m)$ is determined by fixing a divisor of degree m on a line.

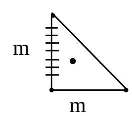


Figure 2.12.

First, assuming that the fixed point P is $[0, 0, 1]$ and then identifying $\mathcal{L}_m(m) \cong K[X, Y]_m$ one can easily see that $\dim \mathcal{L}_m(m) = v(\mathcal{L}_m(m)) = m$. We note that each point on the line and P determine a unique line in the plane, so the corresponding curve is the product of the m lines. Therefore this curve has degree m and an m multiple point at P.

(2) We claim that the linear system $\mathcal{L}_{m+1}(m)$ is determined by the restriction of a divisor of degree $m+1$ on two lines.

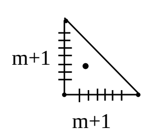


Figure 2.13.

Again we compute the dimension $\dim \mathcal{L}_{m+1}(m) = v(\mathcal{L}_{m+1}(m)) = 2m + 2$ since if $P = [0, 0, 1]$, then the linear system is just $ZK[X, Y]_m + K[X, Y]_{m+1}$.

Let $C = L_i + L_j$ be the divisor of the two lines, $C \equiv 2H$, one gets an exact sequence by taking the restriction map of each curve to the conic C

$$0 \to \mathcal{O}((m-1)H - mE) \to \mathcal{O}((m+1)H - mE) \to \mathcal{O}_C(m+1) \to 0.$$

By Lemma 1, $h^0(\mathcal{O}((m-1)H - mE)) = h^1(\mathcal{O}((m-1)H - mE)) = 0$ so we obtain

$$H^0(\mathcal{O}((m+1)H - mE)) \cong H^0(\mathcal{O}_C(m+1))$$

.

(3) We claim that the linear system $\mathcal{L}_{m+2}(m)$ is determined by the restriction of a divisor of degree $m+2$ on a triagle of lines.

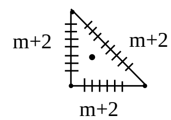


FIGURE 2.14.

The linear system is again $Z^2K[X,Y]_m + ZK[X,Y]_{m+1} + K[X,Y]_{m+2}$ so $\dim \mathcal{L}_{m+2}(m) = v(\mathcal{L}_{m+2}(m)) = 3m+5$. If T is the divisor of the three lines, then $T \cong 3H$, so the restriction to this cubic gives us the following short sequence:

$$0 \to \mathcal{O}((m-1)H - mE) \to \mathcal{O}((m+2)H - mE) \to \mathcal{O}_T(m+2) \to 0.$$

By a similar argument one gets an isomorphism on the level of sections:

$$H^0(\mathcal{O}((m+2)H - mE)) \cong H^0(\mathcal{O}_T(m+2)).$$

$\square$

In the quadric case, we present the information in a table. Let X be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two general points, with exceptional divisors E_1 and E_2, and denote by V the vertical fiber class and by H the horizontal fiber class. We fix two vertical fibers V_1 and V_2, and two horizontal fibers H_1 and H_2, not passing through the two points; our divisor D will be a subdivisor of $H_1 + H_2 + V_1 + V_2$. By Lemma 2.2.2, there are two possibilities for the line bundle M for each divisor D; these are presented in the last two columns of the table.

Lemma 2.2.4. *Using the above notation, the restriction map from $H^0(X, M(D))$ to $H^0(D, M(D)|_D)$ is an isomorphism, for all D and $M(D)$ in the following table:*

	Divisor D	*M(D)-a*	*M(D)-b*
1.	0	$mV + (m-1)H$	$(m-1)V + mH$
2.	V_1 *or* V_2	$(m+1)V + (m-1)H$	$mV + mH$
3.	H_1 *or* H_2	$mV + mH$	$(m-1)V + (m+1)H$
4.	$V_i + H_j$	$(m+1)V + mH$	$mV + (m+1)H$
5.	$V_1 + V_2$	$(m+2)V + (m-1)H$	$(m+1)V + mH$
6.	$H_1 + H_2$	$mV + (m+1)H$	$(m-1)V + (m+2)H$
7.	$H_1 + H_2 + V_i$	$(m+1)V + (m+1)H$	$mV + (m+2)H$
8.	$V_1 + V_2 + H_j$	$(m+2)V + mH$	$(m+1)V + (m+1)H$
9.	$V_1 + V_2 + H_1 + H_2$	$(m+2)V + (m+1)H$	$(m+1)V + (m+2)H$

(We abused notation and denoted the $M(D)$'s using the divisor classes only.)

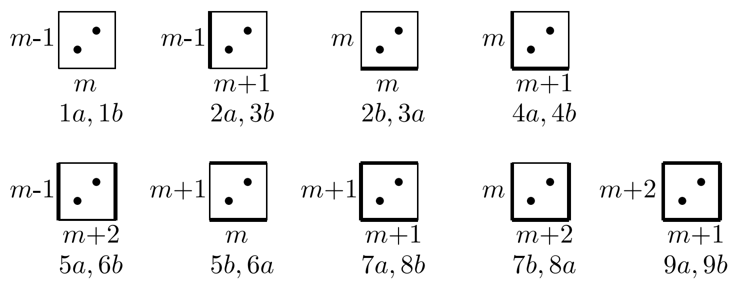


FIGURE 2.15.

We will also need the following useful observation

Lemma 2.2.5. *Consider n surfaces with an interior vertex v, $X_1, ..., X_n$ such that X_i and X_{i+1} intersect along the line L_i. Consider M_i to be complete linear systems on each surface X_i determined by the restriction to each line gives isomorphism*

$$M_i \cong M_i \mid_{L_i}$$

$$M_i \cong M_i \mid_{L_{i+1}} .$$

Then there is a single divisor in the fiber product of elements of M_i that agree on each L_i. Moreover, it is of the form dP_i for some point P_i on each L_i.

PROOF. The composition of the isomorphisms $\ell_i : M_i \mid_{L_i} \rightarrow M_i \mid_{L_{i+1}}$ gives an automorphism $\ell : M_1 \mid_{L_1} \rightarrow M_1 \mid_{L_1}$ which must be determined by an automorphism of a line $\sigma : L_1 \rightarrow L_1$, that fixes the vertex v. There is a one to one correspondence between the sets {fiber product of elements of M_i that agree on each L_i } $\leftrightarrow$ {divisors on $M_1 |_{L_1}$ invariant under ℓ }. So after a change of coordinates σ becomes an automorphism of $\mathbb{P}^1$ that fixes the origin and the point of infinity, so is just multiplication by a constant α. So in the affine patch, (in some coordinate system) one has $\sigma(s) = \alpha s$. So, switching to projective coordinates $\sigma[z,w] = [\alpha z, w]$. Consider polynomials of degree d that are invariant under the morphism σ. We ask that

$$\frac{f(z,w)}{z^d} = \frac{f(\sigma[z,w])}{z^d}.$$

This condition forces f to be a monomial of degree d, of the form
$z^r w^{d-r}$, for $0 \leq r \leq d$.
Notice that if $r > 0$, then the divisor associated to the polynomial $z^r w^{d-r}$ has 0 in it's support. Because 0 is the interior vertex the polynomials on each surface X_i not only they will vanish at 0, but also from the compatibility conditions on each line L_i, they will have the tangent direction determined. Since the interior vertex can not be in the support of the divisors, one gets only one dimensional space of polynomials of degree d invariant under σ, generated by w^d. (So, one gets an invariant polynomial of degree d on each line L_i, and they lift to an unique element of the linear system of the total space). $\square$

Remark 2.2.6. Lemma 2.2.5 is useful when there are cyclical configurations of surfaces that overlap. In this case, in each of the cycles of surfaces, there is a unique section up to scalar satisfying the matching conditions. However these two sections will not agree on the overlap. Hence we conclude that any section satisfying the matching conditions must be zero.

Next, we will apply these lemmas by constructing a degeneration of the d-fold Veronese V_d to a union of planes and quadrics as described above. We will degenerate the bundle $\mathcal{O}(d)$ to a bundle on the degenerate configuration which will have certain degrees on the planes and bidegrees on the quadrics. The general multiple points will degenerate either to one point on a plane or to two general points on a quadric. For higher multiplicities, it is necessary to relate the linear systems on the surfaces and on the double curves. An example will illustrate the argument.

The next theorem is the more general statement, which is slightly better than Nagata's conjecture in this case, but it is weaker than Harbourne-Hirschowitz.

Theorem 2.2.7. *The system $\mathcal{L}_{km}(m^{k^2})$ has the expected dimension, in particular it is empty for $k \geq 4$.*

Proof. As above, we consider the system associated to the line bundle $\mathcal{O}(m)$ on the k-fold Veronese V_k. Degenerating, we form a total planar degeneration to k^2 planes, and on each plane we have the linear system of curves of degree m. We degenerate the k^2 points by putting one in general position on each plane of the degeneration; for example, a $k = 5$ example is illustrated below.

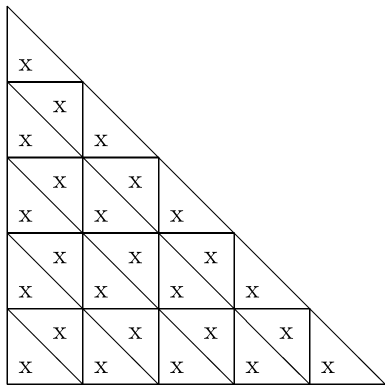


Figure 2.16.

The cases $k = 1, 2$ and 3 can be analyzed separately. For $k \geq 4$, the system is expected to be empty. For $k = 4$ the linear system is empty.

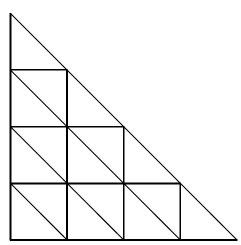


Figure 2.17.

Indeed, there is a unique divisor satisfying the matching conditions on the six planes adjacent to each of the three interior vertices. However for any two of these interior vertices, there are adjacent planes in common (by lemma 2.2.6). The divisors will not agree on these common adjacent planes. Hence the system is empty as expected.

Finally for $k > 4$, if we form the same type of configuration, by induction, the *top* $(k-1)^2$ planes already cannot support a divisor. The system will thus be empty. $\square$

3. Line bundles on quadrics degenerations of the Veronese.

More flexibility can be acquired in the limiting line bundle on the configuration by using the quadrics degeneration of the Veronese that we presented in §2. We recall that this is the triangular configuration of $\binom{d}{2}$ quadrics, meeting along lines, with d planes on the 'hypotenuse' of the configuration. We will coordinatize the configuration, and index the surfaces in the configuration as T_{ij}, with $i \geq 1$, $j \geq 1$, and $i + j \leq d + 1$; the quadrics are the surfaces with $i + j \leq d$, and the planes are the surfaces $T_{i,d+1-i}$. We have that T_{ij} meets $T_{k\ell}$ along a line if and only if either $i = k$ and $|j - \ell| = 1$ or $j = \ell$ and $|i - k| = 1$.

We can form a line bundle on this partial quadrics degeneration by putting a line bundle on each surface such that on each double curve the restriction of the two bundles agree. This can be done by choosing d integers $r_1, r_2, \ldots, r_d$, and for $i + j \leq d$ putting the bundle of bidegree (r_i, r_{d+1-j}) on the quadric T_{ij}; on the plane $T_{i,d+1-i}$ one puts the bundle of degree r_i. This can be conveniently with the following picture referring to the case $d = 5$: This line bundle is the limit of the line bundle

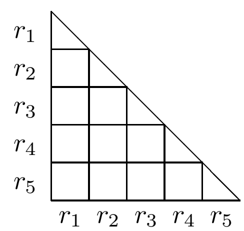


FIGURE 2.18.

$\mathcal{O}_{\mathbb{P}^2}(r)$, with $r = r_1 + r_2 + \ldots + r_d$. We will use Lemmas 2.2.3 2.2.4 and 2.2.5, for proving the following theorem:

Theorem 2.3.1. *The system $\mathcal{L}_{km+1}(m^{k^2})$ has the expected dimension. In particular, it is empty for $k \geq 6$.*

PROOF. First note that the expected dimension is $v = mk(5-k)/2+2$. We will degenerate V_k to a union of k planes and $\binom{k}{2}$ quadrics placing one point on each plane and two points on each quadric and the bundle degeneration is as indicated above: we will see for each value of k which values of $r_1, \ldots, r_k$ is convenient to take. In particular, for $k \leq 4$ we will take $r_1 = m+1, r_2 = \ldots = r_k = m$. As before, cases $k = 1, ..., 5$ can be analyzed separately (see [**4**]).

Case $k = 6$. This is the first case where we must show that the system is empty. Here we consider the following degeneration of the plane into six surfaces, three re-embedded quadrics and three Veronese surfaces, with degrees indicated, that sum to $6m + 1$:

The number of points on each quadric is 8, while the number on each Veronese is 4; note that the total number is 36 as required.

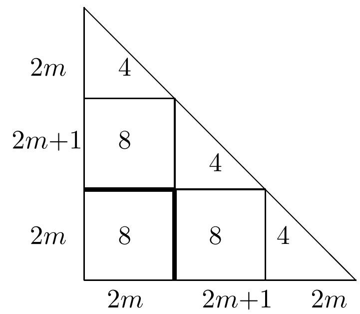


FIGURE 2.19.

We focus on the lower left quadric $T_{1,1}$, where we have the linear system of curves of bidegree $(2m, 2m)$ on a quadric, with 8 points of multiplicity m. This has a space of sections of dimension one, namely a unique divisor, the m-fold curve in the linear system of bidegree $(2, 2)$ through the 8 points. The restriction of this space of sections to both the right double curve and the top double curve has dimension one.

Now consider the quadric $T_{2,1}$ just to the right of this, and consider the restriction to the double curve on the left with $T_{1,1}$. This restriction of sections is onto (the sheaf is the sheaf of degree $2m$ on that vertical curve), and the kernel has dimension one (as a vector space), with the similar analysis as above. Therefore the space of sections here that could agree with an element of the dimension one space of sections coming from $T_{1,1}$ has dimension two, one coming from the restriction and one coming from the kernel.

This same analysis holds for the quadric $T_{1,2}$: there is a dimension two space of sections there that restrict to some element of the dimension one space of sections of the double curve where this quadric meets the lower left quadric.

The space of sections now on these three quadrics has dimension three: 2 each on the two quadrics, but there is a condition that the sections agree at the point of intersection, which is the interior point of the configuration.

Now look at one of the corner planes, e.g., $T_{1,3}$. The system there is of degree $2m$, with four m-fold points. We know that this is the linear system composed with the pencil of conics through the four points. Therefore the restriction of this system to the double line is the system (of vector space dimension $m + 1$) of the intersections with the pencil of conics. If $m \geq 2$, no element of the 2-dimensional space of sections on the adjoining quadric will match with any such element on the double line. (The ambient space has vector space dimension $2m + 1$, and we have the restriction of a 2-dimensional space from the quadric and the $(m+1)$-dimensional space from the plane, which will not intersect away from 0 if $m \geq 2$.)

We conclude that the section must be zero on that corner plane, also by symmetry on the other corner plane; then it must be zero as well on the quadrics, and finally on the center plane $T_{2,2}$.

Case $k \geq 7$. The virtual dimension is $v < 0$ and we must show the system is empty. We use a degeneration with $r_1 = r_2 = r_3 = m$, $r_4 = m+1$, and $r_i = m$ for $i \geq 5$:

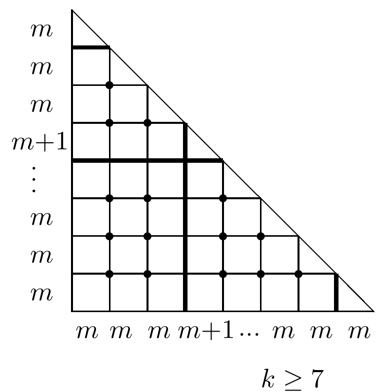


FIGURE 2.20.

The sections on the $3k-12$ lower left quadrics must be zero, using the Remark 2.2.6. Then on the eight surfaces just above these, we must also have zero sections; this applies as well all of the surfaces to the right of these, except the final corner plane. This leaves only the two corner planes $T_{1,k}$ and $T_{k,1}$, and the final plane $T_{4,k-3}$; sections on these are now seen to be zero as well. $\square$

CHAPTER 3

Triple points in $\mathbb{P}^2$

1. Toric varieties and toric degenerations.

In this section we recall a few basic facts about toric degenerations of projective toric varieties we referred to [**15**], for more information on the subject and to [**12**] for relations with tropical geometry.

The datum of a pair $(X, \mathcal{L})$, where X is a projective, n–dimensional toric variety and $\mathcal{L}$ is a base point free, ample line bundle on X, is equivalent to the datum of an n dimensional convex polytope $\mathcal{P}$ in $\mathbb{R}^n$, determined up to translation. Thus we will assume all points of $\mathcal{P}$ have non–negative coordinates (see [**11**], page 72). If $m_i = (m_{i1}, \dots, m_{in})$, $0 \leq i \leq r$, are the $r+1$ integral points of P, we consider the map

$$\phi_{\mathcal{P}} : x \in (\mathbb{C}^*)^n \to [x^{m_0} : \ldots : x^{m_r}] \in \mathbb{P}^r$$

where $x = (x_1, \dots, x_n)$ and

$$x^{m_i} := x_1^{m_{i1}} \cdots x_n^{m_{in}}.$$

The closure of the image of $\phi_{\mathcal{P}}$ is the image $X_{\mathcal{P}}$ of X via the morphism $\phi_{\mathcal{L}}$ determined by the line bundle $\mathcal{L}$. For example, if $\mathcal{P}$ is the triangle $\Delta_d := \{(x,y) : x \geq 0, y \geq 0, x+y \leq d\}$ then X_{Δ_d} is the Veronese surface V_d.

If $\mathcal{P}$ is the rectangle $R_{a,b} := \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\}$ with a, b positive integers, then $X_{R_{a,b}}$ is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{ab+a+b}$ via the linear system $\mathcal{L}_{(a,b)}$ of curves of bidegree (a,b).

If $\mathcal{P}$ is the trapezoid $T_{a,b} := \{(x,y) : x \geq 0, 0 \leq y \leq b, x+y \leq a\}$ with $a > b$ positive integers, then $X_{T_{a,b}}$ is $\mathbb{F}_1$, i.e. the plane blown up at a point p, embedded in $\mathbb{P}^r$, $r = ab + a - b(b+1)/2$, via the proper transform of the linear system of curves of degree a with a point of multiplicity $a-b$ at p.

We consider a *subdivision* $\mathcal{D}$ of $\mathcal{P}$ into convex subpolytopes; i.e. a finite family of n dimensional convex polytopes whose union is $\mathcal{P}$ and such that any two of them intersect only along a face (which may be empty). Such a subdivision is called *regular* if there is a piecewise linear, positive function F defined on P such that:

(i) the polytopes of $\mathcal{P}$ are the orthogonal projections on the hyperplane $z = 0$ of $\mathbb{R}^{n+1}$ of the n–dimensional faces of the *graph polytope*

$$G(F) := \{(x, z) \in \mathcal{P} \times \mathbb{R} : 0 \leq z \leq F(x)\}$$

which are neither vertical, nor equal to P;

(ii) the function F is *strictly convex*, i.e., the hyperplanes determined by each of the faces of $G(F)$ intersect $G(F)$ only along that face.

If there is a regular subdivision $\mathcal{D}$ as above, one can construct a projective degeneration of $X_{\mathcal{P}}$ (parametrized by the affine line $\mathbb{C}$), to a reducible variety X_0 which is the union of the toric varieties X_Q, with Q in $\mathcal{D}$. The intersection of the components X_Q of X_0 is dictated by the incidence relations of the corresponding polytopes: if Q and Q' have a common face R, then X_Q intersects $X_{Q'}$ along the toric subvariety of both determined by the face R.

The degeneration can be described as follows. Consider the morphism

$$\phi_{\mathcal{D}} : (x, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to [t^{F(m_0)}x^{m_0} : \ldots : t^{F(m_r)}x^{m_r}] \in \mathbb{P}^r.$$

The closure of the image of $(\mathbb{C}^*)^n \times \{t\}$, $t \neq 0$, is a variety X_t which is projectively $X_{\mathcal{P}}$. The limit of X_t when t tends to 0 is the variety X_0. X_0 is the union of the varieties X_Q, with $Q \in \mathcal{D}$. Indeed, suppose that $F|_Q$ is the linear function $a_1x_1 + \ldots + a_nx_n + b$. First we act with the torus in the following way:

$$(x_1, \ldots, x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to (t^{-a_1}x_1, \ldots, t^{-a_n}x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^*.$$

Then we compose with $\phi_{\mathcal{D}}$, and we get

$$(x_1, \ldots, x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to [\ldots : t^{F(m_i)}t^{-a_1m_{i1}-\ldots-a_nm_{in}}x^{m_i} : \ldots] \in \mathbb{P}^r.$$

We note that the point in $\mathbb{P}^r$ equals

$$[\ldots : t^{F(m_i)}t^{-a_1m_{i1}-\ldots-a_nm_{in}}x^{m_i} : \ldots] = t^{-b}[\ldots : t^{F(m_i)}t^{-a_1m_{i1}-\ldots-a_nm_{in}}x^{m_i} : \ldots]$$

i.e. the point

$$[\ldots : t^{F(m_i)-F|_Q(m_i)}x^{m_i} : \ldots].$$

Then by by letting $t \to 0$ in the above expression, we see that X_Q sits in the flat limit X_0 of X_t.

We will now prove the existence of a lifting function by iterating an obvious lemma. Let X be a toric surface and $\mathcal{P}$ be its associated polytope. Consider $\mathcal{P}_\infty$ and $\mathcal{P}_\in$ to be two disjoint polytopes in $\mathcal{P}$ and X_1 and X_2 their corresponding toric varieties. We let L be a line separating $\mathcal{P}_\infty$ and $\mathcal{P}_\in$ and not containing any integer point.

Lemma 3.1.1. *The toric variety X degenerates into a union of toric varieties two of which are skew.*

Proof. We consider the convex piecewise linear function given by

$$f(x, y, z) = \max\{z, L + z\}$$

Consider the image of the points on the boundary of the polytopes X_1 and X_2 through f. Take now the convex function corresponding to the convex hull of the boundary points separated by L. The function will still be convex and piecewise linear, therefore we get a regular degeneration. We consider now the toric varieties associated to each polytope, and since $\mathcal{P}_\infty$ and $\mathcal{P}_\in$ are disjoint, we obtain that two of the toric varieties, namely X_1 and X_2 are skew.

For example, in the picture below we have four polytopes, two of which are disjoint. The corresponding degeneration will contain four toric varieties, two of them X_1 and X_2, being skew.

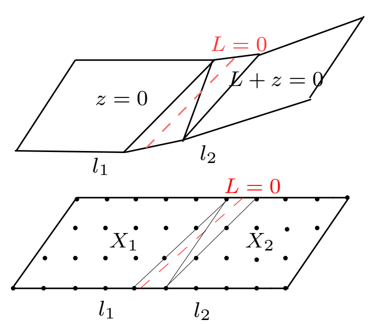


Figure 3.1.

∎

It is easy to see how we iterate this process. We regard X_2 as a surface independent of X, and we let M be a line cutting the polytope associated to X_2 and not containing any of its interior points. Then X_2 degenerates into a union of toric surfaces, two of which are skew, Y_2 and Y_3.

We conclude that X degenerates into nine toric surfaces three of which X_1, Y_2 and Y_3 being skew, as the picture indicates.

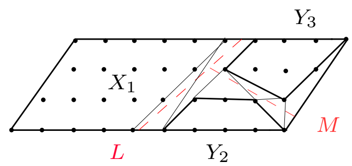


FIGURE 3.2.

Later on, we will ignore the varieties lying in between the disjoint ones; they are only important for the degeneration and not for the analysis itself.

2. Notation and Terminology.

The group $SL_2^{\pm}(\mathbb{Z})$ acts on the column vectors of $\mathbb{R}^2$ by left multiplication. This induces an action of $SL_2^{\pm}(\mathbb{Z})$ on the set of convex polytopes $\mathcal{P}$ by acting on its enclosed points. We will denote $\mathcal{P}'$ to be the image of $\mathcal{P}$. For example, the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ acts on the polytope $\mathcal{P}$ by sending every point $\begin{pmatrix} x \\ y \end{pmatrix}$ enclosed by $\mathcal{P}$ to $\begin{pmatrix} x-y \\ y \end{pmatrix}$. Note that the points on the base level $\begin{pmatrix} x \\ 0 \end{pmatrix}$ are fixed by this action while the points on the first level $\begin{pmatrix} x \\ 1 \end{pmatrix}$ will be shifted by 1, the points on the second level will be shifted by 2 etc. Denoting by $\mathcal{P}'$ the resulting polytope, we have that $\mathcal{P}$ and $\mathcal{P}'$ are congruent. We will say we shift a polytope by n when we repeat this operation n times, i.e. when we act by the matrix $\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$. Shifting left or right depends on the sign of n. In a similar way rotation by an angle of π corresponds to the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; reflection to the action of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and translation corresponds to the the action of $\mathbb{Z}^2$ etc. Orientation preserving lattice equivalences form a group, the semidirect product $SL_2(\mathbb{Z})$ and $\mathbb{Z}^2$. $SL_2^-(\mathbb{Z})$ corresponding to orientation reversing lattice equivalences.

Let $\#(\mathcal{P})$ denote the number of integer points enclosed by the polytope. We recall a useful formula that is similar to Riemann-Roch for toric surfaces ([**11**], page 113):

Remark 3.2.1. *Pick's Formula.*

$$\#(\mathcal{P}) = Aria(\mathcal{P}) + Perimeter((\mathcal{P}))/2 + 1$$

Next, we will classify all convex polytopes enclosing six lattice integer points modulo the actions described above. For more connections with toric geometry the interested reader can consult [**14**], [**22**] or [**23**]. We first start with a definition.

Definition 3.2.2. *We say the polytope $\mathcal{P}$ is in standard position if*

(1) *It contains $O = (0, 0)$ as a vertex*
(2) *OS is a vertex where $S = (0, m)$ and m is the largest edge length*
(3) *OP is an edge where $P = (p, q)$ and $0 \leq p < q$*

Remark 3.2.3. *Every polytope has a standard position.*

Indeed, we first choose the longest edge and then we translate one of its vertices to the origin. We will now rotate the polytope to put the longest edge on the positive side of the x axis and then we shift it such that the adjacent edge lies in the upper half of the first quadrant. Indeed, if OP is an edge with $P = (s, q)$ and $s \geq q$; then $s = mq + p$ for $0 \leq p < q$ so we shift left by m. We will call this procedure normalization.

It is easy to see that the standard position of the polytope may not be unique, it depends on the choice of the longest edge, and of the choice of the special vertex that becomes the origin.

We can now begin the classification of the polytopes in standard position according to m (the maximum number of integral points lying on the edges of the polytope), and also according to their number of edges, n. Obviously, the polytopes $\mathcal{P}$ will have at most six edges, and at most five points on an edge, so we get the inequalities $n \leq 6$ and $m \leq 5$. We will denote by R the point $(0, 1)$. Obviously, $R \in \mathcal{P}$

3. The Classification of Polytopes.

Remark 3.3.1. *If $m \neq 5$, then $M = (1, 1) \in \mathcal{P}$*

(1) *Indeed, knowing that $p < q$ and first assuming $p \neq 0$, then $M = (1, 1)$ is inside the region enclosed by the lines PO and PR.*
(2) *If $p = 0$ and q is at least 2 then $m \geq 2$ and by convexity $\mathcal{P}$ contains the point $(1, 1)$. If $p = 0$ and $q = 1$ then PR contains at most 4 points (since $m \leq 4$) so $\mathcal{P}$ has at least one more vertex, $Q = (s, t)$. If $s < t$ apply the previous analysis for QOR; if $s > t$ apply it for QPO while if $s = t$, then $(1, 1) \in OS \subset \mathcal{P}$.*

We will therefore start the classification of all the convex polytopes $\mathcal{P}$, that contain the points O, R and M and the next vertex is the point $P = (p, q)$ where $p < q$ and $p, q \geq 0$.

(1) $m = 5$ so no interior points. There is only one other vertex P that after shifting can be assumed to be on the y axis. We conclude the only possibility is

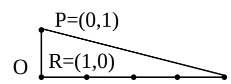


FIGURE 3.3.

(2) $m = 4$ so the longest edge has 4 vertices.

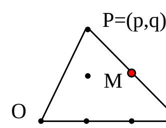


FIGURE 3.4.

We claim that $P = (p, q) = (0, 1)$. Indeed, assume $p \neq 0$. Since the point $M = (1, 1)$ is in the interior, we conclude that $q < 2$ otherwise the polytope contains $(2, 1)$. Therefore $q = 1$ and $(1, 1)$ is a vertex, so shifting by one allows us to assume that P is $(0, 1)$. Remark 3.3.1 case (2) gives us that $\mathcal{P}$ needs to contain the point $(1, 1)$.

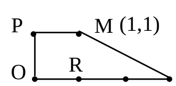


FIGURE 3.5.

(3) $m = 3$. As before we have that $\mathcal{P}$ contains $O, R, M, N = (0, 2)$ and $P = (p, q)$.

(a) $p = 0$

(i) $(2, 0)$ is in the polytope so $\mathcal{P}$ corresponds to the projective space $\mathbb{P}^2$

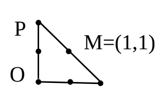


FIGURE 3.6.

(ii) $(0, 1)$ is the next vertex. We distinguish two subcases:

(A) $(1,1)$ is on the next edge so the polytope $\mathcal{P}$ corresponds to the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$

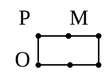


FIGURE 3.7.

(B) Next edge is at a higher angle, and since $(2,1) \notin \mathcal{P}$ we get $p < 2$. So $p = 1$ and $q = 2$

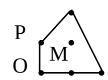


FIGURE 3.8.

(b) $p = 1$ so $q \leq 3$.

(i) $q = 2$ then $(1,2)$ is the last vertex. Remark 3.3.1 case (2) with the origin translated at the point $(0,1)$ gives $(2,1)$ is in $\mathcal{P}$ also.

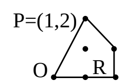


FIGURE 3.9.

Note that this is a reflection of the polytope from Case 3a)iiB, so the two polytopes are equivalent and are both in standard position.

(ii) $q = 3$

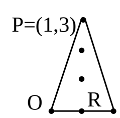


FIGURE 3.10.

(c) $p \geq 2$. We have $q \geq 2$ and by convexity, $(2,1), (2,2) \in \mathcal{P}$ and these are all six points. After shifting we obtain $P = (0,2)$ that contradicts the hypothesis $p \geq 2$

(4) $m = 2$. The classification of the polytopes will now depend on the number their edges, n.

(a) $n = 3$. For this we use Pick's formula (see [**11**] page 113) to get $6 = 1 + \frac{|\mathcal{P}|}{2} + area = 1 + \frac{3}{2} + area$, we get that $area = \frac{q}{2} = \frac{7}{2}$, so $q = 7$. We have to classify all polytopes $\mathcal{P}_p$ with the vertex at $P = (p, 7)$ and $p \leq 7$. Note that the assumption $m = 2$ imposes

$p \neq 0, 1$.

By shifting and reflecting we find that $\mathcal{P}_6$ and $\mathcal{P}_2$ are congruent, and similarly, $\mathcal{P}_5$ and $\mathcal{P}_3$. We also have that $\mathcal{P}_4 \equiv \mathcal{P}_2$. Indeed, since the matrix $\begin{pmatrix} -2 & 1 \\ -7 & 3 \end{pmatrix}$ takes the set

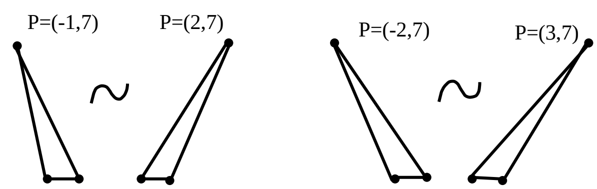


FIGURE 3.11.

of points $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$ to the set $\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.
so they differ by an element of $SL_2(\mathbb{Z})$.

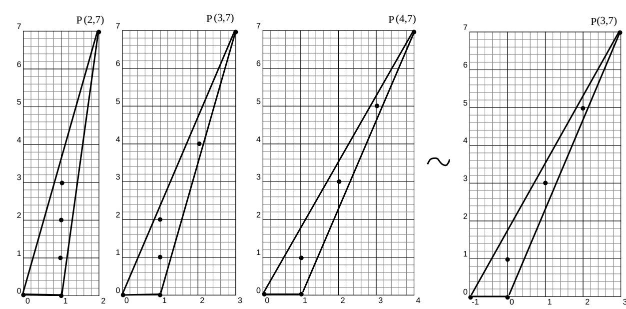


FIGURE 3.12.

The distinct polytopes that we obtain in this case are $\mathcal{P}_2$ and $\mathcal{P}_3$

FIGURE 3.13.

(b) $n = 4$ so 2 interior points.

(i) The first chart is smooth so P is $(0, 1)$. We denote by $Q = (s, t)$ the last vertex of the polytope that also encloses the point $(1, 1)$. $6 = 1 + \frac{4}{2} + area$, so $area = \frac{s+t}{2} = 3$. Since $s + t = 6$ we have two possibilities

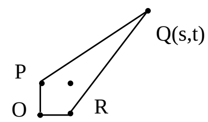


FIGURE 3.14.

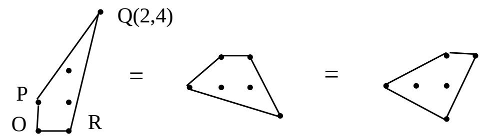


FIGURE 3.15.

(A) $Q = (2, 4)$
(B) $Q = (3, 3)$

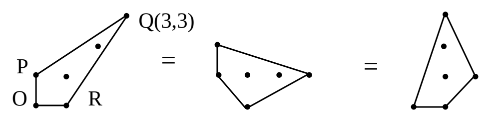


FIGURE 3.16.

(c) $n = 4$

(i) The triangle OPR has no points in interior and 4 points on the boundary. We get

$4 = 1 + \frac{4}{2} + area \Rightarrow area = q = 2$. $p \neq 0, 2$ since $m = 2$, so $P = (1, 2)$ is the next vertex of the polytope and $(1, 1)$ is an interior point. Let $Q = (s, t)$ be the last vertex. $6 = 1 + \frac{4}{2} + area \Rightarrow 3 = area = 1 + \frac{2(s-1)}{2} \Rightarrow s = 3$ and $t \leq 6$ If $t = even$ then the edge OR contains one more integral point $(2, \frac{t-2}{2} + 2)$ that contradicts the assumption $m = 2$. We distinguish two cases for t: $\{3, 5\}$. For

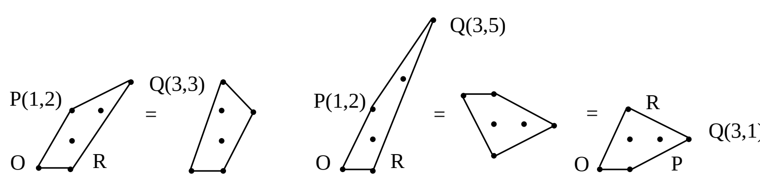


FIGURE 3.17.

$t = 4$ we obtain the polytope from Case $4biA$

(ii) The triangle OPR has one interior point and 3 points on the boundary. We get $4 = 1 + \frac{3}{2} + area \Rightarrow area = \frac{q}{2} = \frac{3}{2}$ and since $p \neq 0, 1, 3$ we get $P = (2, 3)$. Then $6 = 1 + \frac{4}{2} + (\frac{3}{2} + A)$ then $A = \frac{3}{2}$.

If $Q=(s,t), s>2$ is the next vertex then PQ has the equation

$x = 2+(y-3)\frac{s-2}{t-3}$. It intersects the line $x=1$ at the point $(1, 3+\frac{3-t}{2-s})$. $A = (3+\frac{3-t}{s-2})\frac{s-1-1}{2} = \frac{3}{2}$. Then $s = 2+\frac{t}{3}$ and by convexity we have $\frac{t}{s} < \frac{3}{2}$ so

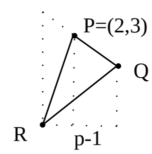


FIGURE 3.18.

$s<4$ and therefore $Q=(3,3)$. If $s<3$ then $s=2$ and RP contains then 3

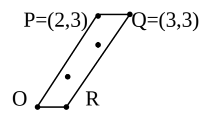


FIGURE 3.19.

points that contradicts our assumption.

(iii) The triangle OPR has 5 points on the boundary and no interior point. We get $5 = 1+\frac{5}{2}+area \Rightarrow area = \frac{q}{2} = \frac{3}{2}$ and since $p=1$ we get $P=(1,3)$. Let $Q=(s,t)$ the last vertex with $0<t<6$.

Then $6 = 1+\frac{4}{2}+area$ then $area = \frac{3}{2}+\frac{3(s-1)}{2} \Rightarrow s=2$.

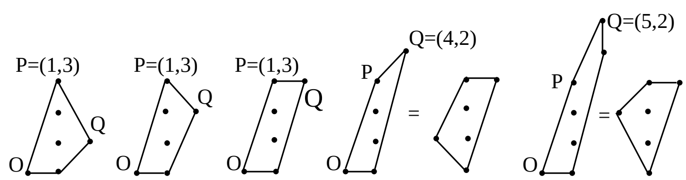


FIGURE 3.20.

All these polytopes were studied before. Indeed, we notice that the polytopes with the vertices at $Q=(2,1)$ and $Q=(5,2)$ are equivalent to the one form Case $4biB$ and same for $Q=(2,2)$ and $Q=(4,2)$ that is the equivalent with Case $4ci$, while if Q is $(2,3)$ we obtain the polytope from Case $4cii$.

(iv) The triangle OPR encloses two interior points. Then $5 = 1+\frac{3}{2}+area \Rightarrow area = \frac{p}{2} = \frac{5}{2}$ so $p=5$ and $1<p<q$ so we get three cases for P: $(2,5),(3,5),(4,5)$

Shifting by 1 and then reflecting we get that $\mathcal{P}_{(4,5)} = \mathcal{P}_{(-1,5)} = \mathcal{P}_{(2,5)}$. Let $Q=(s,t)$ be the last vertex

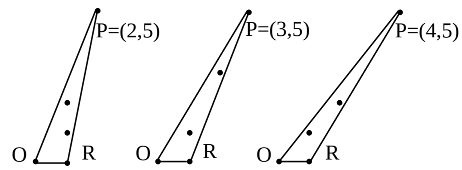


FIGURE 3.21.

(A) If $P(2,5)$ then $6 = 1 + \frac{4}{2} + (\frac{5}{2} + A)$ so $A = \frac{1}{2}$

If $s > 2$ then the line PQ with equation $x = 2 + (y-5)\frac{s-2}{t-5}$ intersects the line $x = 1$ at the point $(1, 5 + \frac{5-t}{s-2})$ so $A = (5 + \frac{5-t}{s-2})\frac{s-1-1}{2} = \frac{5s-t-5}{2} = \frac{1}{2}$.

We get $5s - t = 6$ and $\frac{t}{s} < \frac{5}{2}$ so $t < 6$ and $s < 2.5$

Therefore $s = 2$ and by convexity $Q = (2,4)$.

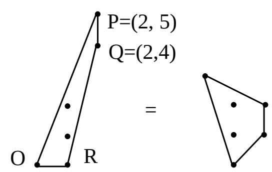


FIGURE 3.22.

This polytope is no new; it was obtained before in Case $4biA$.

(B) If $s > 3$ and $P = (3,5)$ then the line PQ with equation $y = 5 + (x-3)\frac{t-5}{s-3}$ intersects the line $x = 1$ at the point $(1, 5+\frac{5-t}{s-3})$ so $A = (5+2\frac{5-t}{s-3})\frac{s-1-2}{2} = \frac{5s-t-5}{2} = \frac{1}{2}$. We get $5s - t = 6$ and $\frac{t}{s} < \frac{5}{3}$ so $t < 6$ and $s < 3.5$

If $s = 3$ then the triangle PQR encloses 1 point so $s = 2$ and Q is $(2,2)$.

We notice that this is the same with the previous one.

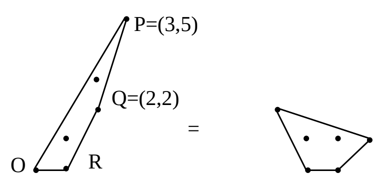


FIGURE 3.23.

(v) $n = 5$ and 1 interior point.

(A) The triangle formed by the three points $(0,0), (1,0), (p,q)$ contains 3 vertices and 1 interior point. Then $4 = 1 + \frac{3}{2} + area$ so $area = \frac{q}{2} = \frac{3}{2}$ so $q = 3$. Since $p \leq q$, $p \neq 0, 1, q$ then $P = (2,3)$ is the only possible case. Let $Q = (s,t)$ be any of the two remaining vertices. $5 = 1 + \frac{4}{2} + area$ so $area = A_1 + A_2 = \frac{3}{2} + A_2 = 2$ and $A_2 = \frac{1}{2}$.

The same analysis as in $4B2$? shows that $\frac{3s-t-3}{2} = \frac{1}{2}$ so $3s - t = 4$. By

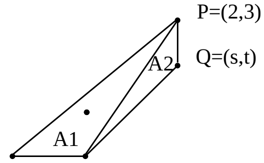


FIGURE 3.24.

convexity $\frac{t}{s} < \frac{3}{2}$ i.e. $t < 4$ and $s \leq 2$ we conclude that $Q = (2,3)$ is the only possibility for the two remaining vertices which is a contradiction.

(B) The triangle $(0,0),(1,0),(p,q)$ contains one more vertex on the edge, so $4 = 1 + \frac{4}{2} + area$ we conclude that $area = \frac{q}{2} = 1 \Rightarrow q = 2$ so $P = (1,2)$.

Let $Q = (s,t)$ be any one of the two other vertices.

Applying Pick's formula for the polytope $ORPQ$ we get

$5 = 1 + \frac{4}{2} + area$ so $area = A_1 + A_2 = 1 + A_2 = 2$ and $A_2 = \frac{2(s-1)}{2}$. We get $s = 2$, and since $\frac{t}{s} < 2$ we have $t < 4$.

We find 3 possibilities for the last two vertices Q and L, $\{(2,3),(2,2),(2,1)\}$ and two set of pairs $\{(2,3),(2,2)\}$, $\{2,2),(2,1)\}$ give two distinct ones that are equivalent.

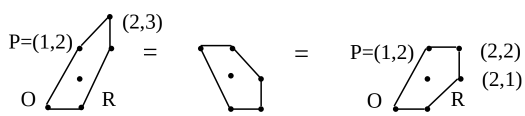


FIGURE 3.25.

(C) The triangle O, P, R contains no other integral point. $4 = 1 + \frac{4}{2} + area \Rightarrow$ $area = \frac{p}{2} = 1$.

After shifting we reduce to the case when $P = (0,1)$.

By convexity the polytope contains the point $(1,1)$. Let $Q = (s,t)$ be any other vertex. If s and t are > 1 then $5 = 1 + \frac{4}{2} + area$ so $1 + \frac{p-1+q-1}{2} = 2$ and therefore $p + q = 4$ so in this case we get the point $Q = (2,2)$. If one coordinates is 1 then PQ encloses three points so $5 = 1 + \frac{5}{2} + area$. We conclude $area = 1 + \frac{s-1}{2} = \frac{3}{2}$ and we obtain two other possibilities for Q $(2,1),(1,2)$. We obtain the same polytopes as in the previous case.

(vi) $n = 6$. Consider the triangle OPR so $3 = 1 + \frac{3}{2} + area$ and $P = (0,1)$. If $Q = (s,t)$ is any other vertex than $OPQR$ has no interior point, therefore $4 = 1 + \frac{4}{2} + area$ so $area = \frac{2st}{2} = st = 1$ therefore $Q = (1,1)$. We get the same value for three remaining vertices which is a contradiction.

We've just proved the following result

Proposition 3.3.2. *Any polytope enclosing six lattice points is equivalent to exactly one from the following list*

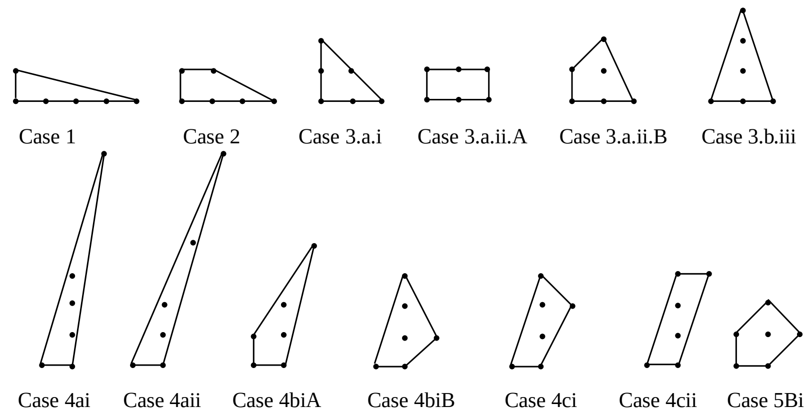


FIGURE 3.26.

We now recall that any rational convex polytope $\mathcal{P}$ in $\mathbb{R}^n$ enclosing a fixed number of integer lattice points defines an n dimensional projective toric variety $X_{\mathcal{P}}$ endowed with an ample line bundle on $X_{\mathcal{P}}$ which has the integer points of the polytope as sections. We get the following result

Corollary 3.3.3. *Any toric surface endowed with an ample line bundle with six sections is completely described by exactly one of the polytopes from the above list.*

4. Triple Point Analysis.

We first observe that six, the number of integer points enclosed by the polytope, represents exactly the number of conditions imposed by the a triple point. We will now classify all polytopes from Proposition 3.3.2 for which their corresponding linear system becomes empty when imposing a triple point. There are two methods for testing the emptiness of these linear systems: an algebraic method and a geometric method. In our case, in order to be efficient and at the same time geometric we will use both of them, so we will briefly describe them below. For the algebraic approach, checking that a linear system is non-empty when imposing a triple point reduces to showing that the conditions imposed by a triple point in $\mathbb{P}^2$ are dependent. For this, one needs to look at the rank of a six by six matrix where the first column represents the sections of the line bundle and the other five columns represent all first and second derivatives in x and y. We conclude that the six conditions are dependent if and only if the matrix doesn't have maximum rank. In order to give a complete classification, we will use a quick algebraic remark to eliminate the non useful polytopes, and a geometric argument to illustrate the emptiness. The geometric method for testing when a planar linear system is empty is to explicitly find it and show that it contains no curve, using $\mathbb{P}^2$ as a minimal model for the surface X and writing its resolution of singularities. More explicitly we consider the projective toric variety X to be the blow up of $\mathbb{P}^2$. We give geometric conditions for its emptiness illustrating our computations by one example, and we obtain five surfaces that pass the emptiness test.

Remark 3.4.1. *The corresponding linear systems of the following polytopes are non-empty when imposing a base point with multiplicity three.*

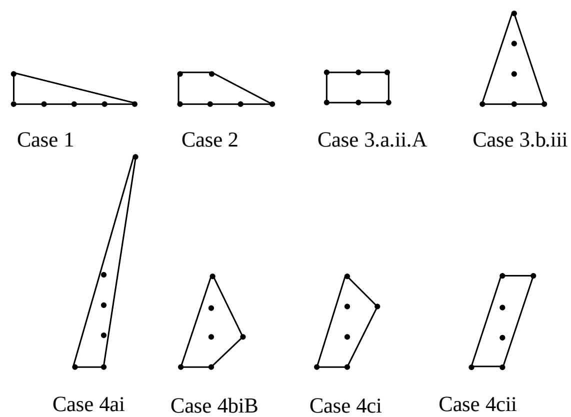


FIGURE 3.27.

PROOF. It is easy to check that the algebraic conditions imposed by at least four sections on a line are always dependent. Indeed, we have two possible cases, if the line of sections is an edge, or if is enclosed by the polytope. For the first case, we can only have sections on two levels so the vanishing of the second derivative in y gives a dependent condition (The same argument applies for Case 3.a.ii.A representing the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$). For the second case we notice that the vanishing of the first derivative in y and the second derivative in x and y give two linearly dependent conditions. $\square$

We will use the Remark 3.4.1 to eliminate the polytopes that don't have the desired property and we now obtain five polytopes for which we will study the corresponding algebraic surfaces and linear systems using toric geometry methods.

We will illustrate these methods by considering the toric surface described by the polytope from Case $4aii$ (see [**11**] Chapter 2). It's fan obtained by dualizing the polytope's angles, consists of three cones generated by rays B, F and J. In this case, each cone represents a singular open subset, so we conclude that the corresponding toric variety obtained by gluing them together is singular. We will consider it's resolution of singularity, by blowing up the singular points. In toric geometry, desingularizing varieties corresponds to a subdivision of the fan such that each cone is a nonsingular open subset. In our example, we subdivide the fan by introducing nine vectors; we will denote all generating vectors with letters from $A, .., L$ (see the picture below).

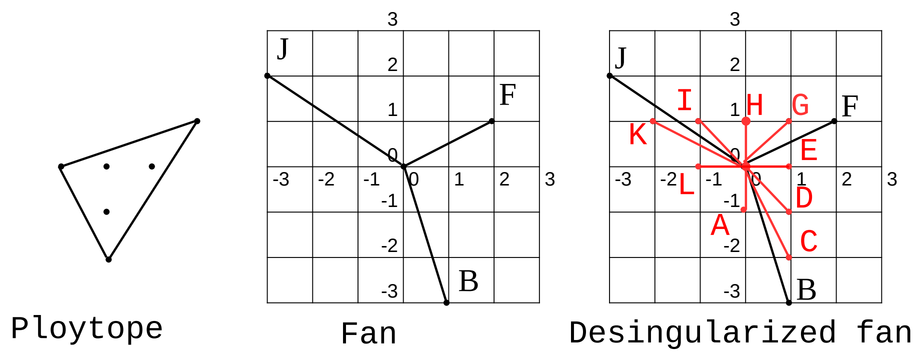


FIGURE 3.28.

Any two adjoint cones represent affine planes that are glued together and the rays become curves that meet in a cycle in the toric variety. In the picture, we illustrate the geometry of the toric variety by specifying the cycle of curves as well as their selfintersection. We will describe the

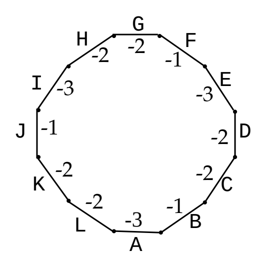


FIGURE 3.29.

toric surface S by considering $\mathbb{P}^2$ as the minimal model, after contracting the set of curves $\{F, J, B\}$, $\{C, K, G\}$, $\{H, L\}$ and D. Denoting by π the compositions of all blow downs we get that the line class on S is $\mathcal{L} = \pi^{-1}(I) = \pi^{-1}(E) = \pi^{-1}(A)$ so we get the that following divisors are congruent $3J+2K+L+H+G+F$, $E+F+H+2G+2F+D+C+B$, $A+B+M+L+J+D+2C+2B$. Furthermore, the divisor that describes the embedding is $C+2D+3F+7F+5G+3H+I$ that corresponds to $4\mathcal{L}-(3B+2C+D)-(3F+2G+H)-(3J+2K+L)$ that represents quartics with three base points that are flex to the line joining any two $\mathcal{L}_4([1,1,1]^3)$. In general, we will use the notation $\mathcal{L}_d([1,1]), \mathcal{L}_d([2,1]), \mathcal{L}_d([1,1,1])$ for linear systems of degree d that pass through a base point with a defined tangent, a double point with a defined tangent or having a flex direction. Since the base points are special, we will analyze the linear system separately. We will blow up at one of the base points and use an elementary transformation at a different base point to get to $\mathbb{P}^1 \times \mathbb{P}^1$. By blowing up the two base points we will get a linear system with two tangent condition while by contracting the line joining them will transform the flex condition to the contracted line into a cusp as the picture indicates. The linear system on $\mathbb{P}^1 \times \mathbb{P}^1$ has now bidegree $(3, 3)$. We will now impose

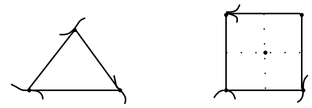


FIGURE 3.30.

a triple point at a general point of $\mathbb{F}_0$, and using a set of transformation similar to the one described above but applied to a different set of base points we return to $\mathbb{P}^2$.

Furthermore there is a correspondence between linear systems on the two surfaces (see [**4**])

$$\mathcal{L}_{a,b}(m) \cong \mathcal{L}_{a+b-m}(a-m, b-m)$$

We apply this for $a = b = m = 3$ and we get a linear system on the projective plane given by cubics $\mathcal{L}_3([2, 1, 1], [1, 1], [1, 1, 1])$. Finally we can consider that the tangency condition of the double point and of the flex point are general and it will facilitate the analysis

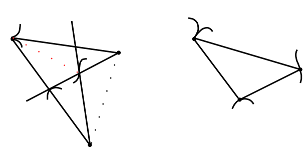


FIGURE 3.31.

The linear system has now degree three and after performing a Cremona transformation we reduce it to $\mathcal{L}_2([1, 1]^3)$ that is empty since there are no conics passing through three points with three given tangents.

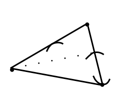


FIGURE 3.32.

We just proved that $\mathcal{L}_4(3, [1, 1, 1]^3)$ is empty. In the same way we analyze the other four linear systems described by the four remaining polytopes and we conclude the emptiness applying birational transformations and splitting off -1 curves and we obtain the following result:

Lemma 3.4.2. *The linear systems corresponding to the following polytopes become empty after imposing a triple point.*

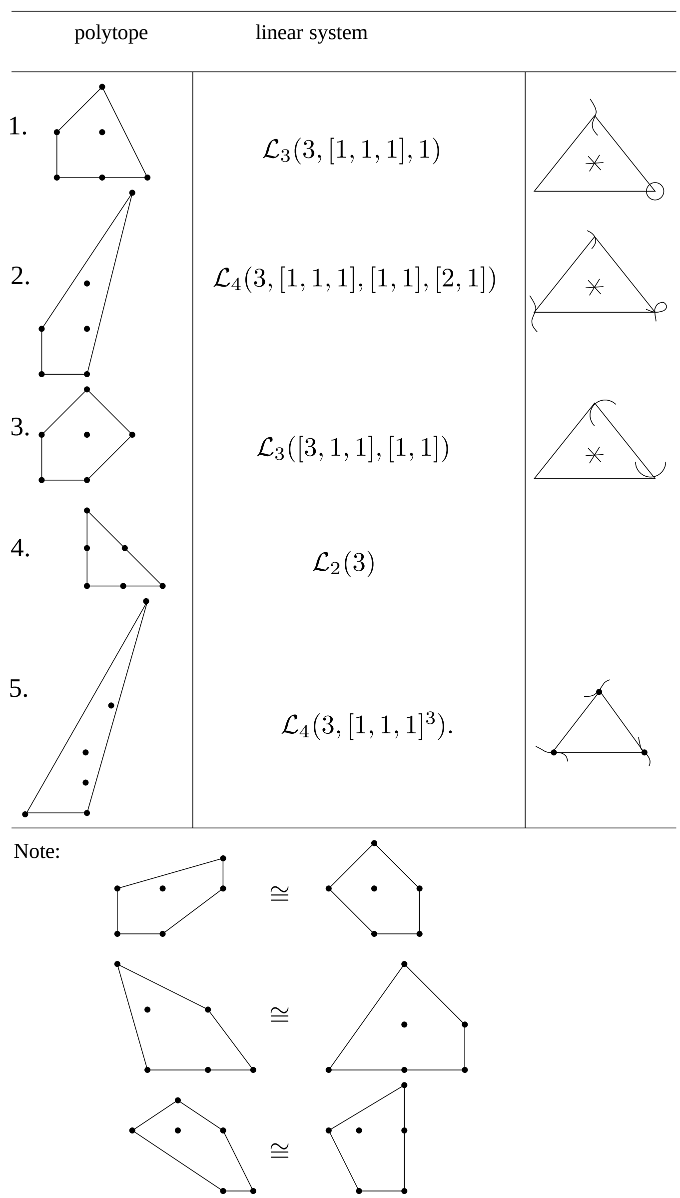


FIGURE 3.33.

One can obtain more polytopes with an empty linear system by rotating or by shifting the main ones by any integer numbers since this won't change the linear system or the surface.

Even though all the linear systems were found from the resolution of singularities of the associated surface we may observe connections with the toric geometry. We predict that the linear system could be detected from the shape of the polygon. Because these surfaces are all toric the plane $\mathbb{P}^2$ is blown up at at most three general points. For example, the first polygon below represents a blown up $\mathbb{P}^2$ embedded by a linear system of degree 3. The removed parts of the embedded plane represent the conditions imposed by the base points of the linear system. In the first picture, we have two general base points, and removing the three sections on a line represents a flex condition imposed by one the points; we can see that it matches the system from the table above. We also remark that the linear system that gives the embedding may not be unique; indeed, we may have different polygons associated to the same surface. For instance, in the example below the second picture represents a $\mathbb{P}^2$ embedded by a linear system of cubics, with two base points and two tangent conditions. We believe that the two linear systems (in this case of the same degree) are equivalent, being connected by a birational transformations of the plane.

FIGURE 3.34.

For the second case, the first polygon represents a blown up $\mathbb{P}^2$ embedded by quartics with three base points; we can see a double point with a fixed tangent, a tangent and a flex point; while the second one can be easily identified with a plane embedded by $\mathcal{L}_4(2,[1],[1,1],[1,1,1])$

FIGURE 3.35.

In the first picture below one can observe a $\mathbb{P}^2$ embedded by cubics with two general tangent base points while the second one should represent an equivalent linear system. We remark that the degree or the number of base points of the systems might not be the same.

FIGURE 3.36.

We also note that the polygon might not necessarily be in standard position.

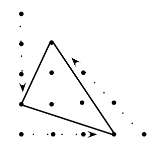


FIGURE 3.37.

From these observations, we predict that Case 1. is a degeneration of Case 3. (since the two tangent base points are in special position) and Case 2. is a degeneration of Case 5; so the three non-degenerated cases correspond to the projective plane embedded by conics, cubics or quadrics with corresponding base points.

Linear systems with six sections and a triple point that become nonempty since curves split out. The base points are not in general position so the linear system consists only of a fixed part. It would be interesting to explain why dependent conditions on the sections level correspond to the splitting of fixed curves. Below we give one example. The linear system that describes the embedding of the

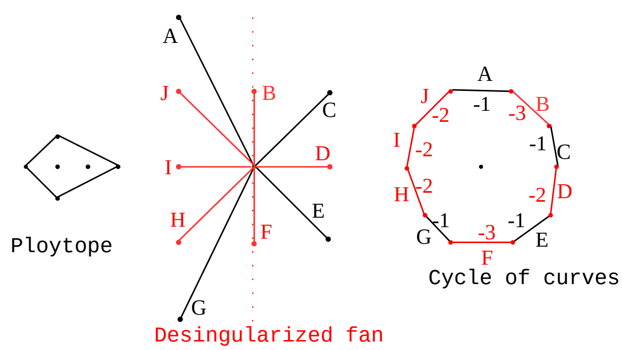


FIGURE 3.38.

plane is $4\mathcal{L} - (2B + 6C + 3D) - (3H + 2I + J) - F$ that represents quartics with a cusp, a flex point and a simple point, in some special position. It is easy to see that $\mathcal{L}_4(3, [2, 1, 1], [1, 1, 1], 1)$ is special, consisting of $B, \bar{G}$ and $2L$. We can also read the linear system from the shape of the polytope.

5. Triple Points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In this section we will present how we can use the results from section [**?**] to the triple point interpolation problem in $\mathbb{P}^1 \times \mathbb{P}^1$. We will only prove the most difficult case when the linear systems in $\mathbb{P}^1 \times \mathbb{P}^1$ with virtual dimension -1 are empty, the general case will follow by induction, but it was already proved in a similar way using algebraic methods by T. Lenarcik in [**16**]. We will use some of

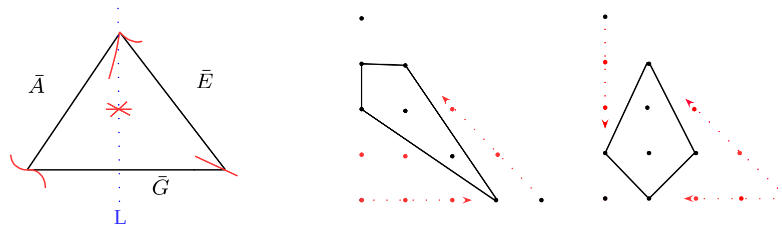


FIGURE 3.39.

this results for the induction argument in the $\mathbb{P}^2$ case. Using semicontinuity arguments it is enough to prove the results for linear systems with virtual dimension -1 and we conclude the emptiness since the union of all the surfaces span a maximal dimensional space (see [**4**]).

Lemma 3.5.1. *Linear systems of bidegree* $(5, n), (11, n), (2, 4n + 3), (8, 2n + 1)$ *and an arbitrary number of triple points have the expected dimension, for* $n \geq 2$.

PROOF.

- For any linear systems of bidegree $(5, n)$ we find a skew $n+1$ set of surfaces and we place each of the $n+1$ triple points in one of the surfaces. We denote the degenerations presented below as C_5^5, C_5^6, C_5^8 and C_5^3 For every $n > 2$ take $i \in \{3, 5, 6, 8\}$ such that $\frac{n-i}{4}$

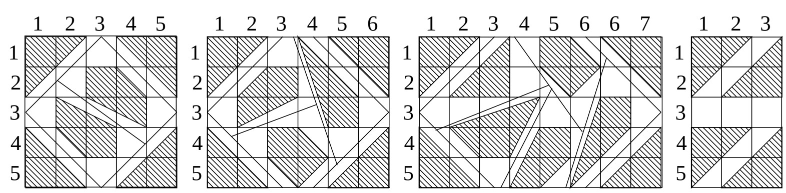


FIGURE 3.40.

  is an integer, k. For any arbitrary n we consider the degeneration $C_5^n = C_5^i + kC_5^3$.
- For linear systems of bidegree $(11, n)$ and n triple points we find a skew $2n + 2$. We denote the degenerations presented below by C_{11}^2, C_{11}^3, and C_{11}^4. For every $n > 2$ take $i \in \{2, 3, 4\}$ such that $\frac{n-i}{3}$ is an integer, k. For any arbitrary n we consider the degeneration $C_{11}^n = C_{11}^i + kC_{11}^2$.
- For curves of bidegree $(2, 4n + 3)$ we consider the degeneration C_2^{4n+3} given by $(n + 1)C_2^3$ (in particular, $C_2^{11} = 3C_2^3$) and for C_8^{2n+1} we use combinations of C_8^3 and C_8^5

□

Corollary 3.5.2. *Linear systems in* $\mathbb{P}^1 \times \mathbb{P}^1$ *with triple points of virtual dimension* -1 *are empty.*

PROOF. We have to prove the statement for linear systems of bidegree $(6k - 1, n)$ and $(3k - 1, 2n - 1)$. We distinguish two cases if k is even $k = 2k'$ we use the degeneration $C_{12k'-1}^n = k'C_{11}^n$;

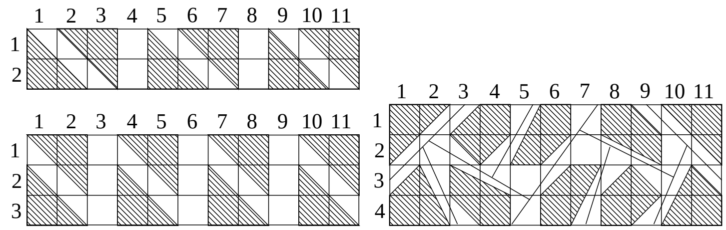


FIGURE 3.41.

while if k is odd of the form $2k'+1$ we use $C^n_{12k'+5} = C^n_5 + k'C^n_{11}$, for $n \neq 4$. For $n = 4$ we use the following degeneration for C^4_{17} and we generalize this case by adding C^4_{11} blocks For the

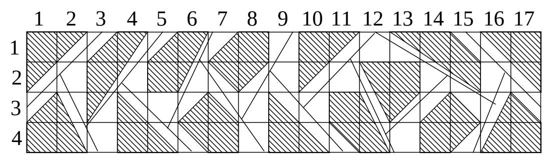


FIGURE 3.42.

bidegree $(3k-1, 2n-1)$ we reduce to the case when k is odd of the form $2k'+1$ and depending on the parity of k', if $6k' = 6+12r$ we use the degeneration $C^{2n-1}_8 + rC^{2n-1}_{11}$ while $6k' = 12r$ we use $C^{2n-1}_5 + C^{2n-1}_8 + (r-1)C^{2n-1}_{11}$ and we put $2n$ points in the first block and $3n$ and $2(r-1)2n$ respectively. $\square$

Remark 3.5.3. *The theorem doesn't hold for* $\mathcal{L}_{(5,4)}(3^5)$ *and for* $\mathcal{L}_{(4k+1,2)}(3^{2k+1})$ *that are nonempty although they have virtual* -1*. Indeed, the first one is cremona equivalent to the planar linear system* $\mathcal{L}_3(1,2,3^4)$ *that is nonempty consisting of a degenerated conic and a line; while the second one has as a fixed divisor* $2k$ *lines and a curve equivalent to the double line* $\mathcal{L}_2(-1,0,2^2)$.

6. Triple points in $\mathbb{P}^2$.

We denote by V_d the image of the Veronese embedding $v_d : \mathbb{P}^2 \to \mathbb{P}^{d(d+3)/2}$ that transforms the plane curves of degree d to hyperplane sections of the Veronese variety V_d. We degenerate V_d into a union of disjoint special surfaces and ordinary planes and we place one point on each one of the disjoint surfaces. The surfaces are chosen such that the restriction of a hyperplane section to each one of them to be linear system that becomes empty when we impose a triple point. We conclude that any hyperplane section to V_d needs to contain all disjoint surfaces, and in particular all of the coordinate points of the ambient projective space covered in this way. Therefore if V_d degenerates exactly into a union of disjoint special surfaces and planes (or quadrics) with no points left over we conclude that the desired linear system is empty.

Theorem 3.6.1. *$\mathcal{L}_d(3^n)$ has the expected dimension whenever $d \geq 5$.*

Proof. Enough to prove the theorem for the number of triple points for which the virtual is -1 so in that case we claim that the linear system is empty. An easy computation shows that $\binom{d+2}{2} \equiv 0 \bmod 6$ if $d \equiv \{2, 7, 10, 11\} \bmod 6$; $\binom{d+2}{2} \equiv 1 \bmod 6$ if $d \equiv \{0, 9\} \bmod 6$; $\binom{d+2}{2} \equiv 3 \bmod 6$ if $d \equiv \{1, 4, 5, 8\} \bmod 6$ and $\binom{d+2}{2} \equiv 4 \bmod 6$ if $d \equiv \{3, 6\} \bmod 6$.

We will use the induction step $V_{12(k+1)+j} = V_{12k+j} + kC^{11}_{11} + C^{11}_{j+1} + V_{10}$ with $j = 1, ..., 12$, $k \geq 0$, $(i, j) \neq (1, 4)$ and to finish the proof we present the degenerations of V_j if $j \leq 12$.

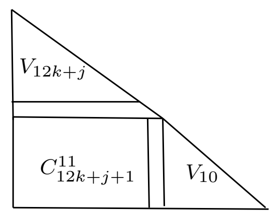


Figure 3.43.

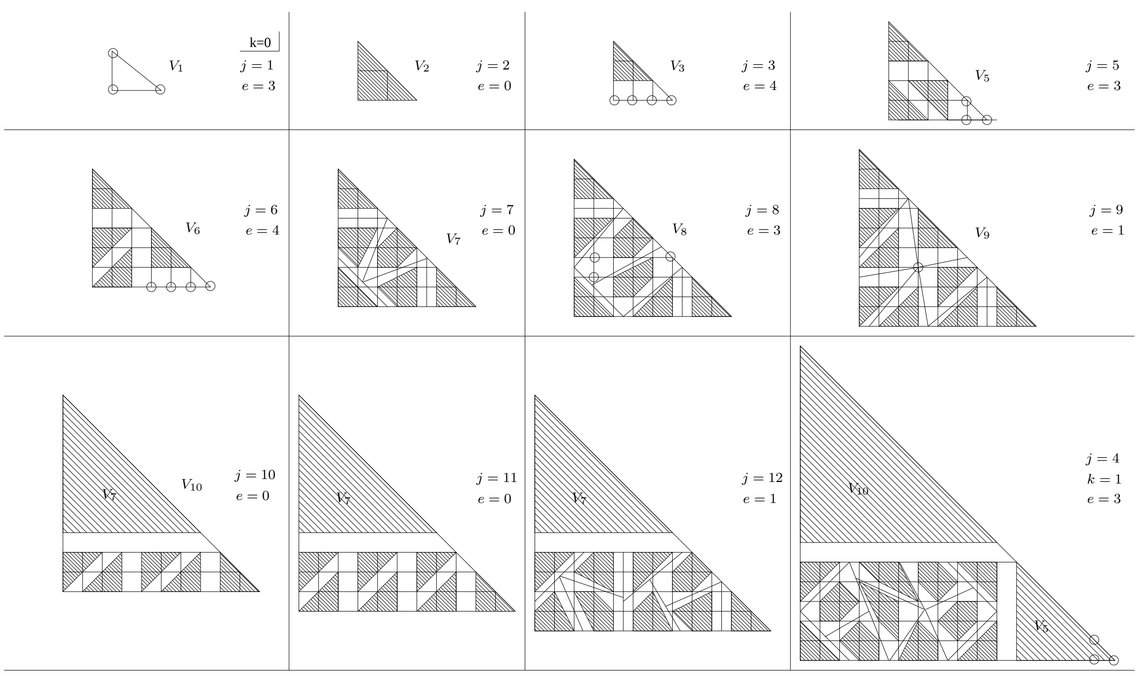


Figure 3.44.

$\square$

Remark 3.6.2. Notice that $\mathcal{L}_4(3^2)$ consists of quartics with two triple points and the expected dimension is 2. This linear system has a fixed part, the double line through the two points and a movable part $\mathcal{L}_2(1^2)$ i.e. conics through two points, that has dimension 3. A simple argument shows that if $d = 4$, the linear system $\mathcal{L}$ is -1–special (we have a -1–curve, line connecting the 2 points, splitting off twice) and therefore special.

One could mention that case $d = 4$ is also a special case for the double points interpolation problem. Is not hard to see that even if $\mathcal{L}_4(2^5)$ has a negative virtual dimension, it is a -1–special system and therefore nonempty. Indeed, $\mathcal{L}_4(2^5)$ consists of the double conic determined by the 5 general points, so for $d = 4$, $m = 2$ Theorem 1.3.2 doesn't hold.

CHAPTER 4

The Emptiness of the Linear System: $\mathcal{L}_d(m^{10})$

1. Nagata's Conjecture and General Results.

Fix general points in the projective plane and multiplicities $m_1, ..., m_n$. We will denote by $\mathcal{L}_d(m_1^{s_1}, ..., m_n^{s_n})$ to be the linear system of plane curves of degree d having multiplicities at least m_i at s_i of the general points. For the homogeneous case, the linear system $\mathcal{L}_d(m^n)$ has the expected dimension

$$e(\mathcal{L}_d(m^n)) = max\{-1, \frac{d(d+3)}{2} - \frac{nm(m+1)}{2}\}.$$

Nagata's conjecture for 10 points states that if $\frac{d}{m} < \sqrt{10} \approx 3.1622$ then $\mathcal{L}_d(m^{10})$ is empty. In 2004 Harbourne and Roé [**13**] proved that if $\frac{d}{m} < 177/56 \approx 3.071$ then $\mathcal{L}_d(m^{10})$ is empty. Subsequently, in 2008 Dumnicki [**9**] (see also [**1**]) found a better limit $313/99 \approx 3.161616$ combining algebraic arguments with methods developed by Ciliberto-Miranda [**6**] and Harbourne-Roé. The aim of this paper is to present and develop a method for analyzing the emptiness of $\mathcal{L}_d(m^{10})$. We prove that $\mathcal{L}_d(m^{10})$ is empty if $\frac{d}{m} < \frac{117}{37} \approx 3.162162$ by using a degeneration of the plane into a union of nine surfaces. Using the same degeneration of the plane Ciliberto and Miranda proved the non-speciality of $\mathcal{L}_d(m^{10})$ for $\frac{d}{m} \geq \frac{174}{55}$ and, as remarked in that article, one obtains as a consequence the emptiness of $\mathcal{L}_d(m^{10})$ for $\frac{d}{m} < \frac{550}{174} \approx 3.1609$ (see [**8**]).

We remark that our emptiness result implies that the corresponding Seshadri constant for ten points in the plane is at least 117/370; see [**13**].

We will construct a family of planes X_t degenerating to a union of nine surfaces in the central fiber X_0, and Proposition 4.4.1 will give us all possible limits $\mathcal{L}_0$ of the line bundle $\mathcal{L}_d(m^{10})$ on the central fiber of the family X. Such a limit line bundle is a line bundle on each surface, which agree on all of the double curves of the degeneration. We will say that a line bundle on X_0 is *centrally effective* if each individual surface line bundle is effective.

If $\mathcal{L} = \mathcal{L}_d(m^{10})$ is nonempty, then there is a curve in the restriction of $\mathcal{L}$ to the general fiber X_t, so there is a limit curve in the central fiber X_0 as well, and therefore there is a limit line bundle $\mathcal{L}_0$ associated to that limit curve. Since X_0 is the union of surfaces, if $\mathcal{L}_d(m^{10}) \neq \oslash$ we then conclude that this limit line bundle must be centrally effective.

Conversely, suppose that, for a fixed ratio of d/m, one can prove that for any limit line bundle $\mathcal{L}_0$, at least one of the restrictions of $\mathcal{L}_0$ to the surfaces from the central fiber is empty. In other words, suppose that for this fixed ratio, there is no centrally effective limit line bundle on X_0. We conclude that there cannot be a limit curve in $\mathcal{L}_0$, so there is no curve in the restriction to the general fibre X_t as well. Therefore $\mathcal{L}_d(m^{10})$ is empty. In this article we will exploit this centrally effective argument, by constructing the desired degeneration of the plane, and then considering all the possible degenerations $\mathcal{L}_0$ of the bundle $\mathcal{L}_d(m^{10})$.

Everywhere in this article we make the assumption that $\frac{d}{m} < \sqrt{10}$ since we are analyzing the emptiness of $\mathcal{L}_d(m^{10})$. We will briefly introduce the degeneration of the plane into a union of nine surfaces, the geometry of each surface and the notations that we use; for more details the interested reader is encouraged to consult [**8**].

In [**5**] we present the same result, obtained by a shorter observation and using previous results obtained in [**8**]. There we argue that for the emptiness purpose is enough to follow the bundles from the first degeneration. Indeed, we use Remark 4.5.2 to restrict to the case when we have the sharpest bound for emptiness. For every twist, we will only analyze the corresponding ones and we call them extremal bundles.

Moreover, every time we make a 2 throw we introduce two infinitely points $[a_1, b_1], [a_2, b_2]$. The identification of the two curves gives us that $b_1 = b_2 = b$ and the sharpest conditions on the plane (usually denoted by T) gives that $a_1 = a_2 = a$, so with every 2 throw we introduce only two parameters $[a, b]$. We conclude we have the matching conditions from the previous degeneration and two more:

1.) the linear system doesn't intersect the curve we throw

2.) the extremal bundle condition gives that $a = b$.

The second one is obviously independent, while the first one is independent since the curve splits out initially. So with every throw we introduce two parameters and two independent conditions and therefore we conclude the bundle should depend on the same number of parameters as the previous one. By induction we find that in fact the bundle depends on only one parameter, and we claim that this is the one obtained from the first degeneration. Indeed, since it is obvious that this is one of the limit bundles, and furthermore, we have a one dimensional family of line bundles, so for some appropriate value of the parameter we get our favorite one.

This work is more complex, it does not use other results and it gives a complete analysis of the degenerations of the plane; degenerations of the linear systems and a general emptiness analysis of non-homogeneous linear systems with various number of points in special position. We don't regard

this analysis as a consequence of [**8**] but rather as a continuation.
Moreover this method can be applied to the analysis of the emptiness of any linear system, however the only difficulty that one runs into is finding the matching conditions of an n throw; this problem was analyzed by Michele Nesci in his thesis (see [**20**]).

2. The First Degeneration.

Consider $X \to \Delta$ the family obtained by taking the trivial family over a disc $\Delta \times \mathbb{P}^2 \to \Delta$ and blowing up a point in the central fiber. The general fibre X_t for $t \neq 0$ is a $\mathbb{P}^2$, and the central fibre X_0 is the union of two surfaces $V \cup Z$, where $V \cong \mathbb{P}^2$ is a projective plane, $Z \cong \mathbb{F}_1$ is a plane blown up at a point, and V and Z meet along a rational curve E which is the negative section on Z and a line on V (see Figure 4.2).

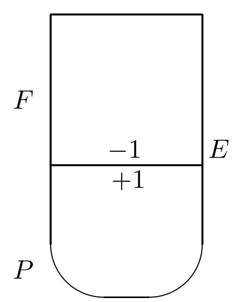


FIGURE 4.1. the degeneration of the plane

We now choose four general points on V and six general points on P. We consider these ten points as limits of ten general points in the general fibre X_t and we blow these points up in the family X. This creates ten surfaces R_i, whose intersection with each fiber X_t is a (-1)-curve, the exceptional curve for the blow–up of that point in the family. We notice that the general fibre X_t of the new family is a plane blown up at ten general points. The central fibre X_0 is the union of V, a plane blown up at four general points, and Z, a plane blown up at seven general points.

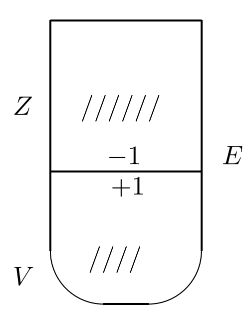


FIGURE 4.2. the degeneration of the blown–up plane

The general fibre X_t for $t \neq 0$ is a plane blown up at ten general points. The central fibre X_0, is the union $V \cup Z$ where:

- V is a plane blown up at four general points;
- Z is a plane blown up at seven general points;
- V and Z meet transversally along a smooth rational curve E which is a (-1)-curve on Z, whereas $E^2 = 1$ on V (it is a line).

Consider the line bundle $\mathcal{L}_0 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{L}_X(-\sum_i mR_i)$, where $\pi : X \to \mathbb{P}^2$ is the natural map. This restricts to $\mathcal{L}_d(m^{10})$ on the general fibre, whereas on the central fibre it is $\mathcal{L}_0(m^4)$on V and $\mathcal{L}_d(0, m^6)$ on Z. (the first base point of the linear system $\mathcal{L}_d(a, m^6)$ denotes the multiplicity of the negative section of $\mathbb{F}_1$, while the other six represent the six blown up points of the ruled surface $\mathbb{F}_1$.)

We will consider all the possible twistings of $\mathcal{L}_0$ by a multiple of Z. Namely, we choose a parameter a, and define

$$\mathcal{L} := \mathcal{L}_0 \otimes \mathcal{O}_X(aZ).$$

We will denote by $\mathcal{L}_V$ and $\mathcal{L}_Z$ the restrictions of $\mathcal{L}$ to V and Z; these bundles have the form

$$\mathcal{L}_V = \mathcal{L}_a(m^4), \quad \mathcal{L}_Z = \mathcal{L}_d(a, m^6).$$

3. The second degeneration.

We will consider the case when the (-1)-curve, in our case the cubic $C \in \mathcal{L}_3(2, 1^6)$, meets the double curve E in two points p_1 and p_2. We assume that C lies on the component V and that the restricted system $\mathcal{L}_V$ has the property that $\mathcal{L}_V \cdot C = -k < 0$. Blow up C, obtaining the ruled surface T, which is isomorphic to $\mathbb{F}_1$; T meets V along C, and this is also the negative section of T. The blow–up will create on the surface Z two exceptional divisors G_1 and G_2. These G_i are also fibers of the ruling of T.

Now blow up C again, creating the ruled surface S. This time $S \cong \mathbb{P}^1 \times \mathbb{P}^1$; S meets V along C, and it meets T along the negative section. The blow–up effects the blow–up surface Z, creating two more exceptional divisors F_1 and F_2 which are (-1) curves on Z. By abusing notation we denote by G_1, G_2 their proper transforms that are now (-2)–curves. The surface S now occurs with multiplicity two in the central fiber of the degeneration, since it was obtained by blowing up a double curve.

We may now blow S down the other way. This contracts C on the surface V, and contracts the negative section of T, so that T becomes a $\mathbb{P}^2$ (by abusing notation, we still denote by T its image after the contraction of S). The image of the surface Z has the two curves F_1 and F_2 identified. We

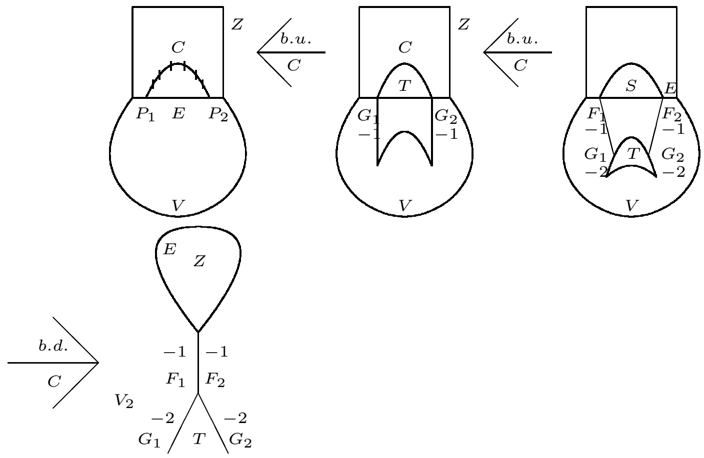


FIGURE 4.3.

introduced two pairs of infinitely near points and we denote assigning multiplicities to each pair by $[a, b]$, indicating a multiple point a and an infinitely near multiple point b, namely $-a(F_i+G_i)-bF_i$. Also note that $F_i + G_i$ is also a curve with self-intersection -1.

The bundle on Z can be interpreted in the geometry of Z where two new compound multiple points have been created, two pairs of infinitely near points that we will denote by $[m_1, m_2]$, indicating a multiple point m_1 and an infinitely near multiple point m_2.

We refer to this operation as a 2-*throw* (of C on V).

Note that in a 2–throw, if the two points p_1 and p_2 lie on the same component of the double curve E, then the curve E becomes a nodal curve, and the construction results in a non–normal component of the degeneration, because of the identification of F_1 and F_2. However this presents no real problems in the analysis; the central fiber, all linear system computations on components are done on their normalizations.

In our case we will blow up the cubic $\mathcal{L}_3(2, 1^6)$ twice, analyze the four bundles and then contract S. Consider all the possible bundles on the four surfaces V, T, S, and Z, such that the limit bundle is $\mathcal{L}_d(m^{10})$. The bundle on V is at the form $\mathcal{L}_\delta(m^4, [a, b], [a, b])$ where δ, a, b are parameters.

We will write down directly the matching conditions for the bundles on the three surfaces V, Z and T (with the cubic contracted to a point) and we will use them for the matching conditions for the third one.

From the beginning we will consider three surfaces T, V, Z where T is just a plane, V is a plane blown up 8 times, twice infinitely near, and Z is a plane blown up six times, and three general bundles on them. We get that the surface V has the four multiplicities equal to m. Because we started with three surfaces we expect to find two free parameters-so choose a_i, b_i and q_i, considering again d and m arbitrary but fixed. We have the general form of the three bundles

- $\mathcal{L}_{\mathcal{V}} = \mathcal{L}_{\deg V}(m^4, [a_1, b_1], [a_2, b_2])$
- $\mathcal{L}_{\mathcal{Z}} = \mathcal{L}_{\deg Z}(q_1, ..., q_6)$
- $\mathcal{L}_{\mathcal{T}} = \mathcal{L}_{\deg T}$

We have four matching conditions:

(1) $V \bigcap T$. Consider the intersection of S and Z that is a fiber on S and the cubic on Z. We start with a bundle on Z that does not meet the cubic, and since $\mathcal{L}_{\mathcal{Z}}$ and $\mathcal{L}_{\mathcal{S}}$ agree on the double curve we get that $\mathcal{L}_{\mathcal{S}}$ is a horizontal bundle. On the other hand, the intersection of S with V forces $\mathcal{L}_{\mathcal{S}}$ to have bidegree $(b, 0)$ (since is also horizontal). $\mathcal{L}_{\mathcal{T}}$ meets a fiber $a - b$ times (since it has to agree with V) and meets the negative section B 0 times (since the bundle on S is horizontal). $\mathcal{L}_V G_i = \mathcal{L}_T \mathcal{L}_1$. We get

$$\deg T = b_1 - a_1 = b_2 - a_2.$$

(2) Since the two curves F_1 and F_2 are identified we obtain $b_1 = b_2 = b$ and from (1) $a_1 = a_2 = a$.

(3) Multiplicity on the surface Z. We notice that the multiplicity on the linear system before contracting the cubic becomes a line in the new one (Cremona $167 - 123 - 145$)

0	0	0	0	0	0	0	-1
1	1	0	0	0	0	1	0
1	1	0	0	0	0	1	0
1	1	0	0	0	0	1	0

TABLE 1.

We get that $q_1 = q_2 = ... = q_6$ since

$$m - (a + b) = \mathcal{L}_Z \mathcal{L}_1(1) = \deg Z - q_i.$$

(4) $Z \bigcap V$. V still intersects Z along $E = \mathcal{L}_1([1, 1]^2)$. On Z though, we need to find the image of B after we contract the cubic. (Cremona $167 - 123 - 145$)

0	-1	0	0	0	0	0	0
1	0	1	1	0	0	0	0
2	1	1	1	1	1	0	0
3	2	1	1	1	1	1	1

TABLE 2.

The condition is $\mathcal{L}_Z\mathcal{L}_3(1^6) = \mathcal{L}_V\mathcal{L}_1([1,1]^2)$ so we obtain

$$3\deg Z - 6q = \deg V - 2(a+b).$$

(5) Degree. It is easy to see that by pulling back a line we get a line on V, a fiber on Z, a line on T and two fibers on S. After contracting the cubic the fiber on Z is mapped to a general line. Indeed, before we had a fiber on Z, of the form $\mathcal{L}_1(1,0^6)$ that has the same form after we contract the cubic, and since the first multiplicity represents a contracted cubic we are left with a general line $\mathcal{L}_1$.

$$d = \deg V + \deg Z + a + b$$

Solving these equations with parameters a and b we obtain the following result

Proposition 4.3.1. *If we fix d and m, then all limits of the bundle $\mathcal{L}_d(m^{10})$ are of the following form for some a and b*

- $\mathcal{L}_{\mathcal{V}} = \mathcal{L}_{\frac{3d}{2}-3m+\frac{a+b}{2}}(m^4, [a,b], [a,b])$
- $\mathcal{L}_{\mathcal{Z}} = \mathcal{L}_{3m-\frac{d}{2}-\frac{3(a+b)}{2}}((2m-\frac{d}{2}-\frac{a+b}{2})^6)$
- $\mathcal{L}_{\mathcal{T}} = \mathcal{L}_{a-b}$

4. The third degeneration.

For the third degeneration we will perform a sequence of 2–throws:

(1) Six disjoint curves, two conics $\mathcal{L}_2(1^4, [1,0], [0,0])$ and four quadrics $\mathcal{L}_4(2^3, 1, [1,1]^2)$ on V .

By executing the six 2–throws we introduce six planes that we will denote by U_1, U_2; and $Y_1, ..., Y_4$ respectively. We now explain how the geometry of all the surfaces changes after these throws. By blowing up twice the four quartics Q_j and contracting, V becomes more complicated with 16 additional blow ups, eight of them infinitely near. Overall we have nine surfaces and the general form of the line bundles on them is: On the surface V we denote by Q_i the four disjoint quartics $\mathcal{L}_4(2^3, 1, [1,1]^2)$ and by C_1 and C_2 the two disjoint conics $\mathcal{L}_2(1^4, [1,0], [0,0])$ and $\mathcal{L}_2(1^4, [0,0], [1,0])$. We notice that each Q_i and C_j are disjoint and they are six (-1)–curves that can be 2–thrown. Indeed, each quartic Q_i intersects F_1 and F_2 once while a conic C_j intersects the double curves G_i

and E once. Throwing the four quartics and the two conics Q_i, C_j we will introduce six other new surfaces i.e. planes that we will denote by Y_i and U_j.

Furthermore, imposing that the nine line bundles on the degenerating plane form a limit of the bundle $\mathcal{L}_d(m^{10})$ on $\mathbb{P}^2$, and also imposing matching conditions that all the line bundles agree on the intersection curves, we obtain a general form for the linear systems on each of the nine surfaces depending on eight parameters.

Also, in our computation, we will use the Cremona transformations $123 - 458 - 467 - 123$

$$\mathcal{L}_2(1^4, [1,0]) \leftrightarrow \mathcal{L}_0([0,-1])$$

$$\mathcal{L}_2(1^4, [0,1]) \leftrightarrow \mathcal{L}_0([-1,0])$$

$$\mathcal{L}_4(2^4, [1,1]) \leftrightarrow \mathcal{L}_0([-1,-1])$$

$$\mathcal{L}_4(1, 2^3, [1,1]^2) \leftrightarrow \mathcal{L}_0(-1)$$

$$\mathcal{L}_9(4^4, [2,2]) \leftrightarrow \mathcal{L}_1$$

Again, the conic C_i is a base point for the linear system on V. Also, C_i meets E once, and G_i once since

$$C_iE = \mathcal{L}_2(1^4, [1,0])\mathcal{L}_1([1,1]^2) = 2 - 1 = 1$$

$$C_iG_i = \mathcal{L}_2(1^4, [1,0])\mathcal{L}_0([-1,1]) = 1 - 0 = 1$$

So, by blowing up both of the conics C_i twice contracting them both, and normalizing, both Z and T will inherit four more blow ups, two of them infinitely near. By blowing up twice and contracting the four quartics , V becomes more complicated with additional 16 blow ups, eight of them infinitely near.

- $\mathcal{L}_{\mathcal{V}} = \mathcal{L}_{degV}(n_i^4, [a_i, b_i]^2, [z_i, t_i]^2_{i=1,..4})$

- $\mathcal{L}_{\mathcal{Z}} = \mathcal{L}_{degZ}(q_1, ..., q_6, [c_1, d_1], [c_2, d_2])$

- $\mathcal{L}_{\mathcal{T}} = \mathcal{L}_{degT}([c_1, d_1], [c_2, d_2])$

- $\mathcal{L}_{U_i} = \mathcal{L}_{c_i - d_i}$

- $\mathcal{L}_{Y_i} = \mathcal{L}_{z_i - t_i}$

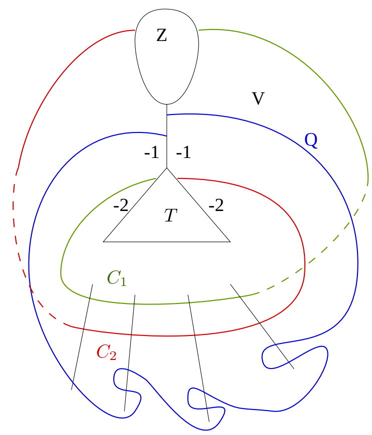


FIGURE 4.4.

In these notations $n_1 = n_2 = n_3 = n_4 = 0$ since they represint the contracted quartics while $b_1 = b_2 = 0$ since they represent the contracted conics. We introduced them in this linear system only for keeping track of the curves transformed after the Cremona transformation.

Mathcing conditions after contracting the quartics and the conics

- 1. $V \bigcap T$.

$$\mathcal{L}_{\mathcal{V}} \mathcal{L}_0([-1,1]) = \mathcal{L}_T \mathcal{L}_1([1,1])$$

We note that the curves $G = \mathcal{L}_0([-1,1])$ are invariant under the Cremona transformations so

$$a_i - b_i = \deg T - (c_i + d_i)$$

Since $b_i = 0$ we obtain $a_i = \deg(T) - (c_i + d_i)$. To see that $c_i + d_i$ is constant, we first argue that a_i are constant. Indeed, the fact that F_1 and F_2 are identified will give after the Cremonas that the two conics $\mathcal{L}_2(1^4, [1,0])$ are identified and therefore their intersection with $\mathcal{L}_V$ is the same, i.e.

$2\deg(V) - (n_1 + ... + n_4) - a_1 = 2\deg(V) - (n_1 + ... + n_4) - a_2$ so $a_1 = a_2$.

This proves that $c_1 + d_1 = c_2 + d_2 = e$.

- 2. Multiplicity on Z

From the second degeneration, the condition for the multiplicity on Z was

$$m = \deg Z - q + (a + b).$$

Here $\deg Z - q$ represents the intersection of $\mathcal{L}_Z$ with the line $\mathcal{L}_1(1)$ that also passes through the contracted cubic (that is not a base point of $\mathcal{L}_Z$) while $a+b$ represents the intersection of $\mathcal{L}_V$ with $\mathcal{L}_0([-1,-1]) = 2F + G$

The four quartics intersect both curves F_i at one point each, and the conics intersect only one curve G at a point. After blowing the conics and quartics up, each F_i will get blown up four times infinitely near (multiplicities are denoted by $z_i + t_i$; we count them double for $2F$) and each G_i once (multiplicity is denoted by $c_1 + d_1 = c_2 + d_2 = e$). After a Cremona transformation, F becomes the conic $\mathcal{L}_2(1^4, [1,0])$ and G preserves its form, we conclude that $\mathcal{L}_0([-1,-1])$ becomes $\mathcal{L}_4(2^4, [1,1])$

$$(\deg Z - q_i) + (4 \deg V - 2 \sum_{i=1,2} n_i - a_1 - b_1) + (c_1 + d_1) + 2 \sum_{i=1,\ldots,4} (z_i + t_i) = m$$

These are six equalities and if we subtract any two we obtain $q_1 = \ldots = q_6 = q$

- 3. $V \bigcap Z$.

The cubic on Z is now blown up twice

$$\mathcal{L}_Z \mathcal{L}_3(1^6, [1,1]^2) = \mathcal{L}_V \mathcal{L}_1([1,1]^2)$$

$$(3 \deg Z - 6q) - (c_1 + d_1 + c_2 + d_2) = \deg V - (a_1 + a_2 + b_1 + b_2)$$

- 4. Multiplicity on V

Consider the four quartics and two conics that we want to throw for the third degeneration $\mathcal{L}_4(2^3, 1, [1,1]^2)$ and $\mathcal{L}_2(1^4, [1,0])$

We note that through each multiple point there are three quartics that are doubled at it and one that simply passes through the point; and the two conics passing through the point. We conclude that after we blow all the quartics up, each -1 curve with the old multiplicity m, gets blown up seven times infinitely near ($7 = 1 + 2 + 2 + 2$, the new multiplicities introduced are $(z_i + t_i)$ and $2 \sum_{i \neq j} (z_j + t_j)$); and from the conics, it gets blown up twice more, infinitely near (the new multiplicities introduced are $\sum_{i=1,2} (c_i + d_i)$)

Now performing the Cremona transformation, $\mathcal{L}_0(-1)$ becomes $\mathcal{L}_4(1, 2^3, [1,1]^2)$. We get the following

$$(4\deg V - n_1 - 2(n_2+n_3+n_4) - 2\sum_{i=1,2}(a_i+b_i)) + \sum_{i=1,2}(c_i+d_i) + (z_i+t_i) + 2\sum_{i\neq j}(z_j+t_j) = m$$

Note that we have four conditions and subtracting any two of them we get that the sum $z_i + t_i$ needs to be constant for all i. Denote by $e = c_1 + d_1$ and $f = z_i + t_i$.

- 5. Degree

  The condition for the degree used to be

$d = \deg V + \deg Z + a + b = \deg V + \deg Z + 2b + \deg T$ since the pull back of a line used to be a line on V, a line on Z, a line on T and two fibers: $2F = 2\mathcal{L}_0([0,-1])$ The third degeneration doesn't affect the general class of a line in Z or T, so the intersection will still be $\deg(V)$ and $\deg(T)$. It will affect class line of the surface V, and also the curves F_i.

A general line on V intersects each at the quartic four times and each of the conic twice, so it will intersect the four quartics $16 = 4*4$ times and the two conics $4 = 2*2$ times. After blowing all the curves up the pull back of a line on V will be a line on V plus $16 = 4*4$ other curves of type $\mathcal{L}_0([-1,-1]))$ (the ones that intersect surfaces Y_i) and $4 = 2*2$ others of type $\mathcal{L}_0([-1,-1]))$ (intersecting the surfaces U_i). We now perform a Cremona so the line on V becomes $\mathcal{L}_9(4^4,[2,2]^2)$ and F changes into the conic $\mathcal{L}_2(1^4,[1,0])$ We also agreed that F will get blown up 4 times infinitely near (by point 3) and furthermore F changes into the conic $\mathcal{L}_2(1^4,[1,0])$; so $2F$ will get blown up 8 times and it changes into $\mathcal{L}_4(2^4,[2,0]) + 2*4*\mathcal{L}_0([-1,-1])$. Gathering together all these observations we obtain

$$(9\deg V - 16n - 4(a+b)) + \deg Z + \deg T + 2(2\deg V - 4n - a) + 16f + 4e + 8f = d$$

By plugging in $c_i + d_i = e$, $a_1 = a_2 = a$, $q_i = q$ and $z_i + t_i = f$ we get

- 1. $a - b = \deg T - e$
- 2. $3\deg Z - 6q - 2e = \deg V - 2(a+b)$
- 3. $(degZ - q) + (4\deg V - 8n - a - b) + e + 8f = m$

- 4. $4\deg V - 7n - 2(a+b) + 2e + 7f = m$

- 5. $(9\deg V - 16n - 4(a+b)) + \deg Z + \deg T + 2(2degV - 4n - a) + 16f + 4e + 8f = d$

Substituding $b = n = 0$ we get

- 1. $a = \deg T - e$

- 2. $3\deg Z - 6q - 2e = \deg V - 2a$

- 3. $(\deg Z - q) + (4\deg V - a) + e + 8f = m$

- 4. $4\deg V - 2a + 2e + 7f = m$

- 5. $(9\deg V - 4a) + \deg Z + \deg T + 2(2\deg V - a) + 16f + 4e + 8f = d$

Solving this linear system for $\deg T$, $\deg V$, $\deg Z$, a and f we obtain the following proposition

Proposition 4.4.1. *If we fix d and m, then all limits of the linear system $\mathcal{L}_d(m^{10})$ are of the following form for some integer values of the parameters z_i, q, x, y and e:*

- $\mathcal{L}_Z = \mathcal{L}_{3q-3m+d}(q^6, [x, e-x], [y, e-y])$

- $\mathcal{L}_V = \mathcal{L}_{-q-41m+13d}([-2q - 16m + 5d + e, 0]^2, [z_i, -6d + 19m - z_i]^2_{i\ i=1,..,4})$

- $\mathcal{L}_T = \mathcal{L}_{-2q-16m+5d+2e}([x, e-x], [y, e-y])$

- $\mathcal{L}_{U_1} = \mathcal{L}_{2x-e}$

- $\mathcal{L}_{U_2} = \mathcal{L}_{2y-e}$

- $\mathcal{L}_{Y_i} = \mathcal{L}_{2z_i-19m+6d}$.

5. The Emptiness of the nine linear systems on the central fiber.

We notice that the systems on Z and V are very complex so it's difficult to make a detalied analysis of the emptiness without additional constraints on the parameters. Therefore we will assume that all the eight linear systems $\mathcal{L}_T, \mathcal{L}_{U_i}, \mathcal{L}_{Y_i}$ and $\mathcal{L}_V$ are nonempty, and with these constraints we will obtain sufficient conditions that make $\mathcal{L}_Z$ empty. The most complicated linear systems $\mathcal{L}_Z$ and $\mathcal{L}_V$ will be studied separately.

First we notice that there are obvious necessary and sufficient conditions for the linear systems on T, U_i and Y_i to be nonempty:

Lemma 4.5.1. *The linear systems $\mathcal{L}_T, \mathcal{L}_{U_i}, \mathcal{L}_{Y_i}$ are nonempty if and only if* (1), (2), *and* (3) *hold*

(1) $\mathcal{L}_T \neq \oslash \Leftrightarrow deg\mathcal{L}_T \geq e \Leftrightarrow -2q - 16m + 5d + 2e \geq e \Leftrightarrow e \geq 2q + 16m - 5d$

(2) $\mathcal{L}_{U_i} \neq \oslash \Leftrightarrow deg\mathcal{L}_{U_i} \geq 0 \Leftrightarrow x \geq \frac{e}{2}, y \geq \frac{e}{2}$

(3) $\mathcal{L}_{Y_i} \neq \oslash \Leftrightarrow deg\mathcal{L}_{Y_i} \geq 0 \Leftrightarrow z_i \geq \frac{19m}{2} - 3d$

Proof. Indeed, 2 and 3 are obvious since the surfaces U_i and Y_i are just planes. To prove 1 we notice that T is a plane blown up four times and there with two tangent lines of the form $\mathcal{L}_1([1,1],[0,0])$ and $\mathcal{L}_1([0,0],[1,1])$ meeting at a point. Both of them are (-1)–curves so their sum is a fiber of the ruling therefore it moves. We get a contradiction since both curves are fixed part of the linear system. We conclude that $\mathcal{L}_T \neq \oslash$ if and only if the two lines don't split off. □

Remark 4.5.2. *One can easily observe that for the study of $\mathcal{L}_Z$ and $\mathcal{L}_V$ it suffices to consider only the boundary cases. Indeed, we first notice that the parameters e, x, y and z_i describe the multiplicity of some points that are infinitely near. An easy computation shows that an $[m+k, m-k]$ point imposes the same conditions as an $[m,m] - kG$ point. We conclude that we get the sharpest bound of the degree by imposing the mildest conditions on the parameters i.e. when the parameters e, x, y and z_i reach the lower bound. This enables us to assume*

- $e = 2q + 16m - 5d$
- $x = \frac{e}{2}, y = \frac{e}{2}$
- $z_i = \frac{19m}{2} - 3d$

Applying this remark in Proposition 4.4.1 we get the linear systems

- $\mathcal{L}_Z = \mathcal{L}_{3q-3m+d}(q^6, [\frac{e}{2}, \frac{e}{2}], [\frac{e}{2}, \frac{e}{2}])$
- $\mathcal{L}_V = \mathcal{L}_{-q-41m+13d}([\frac{19m}{2} - 3d, \frac{19m}{2} - 3d]^8)$

- $\mathcal{L}_T = \mathcal{L}_e([\frac{e}{2}, \frac{e}{2}], [\frac{e}{2}, \frac{e}{2}])$

- $\mathcal{L}_{U_i} = \mathcal{L}_0$

- $\mathcal{L}_{Y_i} = \mathcal{L}_0$.

We will first analyze the system on V, $\mathcal{L}_V$. The Picard group for V will have rank 19, so the analysis of the emptiness of the linear system on V is expected to be difficult. Nevertheless eight pairs of points will be in special position and this will facilitate the analysis. Indeed, we notice that after all blow ups the F_i become (-10)–curves and furthermore contracting the six curves on V will ultimately transform them into conics of the form $\mathcal{L}_2([1,0],[1,1]^4)$. We conclude that $\mathcal{L}_V$ is a homogeneous linear system of the form $\mathcal{L}_k([n,n]^8)$ passing through 16 points in special position where not only each of the eight points comes with a tangent condition but also four distinct pairs $[n,n]$ lie on two conics, C_1 and C_2 each one of the form $\mathcal{L}_2([1,0],[1,1]^4)$ meeting in four points. One can notice that next lemma is also true in the case when all the eight pairs $[n,n]$ are general i.e. the proof doesn't depend on the existence of the curves C_1 and C_2 so even though we abused notation (in denoting $\mathcal{L}_V$ as $\mathcal{L}_k([n,n]^8)$) the result also holds for general pairs $[n,n]$.

Lemma 4.5.3. *The linear system* $\mathcal{L}_k([n,n]^8) \neq \emptyset \Leftrightarrow \frac{k}{n} \geq 4$

Proof. We know that for general points $\mathcal{L}_{4n}(n^{16})$ is nonempty therefore for $\frac{k}{n} \geq 4$ $\mathcal{L}_k([n,n]^8)$ becomes nonempty.
It suffices now to show the emptiness when $k < 4n$. Indeed,

$$\mathcal{L}_k([n,n]^8)\mathcal{L}_2([1,0],[1,1]^4) = 2k - 8n < 0$$

C_1 and C_2 split off $x \geq 1$ times

$$Res_1 = \mathcal{L}_k([n,n]^8) - x[C_1 + C_2] = \mathcal{L}_{k-4x}([-x,0]^2,[n-x,n-x]^8)$$

Now F_1 and F_2 will split off x times each in Res_1 so

$$ResRes_1 = \mathcal{L}_{k-4x}([n-x,n-x]^8)$$

This process continues and after $l = [\frac{k}{4x}]+1$ steps $ResRes_l = \mathcal{L}_{k-4lx}([n-lx,n-lx]^8$ becomes empty since it exhausts the degree. $\square$

Corollary 4.5.4. *A necessary and sufficient condition for the nonemptiness of* $\mathcal{L}_V = \mathcal{L}_{-q-41m+13d}([-3d + \frac{19m}{2}, -3d + \frac{19m}{2}]^8)$ *is* $q \leq 25d - 79m$.

Proof. Indeed, applying the Lemma 4.5.3 to $k = -q - 41m + 13d$ and $n = -3d + \frac{19m}{2}$ we get the desired result. □

Later on we will assume that the conditions from Corollary 4.5.4 are also satisfied, i.e. the conditions for the nonemptiness of the linear systems $\mathcal{L}_V, \mathcal{L}_T, \mathcal{L}_{U_i}, \mathcal{L}_{Y_i}$. Assuming them we want conditions that make $\mathcal{L}_Z$ empty.

One of the key ingredient in analyzing linear systems with points in special position is the following lemma that reduces the study of the emptiness of our linear system to the emptiness of the same system with general points, assuming only -1 curves split out. A similar argument can be used for any negative curves splitting out- Lemma 4.5.3 providing just a trivial example why a linear system becomes empty if a corresponding series of negative curves $\Sigma_{j=0}^{\infty} x$ is divergent. In general this won't be the case, since the dimension of linear systems with points in general position will go up.

Lemma 4.5.5. *Consider the linear system* $\mathcal{L} = \mathcal{L}_r(q^i, [s, s])$ *for fixed* $i \geq 0$*, let* C *be the* -1 *curve* $\mathcal{L}_{r_0}(q_0^i, [s_0, s_0 - 1])$ *for some* r_0, q_0 *and* s_0 *and* G *be the* -2 *curve* $\mathcal{L}_0([-1, 1])$ *and assume that*

$$\mathcal{L}_r(q^i, [s, s])\mathcal{L}_{r_0}(q_0^i, [s_0, s_0 - 1]) = -k < 0$$

Then the curve C *splits off* $2k$ *times and the curve* G *splits off* k *times. Furthermore we get the same residual system as if the points were general.*

Proof. We denote by Res_0 the residual system after splitting off C k times

$$\mathcal{L} = kC + \mathcal{L}_{r-kr_0}((q - kq_0)^i, [s - ks_0, s - ks_0 + k])$$

Now G splits off and we will remove G once from Res_0 denoting the residual Res_1

$$Res_1 = Res_0 - G = \mathcal{L}_{r-kr_0}((q - kq_0)^i, [s - ks_0 + 1, s - ks_0 + k - 1])$$

We denote by $ResRes_1$ to be the new residual after we remove C form Res_1

$$ResRes_1 = Res_1 - C = \mathcal{L}_{r-(k+1)r_0}((q - (k+1)q_0)^i, [s - (k+1)s_0 + 1, s - (k+1)s_0 + k])$$

G starts splitting off again, and we repeat these steps p times and we get

$$ResRes_p = \mathcal{L}_{r-(k+p)r_0}((q-(k+p)q_0)^i, [s-(k+p)s_0+p, s-(k+p)s_0+k])$$

Since $GResRes_p = k-p$ this process will stop when $p=k$ and then

$$ResRes_k = \mathcal{L}_{r-2kr_0}((q-2kq_0)^i, [s-2ks_0+k, s-2ks_0+k])$$

and we conclude that $\mathcal{L} = 2kC + kG + ResRes_k$

We end the proof by remarking that if the points are general both -1 curves $\mathcal{L}_{r_0}((q_0)^i, s_0-1, s_0)$ and $\mathcal{L}_{r_0}((q_0)^i, s_0, s_0-1)$ split off k times and the residual Res has the same form as $ResRes_k$

$$\mathcal{L}_r(q^i, s^2) = k\mathcal{L}_{2r_0}((2q_0)^i, 2s_0-1, 2s_0-1) + \mathcal{L}_{r-2kr_0}((q-2kq_0)^i, (s-2ks_0+k)^2)$$

$\square$

Remark 4.5.6. *Assuming that $\mathcal{L}_T$ and $\mathcal{L}_V$ are nonempty and that $\frac{d}{m} < \frac{117}{37}$ then it suffices to analyze $\mathcal{L}_Z$ when $\frac{11q}{2} \leq e$.*
Indeed, $\frac{d}{m} < \frac{117}{37}$ then $25d - 79m < \frac{32m}{7} - \frac{10d}{7}$. The nonemptiness of $\mathcal{L}_V$ implies $q \leq \frac{32m}{7} - \frac{10d}{7}$ i.e. $\frac{11q}{2} \leq 2q + 16m - 5d$ and finally, the assumption on $\mathcal{L}_T$ gives us $\frac{11q}{2} \leq e$.

For our problem we will only need cases 1 and 2 of the lemma 4.5.7. Moreover, in the following statements we will assume that d and m are big enough, so the ratio d/m becomes an integer number. If we claim that for all smaller values, $\mathcal{L}_d(m^k) = \oslash$ d_0 and m_0 with $d_0/m_0 \leq d/m$, then $\mathcal{L}_{d_0}(m_0^k) = \oslash$. Indeed, we observe that if $\mathcal{L}_{d_0}(m_0^k)$ is nonempty then a multiple of this is still nonempty, i.e. $\mathcal{L}_{rd_0}(rm_0{}^k) \neq \oslash$, and hence $\mathcal{L}_d(rm_0{}^k) \neq \oslash$ for any $d \geq rd_0$

Lemma 4.5.7. *Denote by $\mathcal{L} = \mathcal{L}_r(q^6, [\frac{e}{2}, \frac{e}{2}]^2)$, where $r, q, \frac{e}{2}$ are positive integers.*

(1) *If $e \geq 12q$ then $\mathcal{L}_r(q^6, [\frac{e}{2}, \frac{e}{2}]^2) = \oslash \Leftrightarrow r < e$*

(2) *If $\frac{11q}{2} \leq e < 12q$ then $\mathcal{L}_r(q^6, [\frac{e}{2}, \frac{e}{2}]^2) = \oslash \Leftrightarrow r < \frac{12q}{13} + \frac{12e}{13}$*

(3) *If $\frac{84q}{19} \leq e < \frac{11q}{2}$ then $\mathcal{L}_r(q^6, [\frac{e}{2}, \frac{e}{2}]^2) = \oslash \Leftrightarrow r < \frac{48q}{41} + \frac{36e}{41}$*

(4) *If $\frac{25q}{6} \leq e < \frac{84q}{19}$ then $\mathcal{L}_r(q^6, [\frac{e}{2}, \frac{e}{2}]^2) = \oslash \Leftrightarrow r < \frac{36q}{29} + \frac{25e}{29}$*

Proof. We will shortly present the boundary case when we have equality.

(1) Assume that $e \geq 12q$.

If $r < e$ we conclude that $\mathcal{L}$ is empty since the two tangent lines split out and

$$\mathcal{L}_1([1,1],[0,0])\mathcal{L}_1([0,0],[1,1]) = 1.$$

For the case where $r = e$ we need to prove that $\mathcal{L}$ is nonempty. There are six conics of the form $\mathcal{L}_2(1,[1,1],[1,1])$ that split out q times each.

$$\mathcal{L}_e(q^6,[\frac{e}{2},\frac{e}{2}]^2) = q\mathcal{L}_{12}(1^6,[6,6]^2) + \mathcal{L}_{e-12q}([\frac{e-12q}{2},\frac{e-12q}{2}]^2)$$

Since $e \geq 12q$, we get that the residual linear system has a positive dimension $\frac{e-12q}{2}$ so by Lemma 4.5.1 case 3 we obtain $Res = \mathcal{L}_{e-12q}([\frac{e-12q}{2},\frac{e-12q}{2}]^2) \neq \oslash$ since it consists of a pencil of conics.

(2) We will prove that $\mathcal{L}_{\frac{12q}{13}+\frac{12e}{13}}(q^6,[\frac{e}{2},\frac{e}{2}]^2)$ is a zero dimensional linear system.

We notice that 2 (-1) curves of the form $\mathcal{L}_6(1^6,[3,2],[3,3])$ split off $\frac{6q}{13}-\frac{e}{26}$ times each and by lemma 4.5.5 we get

$$\mathcal{L}_{\frac{12q}{13}+\frac{12e}{13}}(q^6,[\frac{e}{2},\frac{e}{2}],[\frac{e}{2},\frac{e}{2}]) = (\frac{12q}{13}-\frac{e}{13})\mathcal{L}_{12}(2^6,[6,5],[6,5])+$$

$$(\frac{6q}{13}-\frac{e}{26})\mathcal{L}_0([-1,1]^2) + \mathcal{L}_{12(\frac{2e}{13}-\frac{11q}{13})}((\frac{2e}{13}-\frac{11q}{13})^6,[6(\frac{2e}{13}-\frac{11q}{13}),6(\frac{2e}{13}-\frac{11q}{13})]^2)$$

Now we notice that if $2e-11q \geq 0$ then the linear system $\mathcal{L}_Z$ is zero dimensional therefore if we lower the degree it will be empty.

However if $2e-11q < 0$ the linear system $\mathcal{L}_Z$ is already empty because the degree of the residual is negative (the six conics $\mathcal{L}_2(1,[1,1],[1,1])$ split off too much).

(3) 3. We start with the bounds for e, $\frac{84q}{19} \leq e < \frac{11q}{2}$ and again we want to prove that $\mathcal{L}_{\frac{48q}{41}+\frac{36e}{41}}(q^6,[\frac{e}{2},\frac{e}{2}]^2)$ has dimension zero.

We notice that six (-1) curves of the form $\mathcal{L}_{16}(4,3^5,[7,7],[7,7])$ split off $\frac{11q}{41}-\frac{2e}{41}$ times each

$$\mathcal{L}_{\frac{48q}{41}+\frac{36e}{41}}(q^6,[\frac{e}{2},\frac{e}{2}],[\frac{e}{2},\frac{e}{2}]) = (\frac{11q}{41}-\frac{2e}{41})\mathcal{L}_{96}((19)^6,[42,42],[42,42]) + Res$$

$$Res = \mathcal{L}_{24(\frac{19e}{82}-\frac{84q}{82})}(4(\frac{19e}{82}-\frac{84q}{82})^6,[11(\frac{19e}{82}-\frac{84q}{82}),11(\frac{19e}{82}-\frac{84q}{82})]^2.$$
$$= (\frac{19e}{82}-\frac{84q}{82})\mathcal{L}_{24}(4^6,[11,11]^2)$$

We notice that $\mathcal{L}_{24}(4^6,[11,11]^2)$ is the linear system form case 2 with $e=22$, $q=4$ and $\frac{e}{q}=\frac{11}{2}$ and $r=\frac{12*4}{13}+\frac{12*22}{13}=24$. From the previous remark we have that the two curves $\mathcal{L}_6(1^6,[3,3],[3,2])$ split off $\frac{12*4}{13}-\frac{22}{13}=\frac{26}{13}=2$

$$\mathcal{L}_{24}(4^6,[11,11]^2)=2\mathcal{L}_{12}(2^6,[6,5]^2)+\mathcal{L}_0([-1,1]^2)$$

We get

$$Res=(\frac{38e}{82}-\frac{168q}{82})\mathcal{L}_{12}(2^6,[6,5]^2)+(\frac{19e}{82}-\frac{84q}{82})\mathcal{L}_0([-1,1]^2)$$

Now we notice that if $e\geq\frac{84q}{19}$ then the linear system $\mathcal{L}_Z$ is zero dimensional therefore if we lower the degree it will be empty.

However if $e<\frac{84q}{19}$ the linear system $\mathcal{L}_Z$ is already empty because the degree of the residual is negative (the sextics $\mathcal{L}_6(1^6,[3,2],[3,3])$ split off too much).

(4) 4. We consider $\frac{25q}{6}\leq e<\frac{84q}{19}$ and we claim that $\mathcal{L}_{\frac{36q}{29}+\frac{25e}{29}}(q^6,[\frac{e}{2},\frac{e}{2}]^2)$ is a linear system of dimension $\frac{84q-19e}{58}$ and if we decrease the degree it becomes empty. First consider the case $deg\mathcal{L}<\frac{36q}{29}+\frac{25e}{29}$ so $\mathcal{L}$ is at the form
$\mathcal{L}_k(q^6,[\frac{e}{2},\frac{e}{2}],[\frac{e}{2},\frac{e}{2}])$ with $k<\frac{36q}{29}+\frac{25e}{29}$. Then $\mathcal{L}$ is empty since there exist two -1 curves that split off and meet. Indeed,

$$\mathcal{L}_k(q^6,[\frac{e}{2},\frac{e}{2}],[\frac{e}{2},\frac{e}{2}])\mathcal{L}_{29}(6^6,[12,12],[13,13])=29k-36q-25e<0$$

$\mathcal{L}_{29}(6^6,[12,12],[13,13])\mathcal{L}_{29}(6^6,[13,13],[12,12])=1.$

The only statement left to prove is that if $deg\mathcal{L}=\frac{36q}{29}+\frac{25e}{29}$ then $\mathcal{L}$ becomes nonempty. First we notice that six (-1) curves of the form $\mathcal{L}_{16}(4,3^5,[7,7],[7,7])$ split off $\frac{6e}{29}-\frac{25q}{29}$ times each

$$\mathcal{L}_{\frac{36q}{29}+\frac{25e}{29}}(q^6,[\frac{e}{2},\frac{e}{2}],[\frac{e}{2},\frac{e}{2}])=(\frac{6e}{29}-\frac{25q}{29})\mathcal{L}_{96}((19)^6,[42,42],[42,42])+Res$$

$$Res=\mathcal{L}_{19(\frac{84q}{19}-e)}(\frac{114}{29}(\frac{84q}{19}-e)^6,[\frac{475}{58}(\frac{84q}{19}-e),\frac{475}{58}(\frac{84q}{19}-e)]^2.$$

$$=\mathcal{L}_{19l}(\frac{114l}{19}^6,[\frac{475l}{58},\frac{475l}{58}]^2)$$

where $l=\frac{84q}{19}-e>0$. Now consider the following Cremona transformations $8,9,10-127-347-567-8,9,10-123-456-123$

$19l$	$\frac{114l}{29}{}^{3}$	$\frac{114l}{29}{}^{3}$	$\frac{475l}{58}$	$\frac{475l}{58}{}^{3}$
$\frac{779l}{58}$	$\frac{114l}{29}{}^{3}$	$\frac{114l}{29}{}^{3}$	$\frac{475l}{58}$	$\frac{152l}{58}{}^{3}$
$\frac{323l}{58}$	$\frac{76l}{29}{}^{3}$	$\frac{76l}{29}{}^{3}$	$\frac{19l}{58}$	$\frac{152l}{58}{}^{3}$
$\frac{190l}{58}$	$\frac{76l}{29}{}^{3}$	$\frac{76l}{29}{}^{3}$	$\frac{19l}{58}$	$\frac{19l}{58}{}^{3}$
$\frac{152l}{58}$	$\frac{38l}{29}{}^{3}$	$\frac{76l}{29}{}^{3}$	$\frac{19l}{58}$	$\frac{19l}{58}{}^{3}$
$\frac{76l}{58}$	$\frac{38l}{29}{}^{3}$	–	$\frac{19l}{58}$	$\frac{19l}{58}{}^{3}$
$\frac{38l}{58}$	–	–	$\frac{19l}{58}$	$\frac{19l}{58}{}^{3}$

TABLE 3.

Therefore we've just proved that if $\frac{25q}{6} < e < \frac{84q}{19}$ then $\mathcal{L}_{\frac{36q}{29}+\frac{25e}{29}}(q^6, [\frac{e}{2}, \frac{e}{2}], [\frac{e}{2}, \frac{e}{2}])$ is a non-empty and nonspecial linear system of dimension $\frac{84q-19e}{58}$.

$\square$

Remark 4.5.8. *Lemma 4.5.5 will enable us to conclude that the statements of Propositions 4.5.1, 4.5.4 and 4.5.7 hold for general points as well.*

Proposition 4.5.9. *If $\frac{d}{m} < \frac{117}{37}$ and all the linear systems $\mathcal{L}_V$, $\mathcal{L}_T$, $\mathcal{L}_{U_i}$ and $\mathcal{L}_{Y_i}$ are nonempty then $\mathcal{L}_Z$ is empty.*

PROOF. We assume that $\frac{d}{m} < \frac{117}{37}$ and that both linear systems $\mathcal{L}_T$ and $\mathcal{L}_V$ are nonempty. By Remark 4.5.6 we have that $e \geq \frac{11q}{2}$ so we distinguish two cases

- If $\frac{11q}{2} \leq e < 12q$ then we claim that the assumptions in the hypothesis make $\mathcal{L}_Z$ is empty Indeed, $\frac{d}{m} < \frac{117}{37}$ implies $25d - 79m < \frac{231m}{3} - \frac{73d}{3}$ and by the non-empyness of $\mathcal{L}_V$ we obtain $q < \frac{231m}{3} - \frac{73d}{3}$ i.e.

$$\frac{27q}{12} - \frac{39m}{12} + \frac{13d}{12} < 2q + 16m - 5d$$

Now we use the hypothesis on $\mathcal{L}_T$ to obtain

$$\frac{27q}{12} - \frac{39m}{12} + \frac{13d}{12} < e$$

i.e.

$$3q - 3m + d < \frac{12q}{13} + \frac{12e}{13}$$

By Lemma 4.5.7 we conclude that $\mathcal{L}_Z$, and therefore $\mathcal{L}_d(m^{10})$, is empty as desired.

- If $e \geq 12q$ and $\mathcal{L}_T$ is nonempty then $\mathcal{L}_Z$ is empty.

Indeed, if $e = 2q + 16m - 5d \geq 12q$ i.e. $q \leq \frac{16m-5d}{10}$ we claim that $e > 3q - 3m + d$ i.e. $q < 19m - 6d$. The last statement is obvious since the following inequality $\frac{16m-5d}{10} < 19m - 6d$ holds $\Leftrightarrow \frac{d}{m} < \frac{174}{55}$.

We conclude that we get the best results by assuming $e < 12q$.

$\square$

Corollary 4.5.10. *If $\frac{d}{m} < \frac{117}{37}$ then the linear system $\mathcal{L}_d(m^{10})$ is empty.*

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