# ANALYTICAL STUDY OF THE MECHANICS OF SCOUR

# FOR TWO-DIMENSIONAL JET

by

Lucien Duckstein,
Yuichi Iwagaki,
George L. Smith,
and
Maurice L. Albertson

Prepared for

U. S. Bureau of Public Roads

Under Contract #CPR11-5504

Engineering Research
Colorado State University
Fort Collins, Colorado

April 8, 1960

CER60GLS12

## ACKNOWLEDGMENTS

The analytical study of the mechanics of scour for both the three-dimensional and two dimensional jet was made at Colorado State University under the sponsorship of the U. S. Bureau of Public Roads. The permission of Mr. Carl Izzard, Chief of the Division of Hydraulic Research, U. S. Bureau of Public Roads, to present these studies is gratefully acknowledged. The writers also wish to thank Mr. Eric Plate for his review, criticism, and assistance in clarification of the basic theory and of the concept of sediment transport for different conditions of scour and deposition; and Dr. A. R. Chamberlain, Acting Dean and Chief, Engineering Research, Colorado State University for his encouragement in preparation of this report.

CONTENTS

## SYMBOLS

| Symbol | Unit | Definition |
|---|---|---|
| $a_o$ | | Coefficient of proportionality between U and X or V and Z. |
| a | | Function defined by Eq 80. |
| a' | | Function defined by Eq 81. |
| A | | Ratio of momentum thickness of boundary layer to thickness of boundary layer $\delta$ . |
| $A_o$ | | Constant in sediment transport equation. |
| $A_1$ | | Function of time having positive value. |
| b | Ft | Tailwater depth or height of outlet from normal boundary or simulated stream bed. |
| b' | Ft | b/sin $\theta$. |
| B | | Ratio of displacement thickness of boundary layer to thickness of boundary layer $\delta$. |
| $B_o$ | Ft | Width of two-dimensional jet at outlet or tailwater surface. |
| $B_1$ | | Function of time having positive value. |
| c | | Function of time having positive value. |
| $c_1$, $c_2$, $c_3$, $c_4$ | | Integral constants. |
| $c_5$ | | Constant. |
| C | | Function defined by Eq 112. |
| C' | | Function defined by Eq 113. |
| d | Ft | Diameter of three-dimensional jet at outlet or tailwater surface. |
| $d_s$ | Ft | Mean diameter of sediment particle. |
| F | | Function defined by Eq 82. |
| $F_o$ | | Function defined by Eq 83. |

SYMBOLS (cont'd)

| Symbol | Unit | Definition |
|---|---|---|
| G | | Function defined by Eq 114. |
| $G_o$ | | Function defined by Eq 115. |
| g | $Ft/sec^2$ | Acceleration due to gravity. |
| h | Ft | Thickness of deflected jet in case of ideal fluid. |
| k | | Constant, 0.0225 for smooth boundary and $m' = 1/7$. |
| L | | Function defined by Eq 58. |
| m | | Exponent in sediment transport equation. |
| m' | | Exponent in power-law velocity distribution. |
| m" | | Constant. |
| M | Ft | Coordinate point relative to X, Z axis. |
| N | Ft | Coordinate point relative to X, Z axis. |
| n | | Exponent defined by $2m'/(1+m')$ and equal to 1/4 for $m' = 1/7$. |
| p | $Lb/ft^2$ | Pressure intensity at a point. |
| $p_o$ | $Lb/ft^2$ | Pressure at the point of stagnation. |
| P | | Function defined by Eq 147. |
| P' | | Derivative of P with respect to (X/b). |
| $q_s$ | $Ft^3/sec/Ft$ | Mass rate of sediment transport per unit width. |
| Q | | Function defined by Eq 176. |
| Q' | | Derivative of Q with respect to (X/b). |
| r | Ft | Radial coordinate parallel to normal boundary |
| R | | Function defined by Eq 77. |
| R' | | Derivative of R with respect to (X/b). |

| Symbol | Unit | Definition |
|---|---|---|
| Re | | Reynolds number of the flow. |
| s | | Ratio $Z/\delta$. |
| S | | Function defined by Eq 109. |
| S' | | Derivative of S with respect to $(X/b)$. |
| t | Sec | Time. |
| $t_{99}$ | Sec | Time when $Z_s = 0.99\ Z_{s\infty}$ |
| u | Ft/sec | Velocity in the boundary layer. |
| U | Ft/sec | Horizontal component of velocity of deflected jet at a point in ideal fluid. |
| $U_o$ | Ft/sec | Horizontal component of velocity of deflected jet at a point N in Fig. 7. |
| $U_b$ | Ft/sec | Horizontal component of velocity at a point for case of viscous flow along X-axis. |
| $U_*$ | Ft/sec | Shear velocity. |
| $U_{*c}$ | Ft/sec | Critical shear velocity. |
| V | Ft/sec | Vertical component of velocity at a point in ideal fluid. |
| $V_b$ | Ft/sec | Value of V at $Z = 0$. |
| $V_o$ | Ft/sec | Vertical component of jet velocity at outlet or tailwater surface. |
| $V_m$ | Ft/sec | Maximum velocity at centerline of jet. |
| $V_{bm}$ | Ft/sec | Value of $V_m$ at $Z = 0$ |
| V' | Ft/sec | Velocity component of inclined jet at a point. |
| $V'_o$ | Ft/sec | Value of V' at outlet or tailwater surface. |
| $V'_b$ | Ft/sec | Value of V' at $Z = b$. |

| Symbol | Unit | Definition |
|---|---|---|
| $V'_m$ | Ft/sec | Maximum velocity of jet at centerline of inclined jet. |
| $V'_{bm}$ | Ft/sec | Value of $V'_m$ corresponding to a point along the X-axis. |
| $V'_{bmo}$ | Ft/sec | Value of $V'_{bm}$ at $Z = b$ and $X = 0$ |
| $W_o$ | Ft/sec | Vertical component of velocity for three-dimensional jet at outlet or tailwater surface. |
| $W_m$ | Ft/sec | Maximum velocity at center of three-dimensional jet. |
| $X$ | Ft | Horizontal coordinate parallel to normal boundary or simulated alluvial stream bed. |
| $X_o$ | Ft | $X$ corresponding to a certain value of the integral constant. |
| $X'$ | Ft | Horizontal coordinate from centerline of inclined jet. |
| $Z$ | Ft | Vertical coordinate perpendicular to deflecting boundary or bed level. |
| $Z'$ | Ft | Distance from bed level, parallel to center line of inclined jet. |
| $Z_M$, $Z_N$ | Ft | Vertical distances from X-axis to points M and N in Fig. 7. |
| $Z_s$ | Ft | Depth of scour ($Z_s = -Z$). |
| $Z_{s\infty}$ | Ft | Final depth of scour. |
| $Z'_s$ | Ft | Inclined value of $Z_s$. |
| $Z'_{s\infty}$ | Ft | Inclined value of $Z_{s\infty}$. |
| $\alpha$, $\beta$ | | Constants in the expression which characterizes the velocity function at every section within the diffusion region of a jet issuing from a non-submerged outlet. |
| $\alpha'$, $\beta'$ | | Value of $\alpha$, $\beta$ for the case of a submerged outlet. |

SYMBOLS (cont'd)

| Symbol | Unit | Definition |
| --- | --- | --- |
| $\gamma$ | | Coefficient in equation for boundary layer thickness for ideal flow. |
| $\gamma'$ | | Coefficient in equation for boundary layer thickness for viscous flow. |
| $\delta$ | Ft | Thickness of boundary layer. |
| $\delta_*$ | | $\delta/X$. |
| $\psi$ | | Function expressed by Eq 45. |
| $\psi_o$ | | $\psi$ at $Z = 0$. |
| $\psi'$ | | Derivative of $\psi$ with respect to $X$. |
| $\psi_1$ | | Function defined by Eq 143. |
| $\lambda$ | | Porosity of sediment. |
| $\nu$ | $Ft^2/sec$ | Kinematic viscosity of water. |
| $\theta$ | Degrees | Angle of jet with respect to the tailwater surface. |
| $\rho$ | $Lb\text{-}sec^2/ft^4$ | Mass density of water. |
| $\sigma$ | $Lb\text{-}sec^2/ft^4$ | Mass density of sediment $\approx$ specific gravity of sediment $\approx$ 2.65. |
| $\tau$ | $Lb/ft^2$ | Intensity of shear. |
| $\tau_o$ | $Lb/ft^2$ | Intensity of shear along boundary or simulated alluvial stream bed. |
| $\tau_c$ | $Lb/ft^2$ | Critical intensity of shear. |
| $\eta$ | | Function defined by Eq 79. |
| $\eta'$ | | Function defined by Eq 106. |
| $\xi$ | | Function defined by Eq 111. |
| $\xi'$ | | Function defined by Eq 180. |

SYMBOLS (cont'd)

| Symbol | Unit | Definition |
| --- | --- | --- |
| $\xi_1$ | | Function defined by Eq 128. |
| $\omega'$ | | Function defined by Eq 159. |
| $\omega$ | | Function defined by Eq 171. |

## FIGURES

ABSTRACT

Using the rectilinear coordinate system the equation of continuity of mass sediment transport is derived, and the continuity equation is used to help in describing mathematically the phenomenon of scour by the continuity equation. The relationship between the shape of the scour hole and its variation with time is investigated for different conditions of scour and deposition. Expressions for distribution of sediment transport along the bed are derived for each condition.

The impingement of a two-dimensional jet on a normal boundary is analyzed by making the assumption that the Bernoulli Equation is valid in the neighborhood of the stagnation point. Plane, potential flow is considered first, followed by flow of a fluid with viscosity. For flow with viscosity from submerged and non-submerged outlets, expressions for the horizontal velocity and shear distribution along the boundary are developed using Bernoulli's theorem and the boundary layer theory.

The variation of the depth of scour is determined for two conditions of outlets by assuming a law of open channel flow for sediment transportation and by using the previously determined shear distributions and the continuity equation. In particular, the variation of scour depth with respect to time and the final depth of scour are described theoretically in terms of dimensionless parameters. It is then shown that the development of the scour hole with respect to time follows the power law and the logarithmic law for the submerged and non-submerged outlets respectively before the final state is reached.

The influence of the angle of the jet is analyzed in the same manner as in the case of a vertical jet. In this case, expressions for the variations of the depth of scour with respect to time and the final depth of scour are also developed

INTRODUCTION

Various aspects of the phenomena of scour has concerned engineers since the advent of the construction of hydraulic structures in or near alluvial waterways. Classical studies on scour and scour control include those of Rouse (1)*, Doddiah (2), Thomas (3), Hallmark (4), Albertson (5), and Smith (6) to name a few. However, these investigations were concerned with only individual facets of the phenomena.

In 1957, Dr. Yuichi Iwagaki, Kyoto University, Kyoto, Japan, came to Colorado State University as a visiting professor in hydraulics. Because of his interest in the knowledge of scour in alluvial material, M. L. Albertson suggested that Dr. Iwagaki study the theoretical aspects of scour caused by a jet of water from the viewpoint of the continuity, momentum, and energy equations.

In developing the theory, Dr. Iwagaki made use of the Bernoulli theorem, experimental and theoretical studies by Schlichting and Truckenbrodt (7), and Truckenbrodt (8) of a jet deflecting on a normal boundary, experimental and theoretical studies by Albertson and others (9) on the diffusion of a submerged jet, and of the sediment transport equation developed by Brown and Laursen (10). Iwagaki's developmental work was based largely on the fundamental ideas put forth by Shields (11), Kalinske (12), and Exner (13). Concurrently with the development of the three-dimensional theory, L. Duckstein, under the supervision of

*Numbers refer to appended references

Dr. Iwagaki, wrote the two-dimensional form of the theory. Editing of both forms of the theory was done by Dr. Maurice L. Albertson and George L. Smith.

The theory in its two- and three-dimensional form is neither complete nor applicable to an analysis of all phases of the scour phenomena. This theory does attempt to consolidate all of the various studies made on this type of scour. Noteworthy is the fact that Iwagaki's work has helped to focus attention on the areas where more fundamental studies are needed before an adequate theory can be developed. In particular, there is the need for determining the critical shear stress developed along a deflecting hydraulically smooth boundary by a normal jet over a wide range of flow conditions.

In addition to the shear stress, it is essential to know the velocity profile within the boundary layer for different bed roughnesses. Once the velocity profile is known, the rate of change of momentum along the boundary can be determined. At present, it seems desirable to describe the scour of alluvial material in terms of the momentum change occurring along the eroding surface of the alluvial bed.

In light of the need of determining the inter-relationship between shear stress, velocity profile, and rate of change of momentum along a deflecting boundary, a fundamental study on the flow characteristics of a circular, submerged jet impinging normally on a smooth boundary was initiated at Colorado State University by the Association of American Railroads. Results of the study are given in a report by M. Poreh (14). The results indicated that exponents in the theory of scour by Iwagaki would need some modification. Furthermore, since this study supported

Iwagaki's theory, a more comprehensive fundamental study on the dynamics of a jet of air impinging upon a normal, smooth boundary was initiated under sponsorship of the National Science Foundation.

At this time, the experimental equipment for the National Science Foundation supported study has been constructed and tested. The primary objectives of the first phase of this research program are, for given fluid characteristics and boundary geometry, as follows:

1. To measure, for a pre-determined range of flow characteristics, the shear stress, velocity profile, and rate of change of momentum along a hydraulically smooth boundary.
2. To measure, for a pre-determined range of flow characteristics, the shear stress, velocity profile, and rate of change of momentum along an artificially-roughened boundary, where the roughness is of a random or statistical nature.

EQUATION OF CONTINUITY FOR SEDIMENT TRANSPORT:

Theoretical Development - To determine the equation of continuity of mass sediment transport, the quantity of sediment transported through the infinitesimal element AA'B'B, Fig. 1, defined by lines AA' and B'B and lines A'B' and AB may be considered. The quantity of sediment transported through the section AA' per unit width per unit of time is

$$(q_s)(1)$$

and through section A'B (See Fig. 1) is

$$\left(q_s + \frac{\partial q_s}{\partial X}\, dX\right)(1)$$

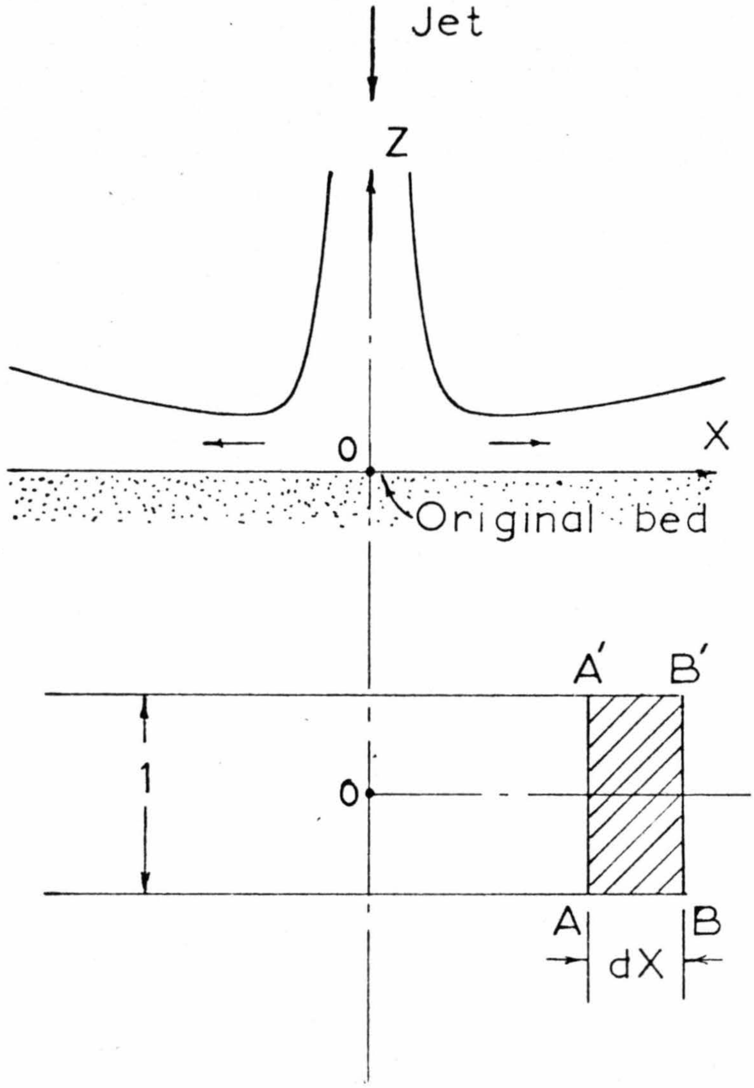


Fig. 1 Rectilinear coordinates for theoretical development of the continuity equation.

in which $q_s$ is the mass rate of sediment transport per unit width from $Z = 0$ to $Z = \infty$.

The difference between these quantities is equal to

$$\frac{\partial q_s}{\partial X} dX \tag{1}$$

The quantity of sediment scoured by the jet per unit of time can be expressed by

$$- \frac{\partial}{\partial t} (Z\ dX)\ (1 - \lambda)\ (1) \tag{2}$$

in which $\lambda$ is the porosity of the sediment.

The equation of continuity of mass sediment transport in rectilinear coordinates is obtained by equating the quantities given by Eqs 1 and 2, that is,

$$(1 - \lambda) \frac{\partial}{\partial t} (Z\ dX) + \frac{\partial q_s}{\partial X} dX = 0$$

or

$$(1 - \lambda) \frac{\partial Z}{\partial t} + \frac{\partial q_s}{\partial X} = 0 \tag{3}$$

This equation has been given by Exner (13) and Iwagaki (15)

<u>Conditions of Application of the Continuity Equation</u> - The relation between the shape of the scour hole and the distribution of the scoured sediment will now be considered for the following conditions of scour and deposition.

1. The condition of $\partial Z/\partial t = 0$ (No scour).

   For this condition Eq 3 becomes

$$\frac{\partial q_s}{\partial X} = 0 \tag{4}$$

   Integration of Eq 4 gives

$$q_s = c_1 \tag{5}$$

   in which $c_1$ is the integral constant.

2. The condition of $\partial Z/\partial t = -c\ (t)$, $c > 0$ (Uniform scour).

From Eq 3

$$\frac{\partial q_s}{\partial X} = (1 - \lambda)\ c$$

or $$q_s = (1 - \lambda)\ cX + c_2 \qquad (6)$$

in which $c_2$ is the integral constant.

3. The condition of $\partial Z/\partial t = c\ (t)$, $c > 0$ (Uniform deposition).

The solution for this condition is obtained by changing the sign of $c$ in Eq 6, or

$$q_s = -(1 - \lambda)\ cX + c_3 \qquad (7)$$

in which $c_3$ is the integral constant (See Fig. 4).

4. The condition of $\partial Z/\partial t = -(A_1 - B_1X)$, $(A_1, B_1 \geqq 0, X < A_1/B_1)$ (Linear scour).

From Eq 3

$$\frac{\partial q_s}{\partial X} = (1 - \lambda)\ (A_1 - B_1X) \qquad (8)$$

Integrating Eq 8 yields

$$q_s = (1 - \lambda)(A_1X - \frac{B_1}{2}X^2) + c_4 \qquad (9)$$

in which $c_4$ is the integral constant.

5. The condition of $\partial Z/\partial t = B_1X - A_1$, $X > A_1/B_1$. The equation for this condition is the same as for the condition of linear scour defined by Eq 9. Fig. 5 also represents this case.

6. The conditions of uniform scour, linear scour, and linear deposition are combined as follows:

$$\partial Z/\partial t = -c\ (t) \quad \text{for} \quad X \leqq X_o \ \text{(Uniform scour)}.$$

$$\partial Z/\partial t = -(A_1 - B_1X) \ \text{for} \ X_o < X < A_1/B_1 \ \text{(Linear scour)}.$$

$\partial Z/\partial t = B_1 X - A_1$ for $X > A_1 B_1$ (Linear deposition).

The expressions giving $q_s$ in this case are as follows:

$$q_s = (1 - \lambda)\, cX + c_2 \quad \text{for} \quad X < X_o$$
$$q_s = (1 - \lambda)(A_1 X - \frac{B_1 X^2}{2}) \quad \text{for } X > X_o \tag{10}$$

The distribution of sediment transport for conditions stipulated in 1 through 6 are illustrated schematically in Figs. 2, 3, 4, 5, and 6. The direction of sediment movement is in the direction of the arrow on each figure.

Limitations of Application of the Continuity Equation - The foregoing mathematical analysis has considered both the physically possible and the physically impossible cases of scour, deposition, and scour and deposition. Therefore, it is essential that the physically possible cases, which have a bearing on the analysis of scour as presented in this report be identified.

First, since $q_s$ is a function of $c_n$ -- integral constant -- the origin must be excluded as a zone of consideration; for if the constant of integration has a finite value, $c_n > 0$, then $q_s$ has a finite value at the origin. To circumvent this discrepancy it is assumed that in the vicinity of the stagnation point ($X = 0$) the constant $c_n$ changes rapidly from 0 at $X = 0$ to $c_n$ at $X = X_o$. This assumption is valid since the constant of integration denotes the initial sediment discharge carried into the zone under consideration by the incoming flow. Due to the finite width of the jet, the discharge cannot be introduced at a single point. Furthermore, since the origin is a point of stagnation, the sediment discharge at the origin must be zero because the velocity is zero.

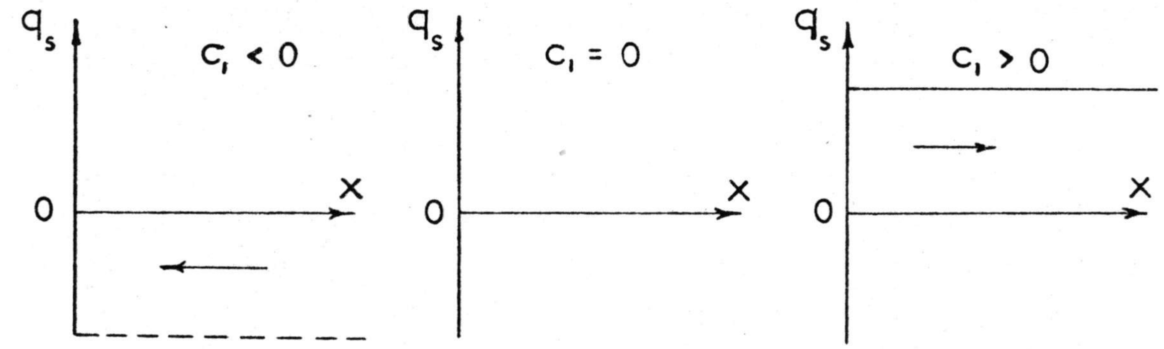


Fig. 2 Distribution of $q_s$ in the case of no scour

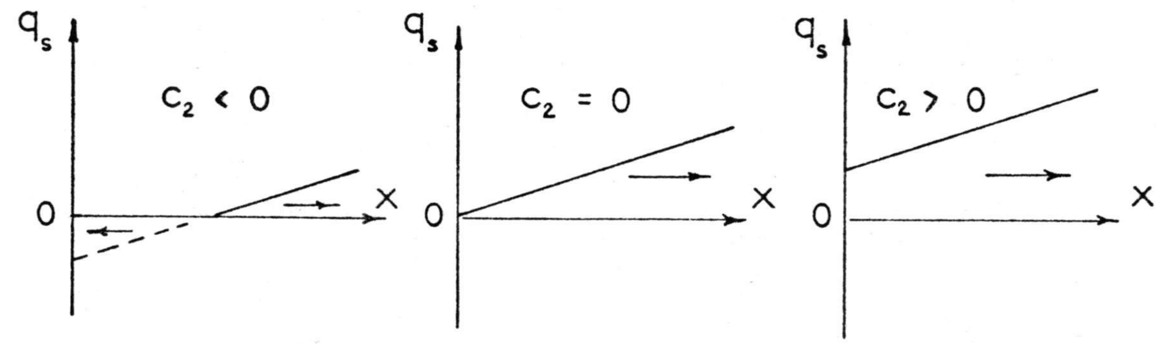


Fig. 3 Distribution of $q_s$ in the case of uniform scour

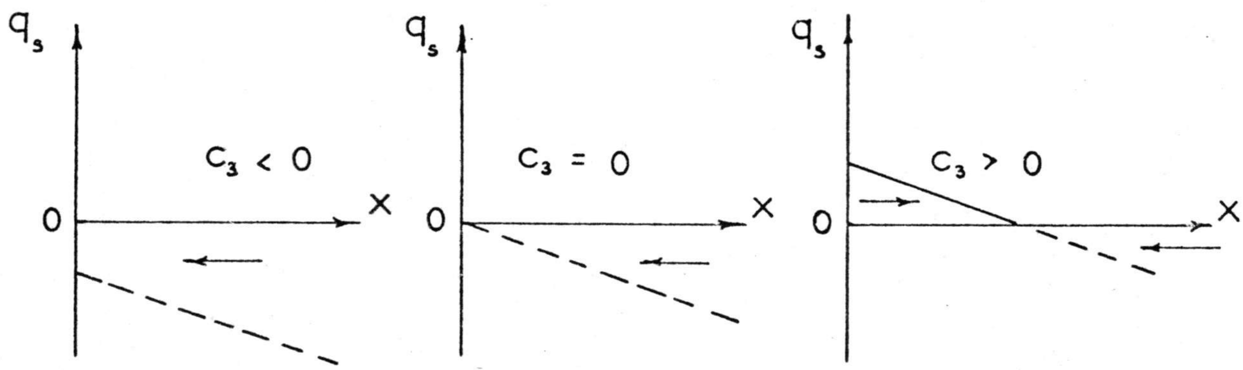


Fig. 4 Distribution of $q_s$ in the case of uniform deposition

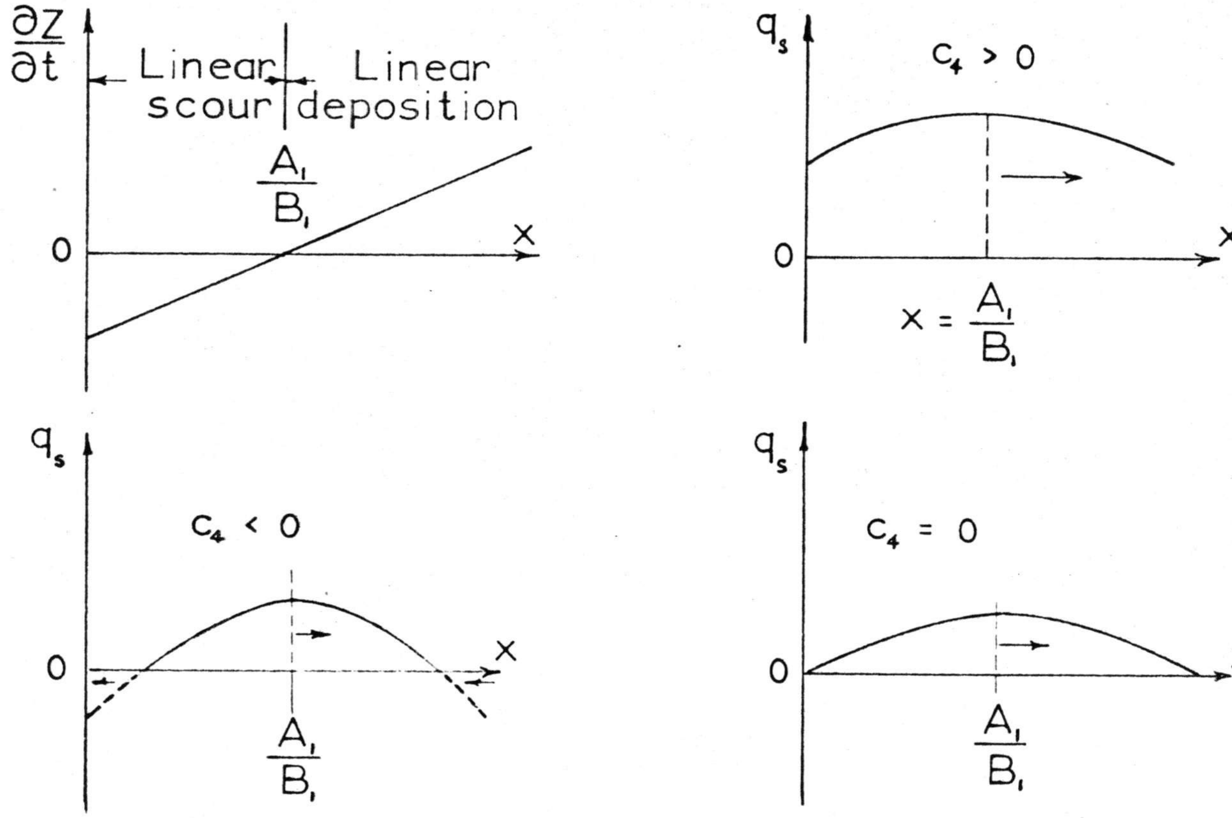


Fig. 5 Distributions of $q_s$ and $\partial z/\partial t$ in the cases of linear scour and linear deposition

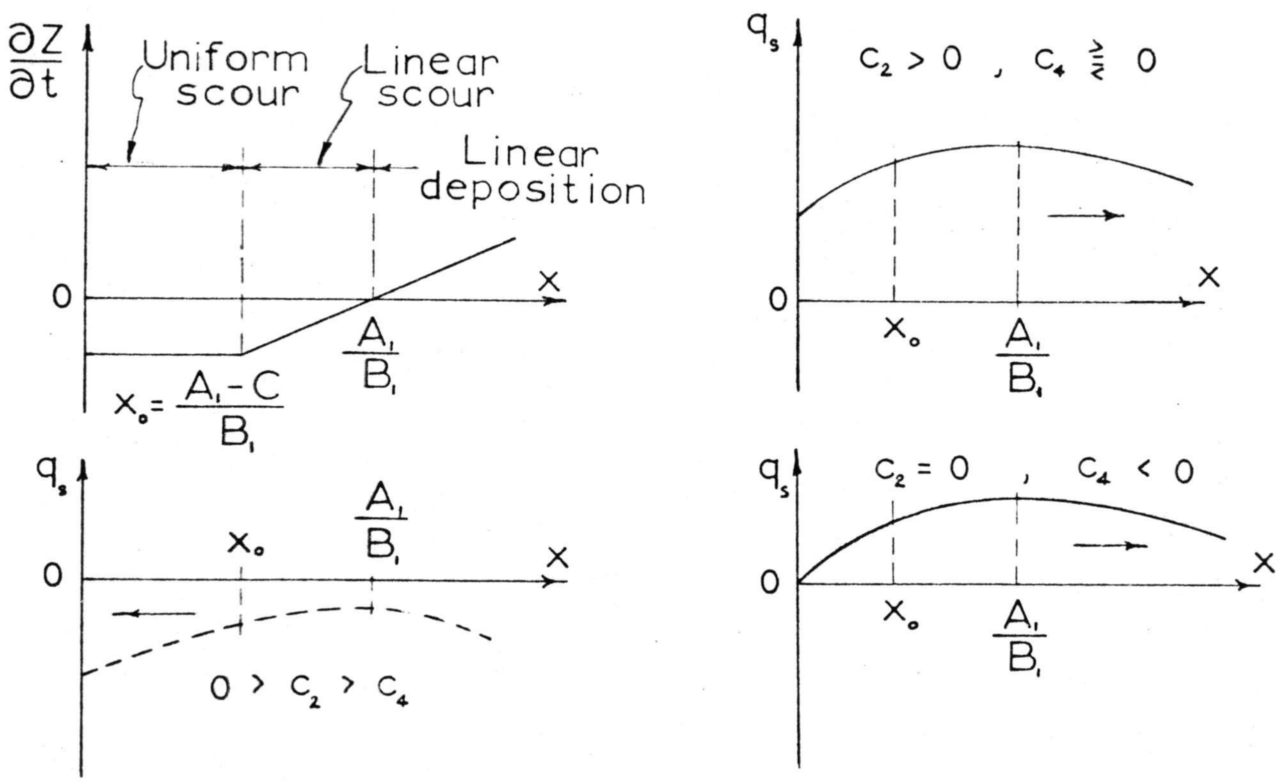


Fig. 6 Distributions of $q_s$ and $\partial z/\partial t$ in the combination of uniform and linear scour and linear deposition

Second, since the constant of integration denotes the initial sediment discharge, it can never become negative as long as the jet is impinging on the boundary. Therefore, all of the foregoing cases in which the direction of sediment movement is toward the oirigin, $c_n \leq 0$, are not relevant to the analysis of this report.

Third, the sediment discharge cannot change sign. That is, as shown in Fig. 4 for the case $c_3 > 0$, only that quantity of sediment contained by the flow can be deposited. Also, if the initial discharge is zero, no sediment can be deposited unless sediment has been scoured from the region near the origin.

IMPINGEMENT OF A TWO-DIMENSIONAL JET OF IDEAL FLUID ON A NORMAL, PLANE BED

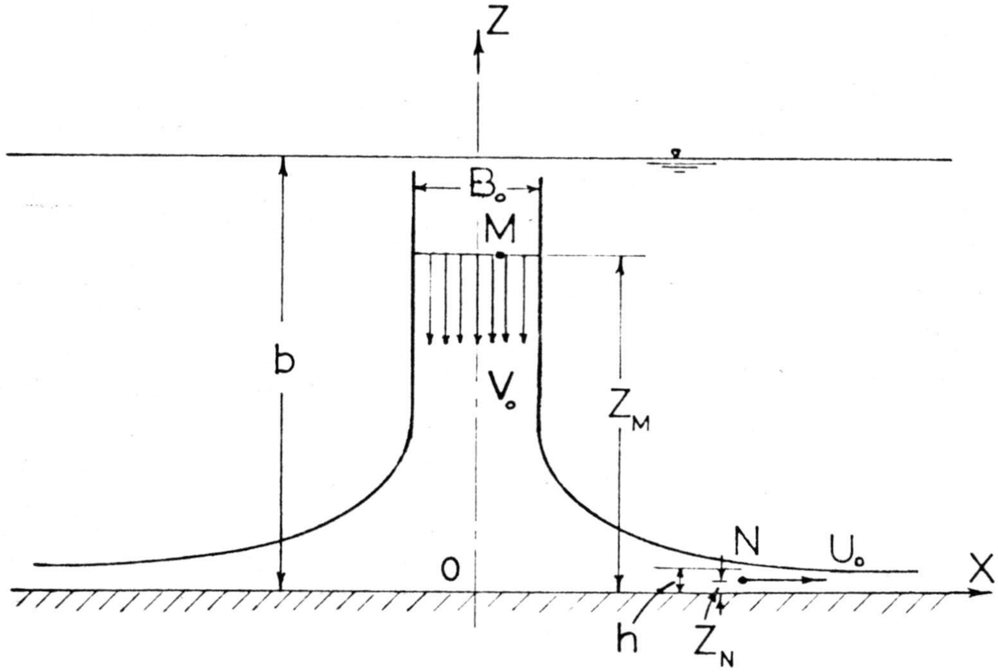


Fig. 7 Schematic drawing of the impingement of a two-dimensional jet of ideal fluid on a normal, plane bed

Ideal Fluid - The impingement of a two-dimensional jet on a non-erodible bed covered by a tailwater of depth b is considered for the case of ideal flow, that is, flow without diffusion or boundary shear (See Fig. 7).

For this case, the Bernoulli Equation applied between points M and N in Fig. 7 gives

$$\frac{V_o^2}{2g} + Z_M + (b - Z_M) = \frac{U_o^2}{2g} + Z_N + (b - Z_n)$$

in which $V_o$ is the velocity component of the jet at a distance $Z_M$ from the bed and $U_o$ is the horizontal velocity component of the deflected jet at a point N located at a great distance from the stagnation point O and at a distance $Z_N$ from the bed.

Bernoulli's equation shows that

$$V_o = U_o \tag{11}$$

From the continuity equation of water mass flux

$$V_o B_o = 2\ U_o\ h \tag{12}$$

and Eq 11, the following relation for the thickness of the deflected jet h is derived:

$$h = B_o/2 \tag{13}$$

This relationship is valid only in the region where the streamlines are parallel to the bed.

The flow within the stagnation region can be approximated by two-dimensional, potential flow with stagnation. For this flow, the vertical and horizontal velocity components V and U can be expressed according to Schlichting (16), by

$$V = -a_o\ Z \tag{14}$$

$$U = a_o\ X \tag{15}$$

in which $a_o$ is a constant. Eqs. 14 and 15 satisfy the equation of continuity

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Z} = 0 \tag{16}$$

For ideal flow, Eq 11 expresses the velocity distribution along the bed where the streamlines are essentially parallel to the bed; whereas, Eq 15 expresses the velocity distribution near the stagnation point. If it is assumed that there is a smooth transition between the two cases, then the velocity distribution along the bed would be as shown in Fig. 8.

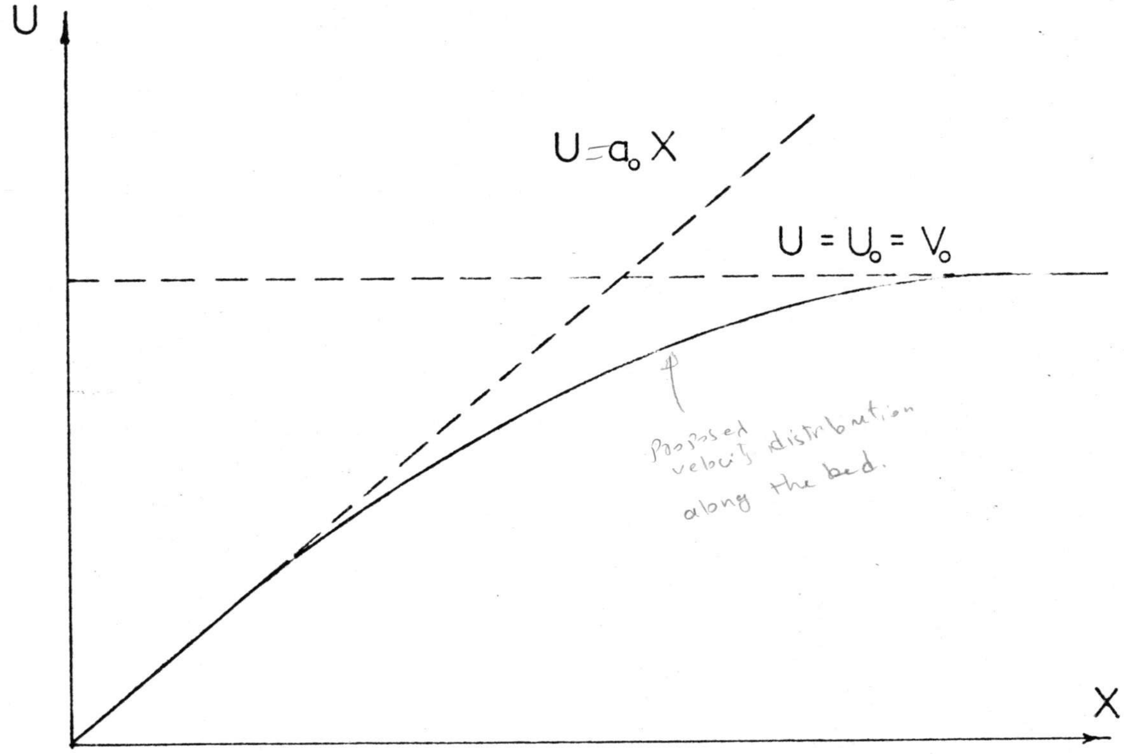


Fig. 8 Distribution of the velocity component U along the bed

IMPINGEMENT OF A NON-SUBMERGED, TWO-DIMENSIONAL JET OF REAL FLUID ON A PLANE BED.

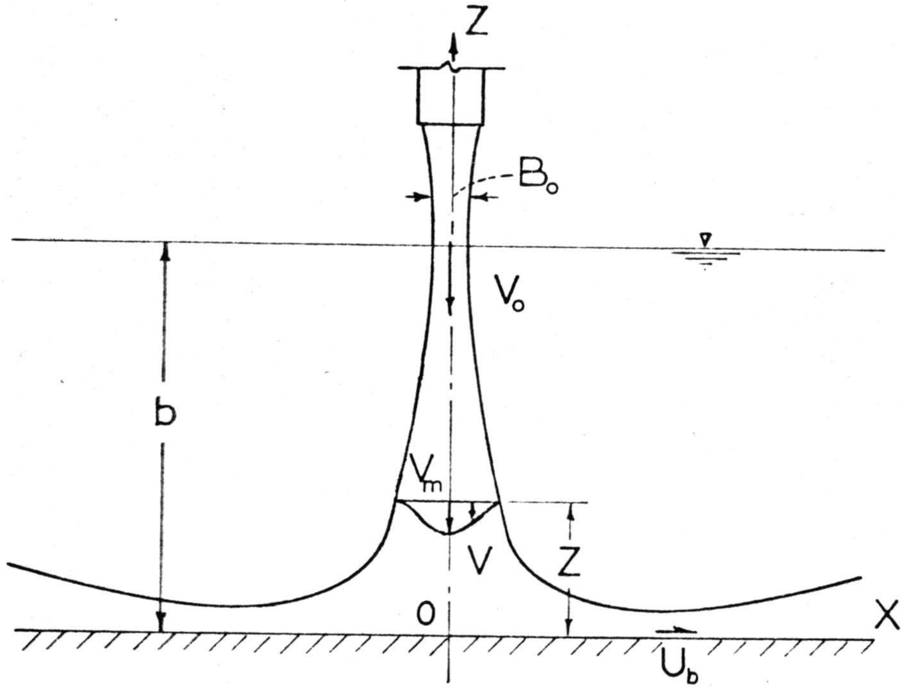


Fig. 9 Schematic drawing of impingement of a non-submerged, two-dimensional jet of real fluid on a normal, plane bed

For real flow (which has viscosity) the analysis of the flow conditions is more complex than for an ideal fluid. The jet of real fluid impinging upon a bed covered by a tailwater of depth b gives rise two types of flow phenomena. One is the development of a boundary layer along the bed through boundary shear; and the other is jet diffusion or more specifically, the development of kinetic energy of turbulence through interaction between the jet and the tailwater.

Boundary Layer: Laminar Flow - For an analysis of the development of the laminar boundary layer use will be made of the results of the experimental study made by L. Howarth (17), who, from an analysis of his experimental data, determined the following relationship

$$\left[\frac{d\ (u/U)}{d\ (\sqrt{a_o/\nu}\ \ Z)}\right]_{Z\ =\ 0} = 1.233 \tag{17}$$

in which $u$ is the velocity in the laminar boundary layer, and $\nu$ is the kinematic viscosity.

Eq 17 can be rewritten as

$$\left[\frac{1}{U}\sqrt{\frac{\nu}{a_o}}\left(\frac{du}{dZ}\right)\right]_{Z\ =\ 0} = 1.233 \tag{18}$$

By definition

$$\tau_o\ (X) = \rho\ \nu\ (du/dZ)_{Z\ =\ 0} \tag{19}$$

Substituting the expression for shear velocity

$$U_*^2 = \tau_o/\rho \tag{20}$$

into Eq 19 gives

$$U_*^2 = \nu\ (du/dZ)_{Z\ =\ 0}\ , \tag{21}$$

and

substituting Eq 18 into Eq 21 yields

$$U_*^2 = 1.233\ (\nu\ a_o)^{\frac{1}{2}}\ U \tag{22}$$

Since $U = a_oX$, Eq 22 can be expressed in dimensionless terms as follows:

$$U_*^2/\nu\ a_o = 1.233\ (a_oX^2/\nu)^{\frac{1}{2}} \tag{23}$$

which shows that $U_*$ is proportional to the horizontal distance $X^{\frac{1}{2}}$ in the case of the laminar boundary layer.

Boundary Layer: Turbulent Flow - For an analysis of the turbulent boundary layer use will be made of the results of the experimental study made by Truckenbrodt (8), who determined the following momentum equation for the boundary layer

$$\tau_o/\rho = \frac{d}{dX}\left[\delta\ U^2 \int_o^1 u/U\ (1-u/U)\ ds\right] + \delta U \frac{dU}{dX} \int_o^1 (1-u/U) ds \quad (24)$$

in which $s = Z/\delta$ and $u$ is the horizontal velocity component in the boundary layer of thickness $\delta$ as shown in Fig. 10.

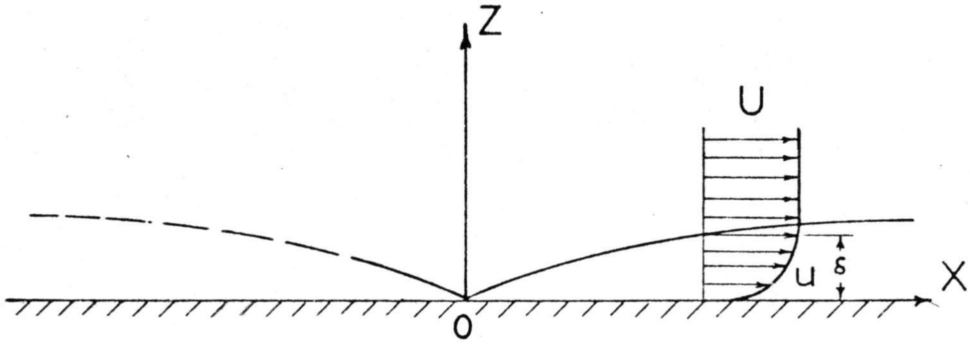


Fig. 10 Boundary layer along the bed

Assume that for a smooth boundary that the velocity profile in the boundary layer can be expressed by

$$u/U = f\ (Z/\delta) = s^{m'} \quad (25)$$

and the shear velocity by

$$\tau_o/\rho\ U^2 = k/(U\delta/\nu)^n \quad (26)$$

in which

$$n = 2m'/1 + m'$$

and $n = 1/4$ when $k = 0.0225$ and $m' = 1/7$.

Substituting the assumptions of Eq 25 and 26 into Eq 24 gives

$$A\ \frac{d\delta}{dX} + (A + \frac{B}{2})\ \frac{\delta}{U^2}\ \frac{dU^2}{dX} = \frac{k}{\left(\frac{U\delta}{\nu}\right)^n} \quad (27)$$

in which

$$A = \int_o^1 s^{m'}\ (1 - s^{m'})\ ds \quad (28)$$

$$B = \int_0^1 (1 - s^{m'})\, ds \tag{29}$$

Integrating Eqs 28 and 29 for $m' = 1/7$ yields

$$A = \frac{m'}{(m' + 1)(2m' + 1)} = \frac{7}{72} \tag{30}$$

$$B = \frac{m'}{m' + 1} = \frac{1}{8} \tag{31}$$

Letting

$$\delta_* = \delta/X \tag{32}$$

Eq 24 becomes

$$AX\, \delta_*^{\,n} \frac{d\delta_*}{dX} + A\, \delta_*^{\,n+1} + \left(A + \frac{B}{2}\right) \delta_*^{\,n+1} \frac{X}{U^2} \frac{dU^2}{dX} = \frac{k}{\left(\frac{UX}{\nu}\right)^n} \tag{33}$$

Since

$$U = a_o X$$

Eq 33 reduces to

$$AX\, \delta_*^{\,n} \frac{d\delta}{dX} + (3A + B)\, \delta_*^{\,n+1} = \frac{k}{\left(\frac{a_o X^2}{\nu}\right)^n} \tag{34}$$

As determined by Truckenbrodt (8), the solution of Eq 34 is

$$\delta_* = \gamma \left(\frac{a_o X^2}{\nu}\right)^{-\frac{1}{1+n}} \tag{35}$$

in which the value of $\gamma$ is determined by substituting Eq 35 into Eq 34 or

$$\gamma = \left[\frac{k}{\left(\frac{n+3}{n+1}\right) A + B}\right]^{\frac{1}{n+1}} \tag{36}$$

Since $n = 1/4$, $k = 0.0225$, $A = 7/72$ and $B = 1/8$, it is found that

$$\gamma = 0.1045 \tag{37}$$

Combining Eqs 15, 26, and 32 yields

$$\frac{\tau_o}{\rho\, (a_o X^2)^2} = \frac{k}{\left(\frac{a_o X^2}{\nu}\right)^n \delta_*^{\,n}} \tag{38}$$

The expression for the shear velocity is obtained by introducing the solution given by Eq 35 into Eq 38 or

$$\frac{\tau_o}{\rho\,(a_o X^2)^2} = k\;\gamma^{-n}\left(\frac{a_o X^2}{\nu}\right)^{-\frac{n}{1+n}} \tag{39}$$

Substituting values for $n$, $k$, and $\gamma$ in Eq 39 gives

$$U_*^2 = 0.175\quad \nu^{1/5}\; a_o^{9/5}\; X^{8/5} \tag{40}$$

which shows that $U_*^2$ is proportional to $X^{8/5}$

Eq 40 can be written in dimensionless form as follows:

$$\frac{U_*^2}{\nu\, a_o} = 0.175\left(\frac{a_o X^2}{\nu}\right)^{4/5} \tag{41}$$

In summary, the shear velocity $U_*$, for a smooth boundary, is proportional to $X^{1/2}$ for the case of a laminar boundary layer and $X^{4/5}$ for the case of a turbulent boundary layer.

Jet Diffusion - An analysis of jet diffusion for the case of flow from a non-submerged outlet impinging on a plane bed covered by a tailwater surface of depth $b$ (See Fig. 9) has been made by Homma (18). Since Homma's investigation was for the three-dimensional case, it is assumed, as demonstrated by Albertson and others (9), that the vertical velocity distribution at any normal section of the zone of flow is similar to that obtained experimentally by Homma. For the three-dimensional jet, Homma determined the following relationships:

$$\frac{W_m}{W_o} = 1.24\; e^{-0.109\,\left(\frac{b\,-\,Z}{d}\right)} \qquad \text{for}\;\; Re < 2.5 \times 10^4 \tag{42}$$

and

$$\frac{W_m}{W_o} = 1.24\ e^{-0.137\left(\frac{b - Z}{d}\right)} \quad \text{for} \quad Re > 3 \times 10^4 \tag{43}$$

in which

$d$ is the diameter of the jet,

$W_o$ is the mean velocity of the jet at the tailwater surface,

$W_m$ is the maximum velocity component along the centerline of the diffused jet, and

$Re$ the Reynolds number of jet flow.

$W_o$, $d$, and $W_m$ correspond to $V_o$, $B_o$, and $V_m$ for two-dimensional jet flow as shown in preceeding Fig. 9.

Assume that

$$\frac{V_m}{V_o} = \alpha\ e^{\beta\left(\frac{b - Z}{B_o}\right)} \tag{44}$$

in which $\alpha$ and $\beta$ are constants. Fig. 11 gives for $\alpha$ the value of 1, and for $\beta$ the value of 0.1.

For the velocity distribution across any normal section of a flow zone of the diffusing jet, the following dimensionless relationship is assumed to be applicable:

$$\frac{V}{V_m} = \phi_1\left(\frac{b - Z}{B_o},\ \frac{X}{b},\ \frac{b}{B_o}\right) \tag{45}$$

in which the function $\phi_1$ must satisfy the following boundary conditions

$$\frac{X}{b} = 0\ , \quad \phi_1 = 1 \tag{46}$$

$$\frac{X}{b} \rightarrow \infty, \quad \phi_1 = 0 \tag{47}$$

$$\frac{b - Z}{B_o} \rightarrow \infty, \quad \phi_1 = 0 \tag{48}$$

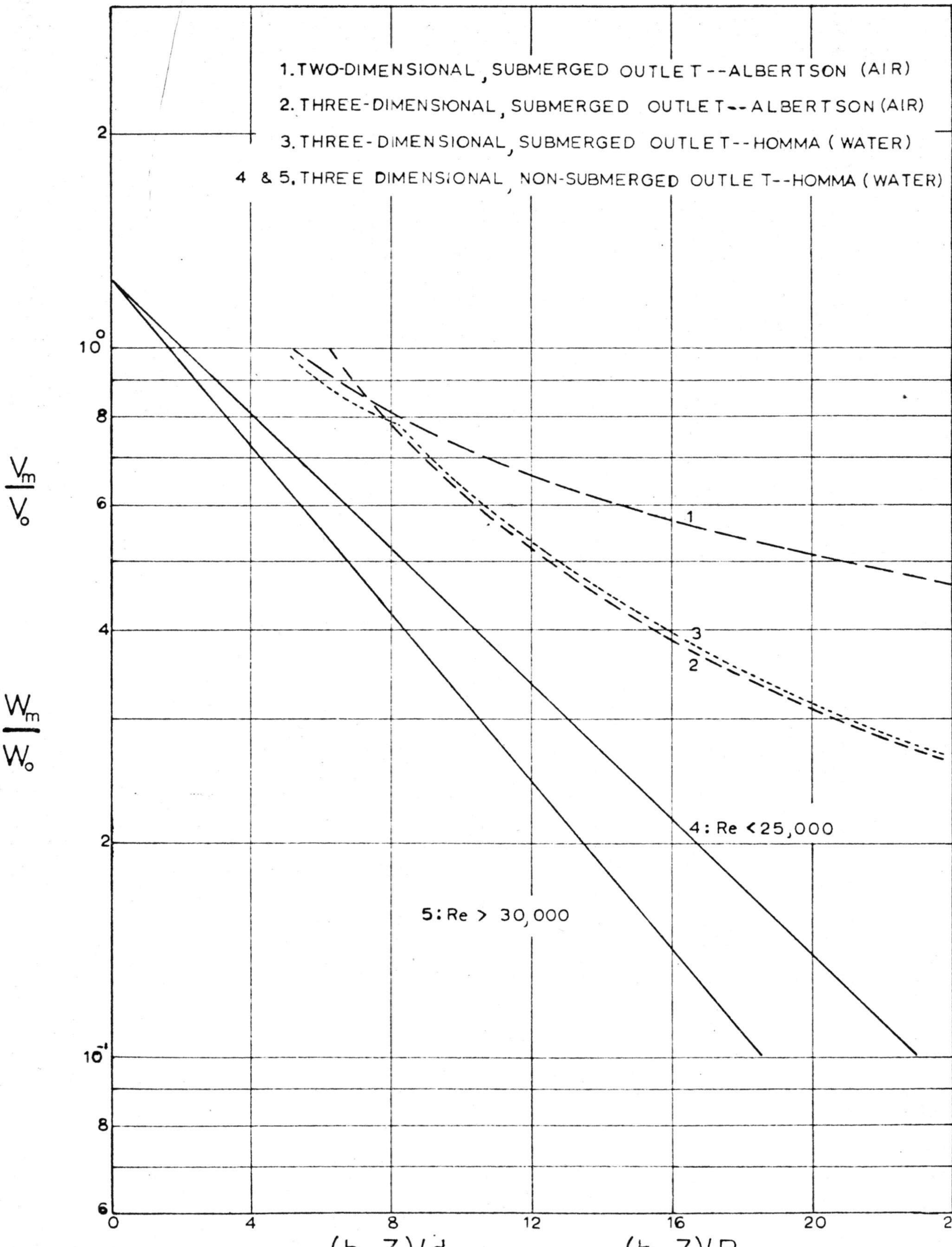


Fig. 11 Deceleration of maximum velocity of diffused jet

For the horizontal velocity distribution along the bed, it is assumed that Bernoulli's equation is applicable along the stagnation stream line. This assumption gives

$$\frac{\rho}{2}\ (U^2 + V^2) + p + \rho g Z = p_o \tag{49}$$

The horizontal velocity component $U_b$ along OX, where V and Z are insignificant, is, according to Eq 49, given by the relation

$$\frac{\rho}{2}\ U_b{}^2 = p_o - p \tag{50}$$

By analogy with the results of Homma's study, it is assumed that the pressure distribution along the X - axis is given by the relation

$$p = \frac{\rho}{2}\ V_b{}^2 + \rho g b \tag{51}$$

in which $V_b$ is the value of V at Z = 0 for the free jet.

The stagnation pressure is given by

$$p_o = \frac{\rho}{2}\ V_{bm}{}^2 + \rho g b \tag{52}$$

Subtracting Eq 51 from Eq 52 obtain

$$p_o - p = \frac{\rho}{2}\ (V_{bm}{}^2 - V_b{}^2) \tag{53}$$

From Eqs 50 and 53 an expression for $U_b$ in dimensionless terms is obtained, or

$$\frac{U_b}{V_{bm}} = \left[1 - \left(\frac{V_b}{V_{bm}}\right)^2\right]^{\frac{1}{2}} \tag{54}$$

in which $V_b/\ V_{bm}$ is determined from Eq 45, for Z = 0, as

$$\frac{V_b}{V_{bm}} = \phi_2\ \left[\frac{b}{B_o}\ ,\ \frac{X}{b}\right]\ = \phi_o \tag{55}$$

Substituting Eq 55 into Eq 54 yields

$$\frac{U_b}{V_{bm}} = (1 - \phi_o{}^2)^{\frac{1}{2}} \tag{56}$$

Eq 56 shows that $U_b \rightarrow V_{bm}$ as $X \rightarrow \infty$ and $\phi_o \rightarrow 0$. This result is not valid since the lateral diffusion of the jet is not taken into consideration. Nevertheless, Eq 55 is assumed to give a correct value for $U_b/V_{bm}$ in the proximity of the stagnation point, and will be used in the following section in computing the boundary layer thickness and shear velocity.

Boundary Layer Development and Shear Velocity - Eq 33 can be written

$$AX\,\delta_*^{\,n}\,\frac{d\delta_*}{dX} + A\,\delta_*^{\,n+1} + \left(A + \frac{B}{2}\right)\delta_*^{\,n+1}\,L = \frac{k}{\left(\frac{U_b X}{\nu}\right)^n} \tag{57}$$

in which

$$L = \frac{X}{U_b^2}\,\frac{dU_b^2}{dX} \tag{58}$$

Eq 57 can be solved only when $L$ is a constant, that is, when $U_b$ is determined by an expression of the following type:

$$U_b = c_5\,X^{m''} \tag{59}$$

in which $c_5$ and $m''$ are constants.

If $L$ is not constant, but if $\phi_o$ in the relation given by Eq 56 is known, Eq 57 can be solved by numerical integration.

Assuming $L$ constant, the solution of Eq 57 can be written as

$$\delta_* = \gamma' \left(\frac{U_b X}{\nu}\right)^{-\frac{n}{1+n}}$$

which is in the same form as Eq 35.

The value of $\gamma'$ is obtained by substituting Eq 60 into Eq 57 or

$$\gamma' = \left[\frac{k}{\frac{L}{2}\left(\frac{2+n}{1+n}A + B\right) + \frac{A}{1+n}}\right]^{\frac{1}{n+1}} \tag{61}$$

Eqs 60 and 61 give the value of the expression for the dimensionless boundary layer thickness $\delta_* = \delta/X$.

An expression for the shear velocity is obtained from Eq 59, which can be rewritten as follows:

$$\frac{U_*^2}{U_b^2} = \frac{k}{\left(\frac{U_b X}{\nu}\right)^n \delta_*^n} \tag{62}$$

Substituting Eq 60 into Eq 62 yields

$$\frac{U_*^2}{U_b^2} = k\,\gamma'^{-n} \left(\frac{U_b X}{\nu}\right)^{-\frac{n}{1+n}} \tag{63}$$

Thus, by introducing the value of $U_b$ given by Eq 56 and the values 1/4 and 0.0225 for $n$ and $k$ respectively into Eq 63, one obtains for the shear velocity $U_*$ the expression

$$\frac{U_*^2}{V_{bm}^2} = 0.0225\ \gamma'^{-\frac{1}{4}} \left(\frac{V_{bm} X}{\nu}\right)^{-\frac{1}{5}} (1 - \phi_o^2)^{\frac{9}{10}} \tag{64}$$

If the function $\phi_o$ is known, the shear velocity distribution can be determined.

## IMPINGEMENT OF A SUBMERGED, TWO-DIMENSIONAL JET OF REAL FLUID ON A NORMAL, PLANE BED

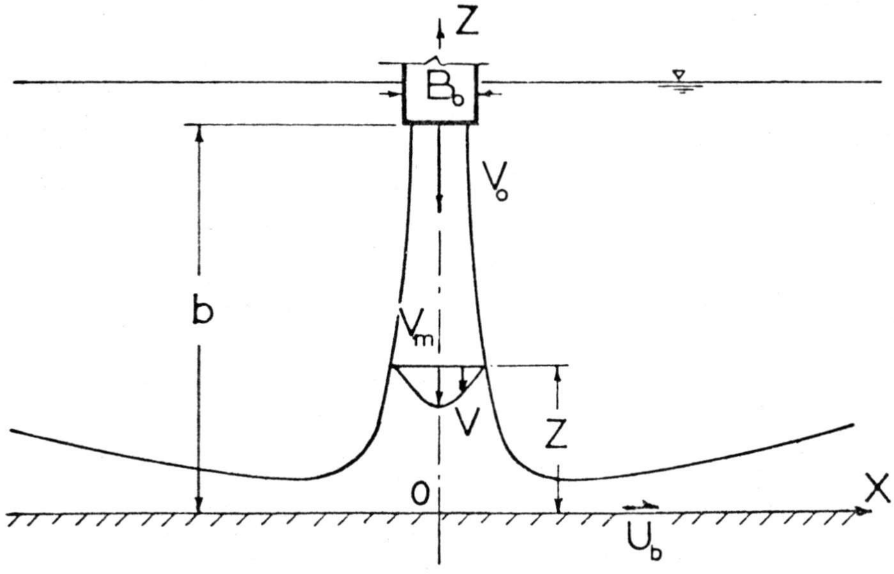


Fig. 12 Schematic drawing of impingement of a submerged, two-dimensional jet of real fluid on a normal, plane bed

Diffusion of Submerged Jet Before Impingement - According to the study by Albertson and others (9) on a two-dimensional, submerged jet without a deflecting boundary, an expression for the distribution of the center-line velocity $V_m$ is given by

$$\frac{V_m}{V_o} = 2.28 \left(\frac{B_o}{b - Z}\right)^{\frac{1}{2}} \tag{65}$$

and for the distribution of longitudinal velocity $V$ in zone of established flow by

$$\frac{1}{2.28} \frac{V}{V_o} \left(\frac{b - Z}{B_o}\right)^{\frac{1}{2}} = e^{-4.24 \frac{X^2}{(b - Z)^2}} \tag{66}$$

Combining Eqs 65 and 66 yields

$$\frac{V}{V_m} = e^{-4.24 \frac{X^2}{(b - Z)^2}} \tag{67}$$

in which $b$ is the distance from the outlet to the boundary as shown in Fig. 12 Equating Eqs 45 and 67 one obtains

$$\phi_1 \left(\frac{b - Z}{B_o}, \frac{X}{b}, \frac{b}{B_o}\right) = e^{-4.24 \left(\frac{X^2}{b - Z}\right)^2} \tag{68}$$

Equating Eqs 55 and 68 gives

$$\phi_o = e^{-4.24 \left(\frac{X^2}{b - Z}\right)^2} \tag{69}$$

An expression for the velocity distribution along the bed -- deflecting boundary -- by means of Eqs 56 and 69 is as follows:

$$\frac{U_b}{V_{bm}} = \left[1 - e^{-8.48 \left(\frac{X}{b}\right)^2}\right]^{\frac{1}{2}} \tag{70}$$

Eq 70 is plotted in Fig. 13 and is assumed to give a correct value for Eq 70 in the proximity of the stagnation point. The dashed line in the figure indicates a more probable velocity distribution when lateral diffusion is considered.

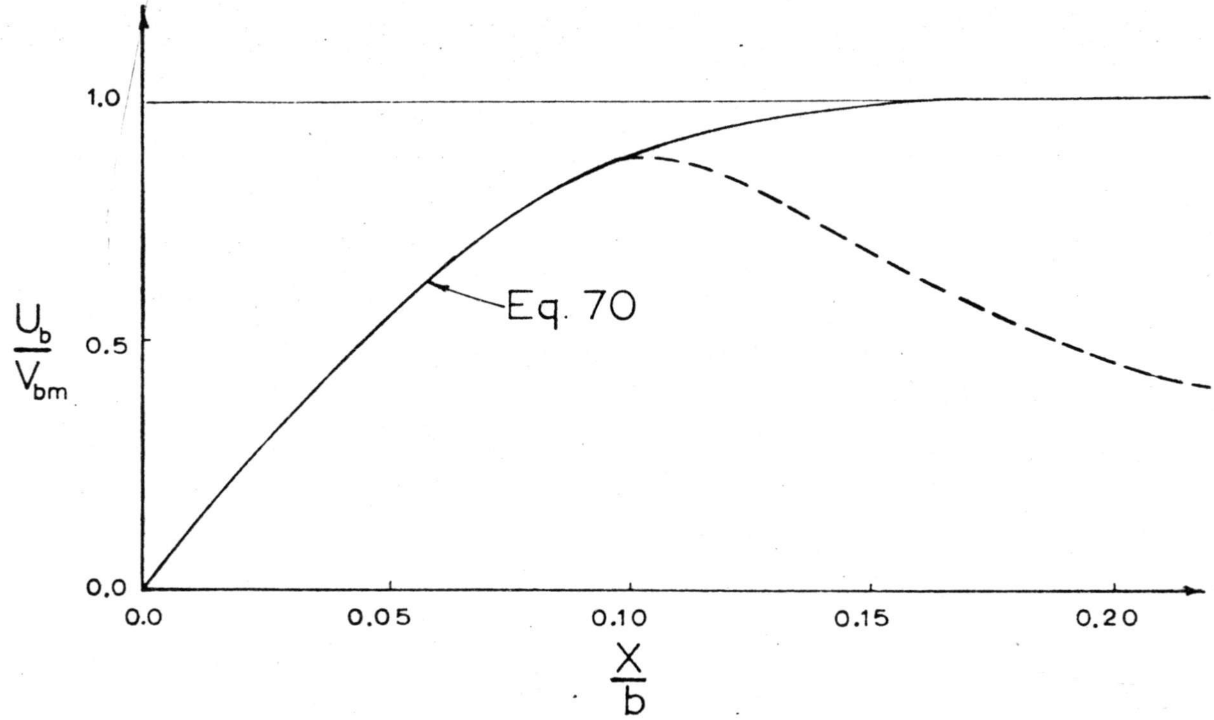


Fig. 13 Horizontal velocity distribution along the bed.

Boundary Layer Thickness and Shear Velocity - Again by means of Eqs 57 and 58, expressions for the boundary layer thickness and shear velocity may be obtained for the case of the submerged outlet. However, in order to determine an expression for L, Eq 70 can be substituted into Eq 58 giving

$$L = 16.96 \left(\frac{X}{b}\right)^2 \frac{e^{-8.48\left(\frac{X}{b}\right)^2}}{1 - e^{-8.48\left(\frac{X}{b}\right)^2}} \tag{71}$$

The variation of L with X/b is given in Fig. 14, which shows that, in the vicinity of the stagnation point, L is approximately constant and equal to 2; whereas for values of X/b > 0.8, L tends rapidly to zero.

Assuming L constant, which is a valid assumption for the proximity of the stagnation point, the boundary layer thickness is given by Eqs 60 and 61, and the shear velocity by Eq 64.

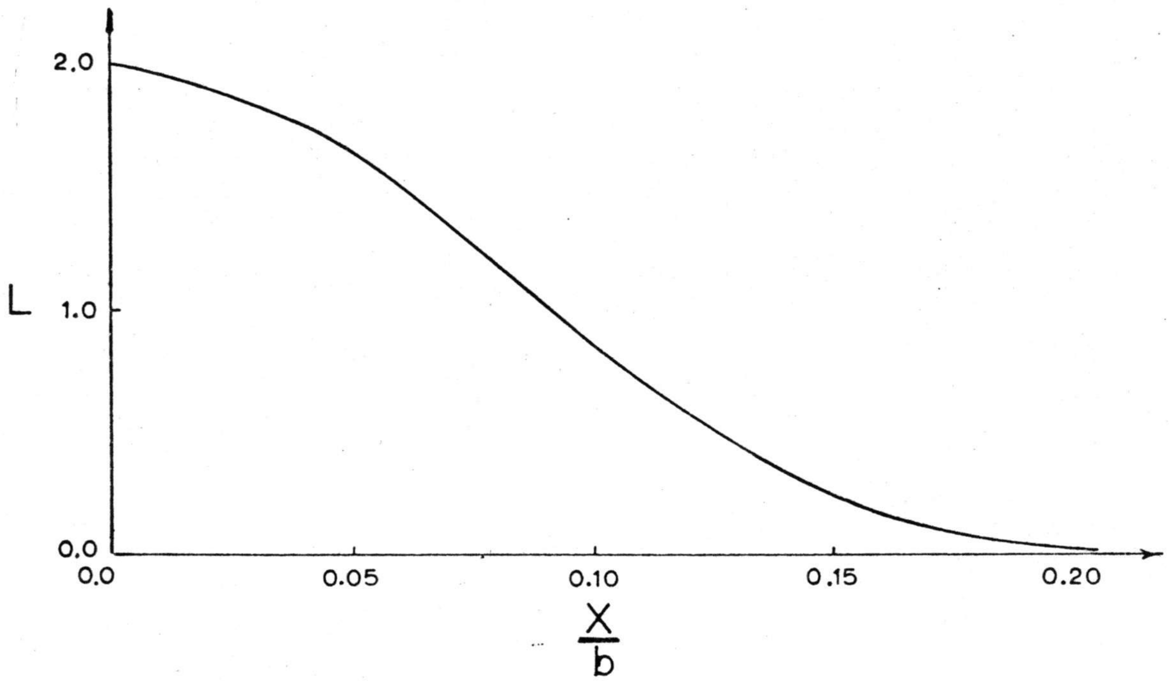


Fig. 14 Variation of L with X/b

Substituting Eq 70 into Eq 64 yields

$$\frac{U_*^2}{V_{bm}^2} = 0.0225\ \gamma'^{-1/4} \left(\frac{V_{bm}X}{\nu}\right)^{-1/5} \left[1 - e^{-8.48\left(\frac{X}{b}\right)^2}\right]^{9/10} \qquad (72)$$

Fig. 15 represents the shear velocity distribution given by Eq 72.

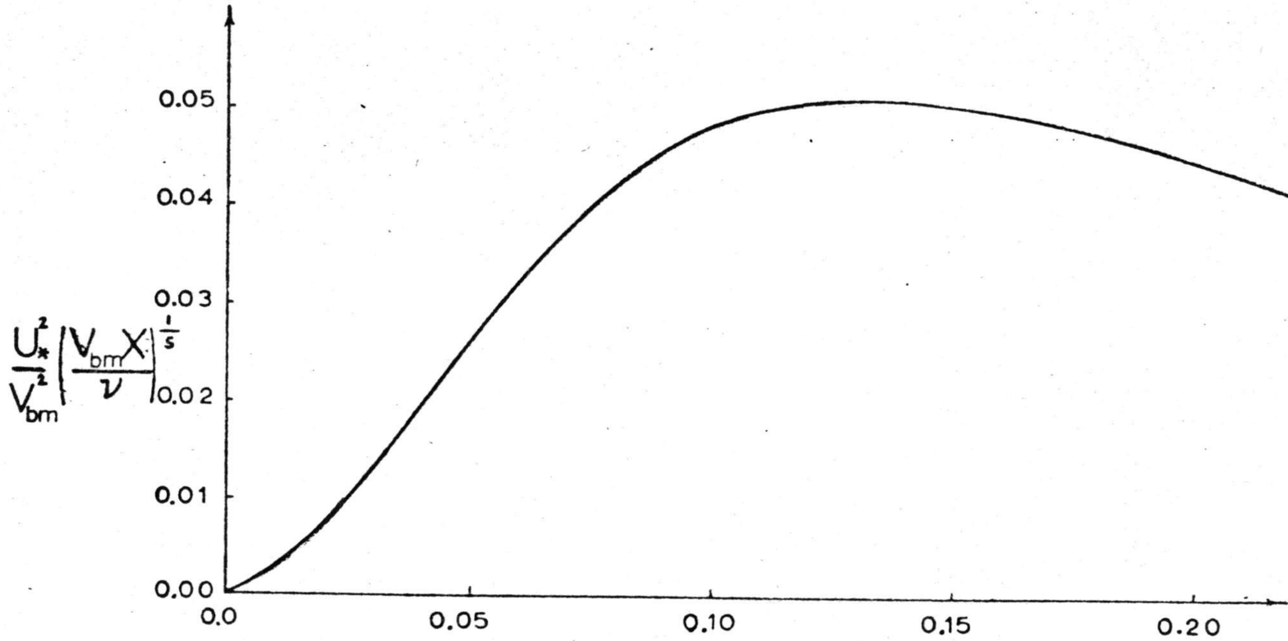


Fig. 15 Shear velocity distribution along the plane bed

IMPINGEMENT OF A NON-SUBMERGED, TWO-DIMENSIONAL, VERTICAL JET OF REAL FLUID ON AN ERODIBLE BED

Rate of Sediment Transport - Brown and Laursen (10), in their analysis of bed-load transport theories as given by Shields (11) and Kalinske (12), show that the rate of sediment transport $q_s$ in open channel flow can be expressed in the following form

$$\frac{q_s}{U_* \, d_s} = A_o \left[\frac{U_*^2 - U_{*c}^2}{V_s^2}\right]^m \tag{73}$$

and

$$V_s^2 = \left(\frac{\sigma}{\rho} - 1\right) g \, d_s \tag{74}$$

In Eqs 73 and 74, $U_{*c}$ is the critical shear velocity, $\sigma$ and $d_s$ the mass density and mean diameter of sediment particles respectively, and $A_o$ and $m$ are constants which are, in general, the function of $U_*^2/V_s^2$. Thus, when $U_*$ is of the same order of magnitude as $U_{*c}$, the value of $m$ increases and the value of $A_o$ decreases with decreasing values of $U_*/V_s$.

When $U_*$ is much larger than $U_{*c}$, $U_* >> U_{*c}$, Eq 73 simplifies to

$$\frac{q_s}{U_* d_s} = A_o \left[\frac{U_*}{V_s}\right]^{2m} \tag{75}$$

<u>Integral Expression for Depth of Scour</u> - Substituting Eq 64 into Eq 73 gives

$$\frac{q_s}{V_{bm}\, d_s} = A_o\, k^{1/2}\, \gamma'^{-1/8} \left[\frac{V_{bm}}{\alpha V_o}\ \frac{\alpha\, V_o b}{\nu}\ \frac{X}{b}\right]^{-1/10} \left[1 - \phi_2^{\,2}\right]^{9/20}$$

$$\cdot \left[\frac{V_{bm}}{V_s}\right]^{2m} \left\{k\, \gamma'^{-1/4} \left[\frac{V_{bm}\, X}{\nu}\right]^{-1/5} (1 - \phi_2^{\,2})^{9/10} - \left(\frac{U_{*c}}{V_{bm}}\right)^2\right\}^m \tag{76}$$

The following quantities are now defined

$$R = k^{1/2}\, \gamma'^{-1/8} \left[\frac{\alpha\, V_o\, b}{\nu}\ \ \frac{X}{b}\right]^{-1/10} (1 - \phi_2^{\,2})^{9/20} \tag{77}$$

$$R' = dR'/d\,(X/b) \tag{78}$$

$$\eta = \frac{9}{10}\,(2m + 1)^2\ \frac{A_o\, \beta}{(1 - \lambda)}\ (\alpha\, F_o)^{2m + 1}$$

$$\cdot \left(\frac{V_o}{V_s}\right)^{2m} R'\, R^{2m} \left(\frac{d_s}{B_o}\ \frac{V_o t}{b}\right) \tag{79}$$

$$a = \frac{U_{*c}}{\alpha\, V_o}\ \frac{1}{R} \tag{80}$$

$$a' = \frac{a}{\sqrt{2m + 1}} \tag{81}$$

$$F = e^{\frac{9}{10}\beta \frac{Z - b}{B_o}} \quad \left(dF/F = \frac{9}{10}\ \frac{\beta}{B_o}\ dZ\right) \tag{82}$$

$$F_o = e^{\frac{9}{10}\beta \frac{b}{B_o}} \tag{83}$$

Eq 76 with the notations of Eq 78 can be written as follows:

$$q_s = A_o\, d_s \frac{V_{bm}}{V_s^{2m}}^{\frac{18m+9}{10}} (\alpha V_o)^{\frac{2m+1}{10}} R \left[R^2 - \frac{U_{*c}^2}{(\alpha V_o)^{1/5} (V_{bm})^{9/5}}\right]^m \tag{84}$$

The equation of continuity of sediment transport given by Eq 3 can be written as follows:

$$(1-\lambda)\frac{\partial Z}{\partial t} + \frac{1}{b}\frac{\partial q_s}{\partial (X/b)} = 0 \tag{85}$$

The derivative of $q_s$ with respect to $X/b$ is obtained from Eq 84, or

$$\frac{\partial q_s}{\partial (X/b)} = A_o\, d_s \frac{V_{bm}}{V_s^{2m}}^{\frac{18m+9}{10}} (\alpha V_o)^{\frac{2m+1}{10}} R' \left[R^2 - \frac{U_{*c}^2}{(\alpha V_o)^{1/5} (V_{bm})^{9/5}}\right]^{m-1} \cdot \left[(2m+1) R^2 - \frac{U_{*c}^2}{(\alpha V_o)^{1/5} (V_{bm})^{9/5}}\right] \tag{86}$$

Introducing the notations of Eqs 80 and 81 into Eq 86 one obtains

$$\frac{\partial q_s}{\partial (X/b)} = A_o\, d_s \frac{V_{bm}}{V_s^{2m}}^{\frac{9}{10}} (\alpha V_o)^{\frac{2m+1}{10}} R'\; R^{2m} (2m+1) \cdot \left[\left(\frac{V_{bm}}{V_o}\right)^{9/5} - a^2\right]^{m-1} \cdot \left[\left(\frac{V_{bm}}{\alpha V_o}\right)^{9/5} - a'^2\right] \tag{87}$$

The value of $V_{bm}$ is given by Eq 44, which can be written:

$$\frac{V_{bm}}{\alpha V_o} = e^{\beta \frac{Z-b}{B_o}} \tag{88}$$

or, with the notation of Eq 82, as follows:

$$\frac{V_{bm}}{\alpha V_o}^{9/10} = F \tag{89}$$

Substituting Eq 89 into Eq 87 gives

$$\frac{\partial q_s}{\partial (X/b)} = A_o \, d_s \, \frac{(\alpha \, V_o)^{2m+1}}{V_s^{2m}} \, F \, R' \, R^{2m} \, (2m + 1) \cdot (F^2 - a^2)^{m-1} \cdot (F^2 - a'^2) \tag{90}$$

Noting that $F$ is a function of $Z$, substitute Eq 90 into Eq 85 and separate the variables which gives

$$\frac{dZ}{F(F^2 - a^2)^{m-1} \, (F^2 - a'^2)} + (2m + 1) \, \frac{A_o \, d_s}{(1 - \lambda) b} \, \frac{(\alpha \, V_o)^{2m+1}}{V_s^{2m}} \, R' \, R^{2m} \, dt = 0 \tag{91}$$

Integrating Eq 91 for given limits, one obtains

$$\int_o^Z \frac{dZ}{F(F^2 - a^2)^{m-1}(F^2 - a'^2)} + \frac{(2m + 1) \, A_o \, d_s}{(1 - \lambda) \, b} \cdot \frac{(\alpha \, V_o)^{2m+1}}{V_s^{2m}} \, R' \, R^{2m} \int_o^t dt = 0 \tag{92}$$

Eq 92 is valid only for $Z \ll b$. If this condition is not fulfilled, then $b$ in the right-hand term of Eq 92 should be replaced by $(b - Z)$; that is, the group $R'(b) \, R^{2m} \, (b)/b$ would be replaced by $R'(b - Z) \, R^{2m} \, (b - Z)/(b - Z)$, which would then be included in the first integral as a function of $Z$. Substituting Eqs 79, 82, and 83 into Eq 92, one obtains

$$\int_{F_o}^{F} \frac{dF}{F^2(F^2 - a^2)^{m-1} \, (F^2 - a'^2)} + \frac{\eta}{(2m + 1) \, F_o^{2m+1}} \tag{93}$$

<u>Variation of Scour Depth for $U_* \gg U_{*c}$</u> - For the case $U_* \gg U_{*c}$, the quantities $a$ and $a'$, which are proportional to $U_{*c}/V_o$, are negligible

and Eq 93 reduces to

$$\int_{F_o}^{F} \frac{dF}{F^{2m+2}} + \frac{\eta}{(2m+1)\,F_o^{2m+1}} = 0 \tag{94}$$

Integrating Eq 94, one obtains

$$\frac{1}{2m+1}\left[F_o^{-(2m+1)} - F^{-(2m+1)}\right] + \frac{\eta}{(2m+1)\,F_o^{2m+1}} = 0 \tag{95}$$

or

$$1 - \left(\frac{F_o}{F}\right)^{2m+1} + \eta = 0 \tag{96}$$

Putting $Z_s = -Z$ and using the expression for $F$ and $F_o$ given by Eqs 82 and 83 respectively, Eq 96 becomes

$$e^{\frac{9}{10}\beta\,(2m+1)\,\frac{Z_s}{B_o}} = 1 + \eta \tag{97}$$

or

$$\frac{Z_s}{B_o} = \frac{10}{\beta\,9(2m+1)}\,\ln\,(1+\eta)$$

in which $\eta$ is a function of the following dimensionless parameters (noting that $m$ and $A_o$ are both functions of $U_*^2/(\sigma/\rho - 1)\,gd_s$):

$$\left(\frac{V_o^2}{(\frac{\sigma}{\rho} - 1)gd_s}\;;\;\frac{X}{b}\;;\;\frac{d_s}{B_o}\;;\;\frac{V_o t}{b}\;;\;\frac{V_o b}{\nu}\;;\;\frac{b}{B_o}\right)$$

The scour depth $Z_s$ given by Eq 97 is an approximation and is valid only when $Z \ll b$.

<u>Final Depth of Scour $Z_{s\infty}$</u> - The final depth of scour $Z_{s\infty}$ will be attained when $U_* = U_{*c}$. For this case $q_s = 0$ and Eq 84 reduces to

$$R = \frac{U_{*c}}{(V_{bm})^{9/10}\,(\alpha\,V_o)^{1/10}} \tag{98}$$

Expressing $V_{bm}$ as a function of $Z$, as given by Eq 88, Eq 98 becomes

$$e^{\beta \frac{9}{10} \frac{Z_{s\infty} + b}{B_o}} = \frac{R \alpha V_o}{U_{*c}}$$

or

$$\frac{Z_{s\infty} + b}{B_o} = \beta \frac{10}{9} \ln \left[\frac{R \alpha V_o}{U_{*c}}\right] \qquad (99)$$

in which $R$ is given by Eq 77.

A better approximation of $Z_{s\infty}$ is obtained if $Z_{s\infty}$ and $b$ are of the same order of magnitude; if $\phi_o$ is written

$$\phi_o \left(\frac{Z_{s\infty} + b}{B_o}, \ \frac{X}{b + Z_s}\right)$$

instead of

$$\phi_o \left(\frac{b}{B_o}, \ \frac{X}{b}\right) \quad ;$$

and if Eq 99 is solved by trial and error

<u>Example of Integration When $U_* = \phi_2 (U_{*c})$</u> - If $m = 2$ then Eq 93 becomes

$$\int_{F_o}^{F} \frac{dF}{F^2(F^2 - a^2)(F^2 - a'^2)} + \frac{\eta}{5F_o^5} = 0 \qquad (100)$$

To integrate the left term, the fraction is first reduced into its several integral parts. Substituting $a'^2 = a^2/5$, as given by Eq 81, obtain

$$\frac{5}{a^4} \int_{F_o}^{F} \frac{dF}{F^2} + \frac{5}{4a^4} \int_{F_o}^{F} \frac{dF}{F^2 - a^2} - \frac{25}{4a^4} \int_{F_o}^{F} \frac{dF}{F^2 - \frac{a^2}{5}} + \frac{\eta}{5F_o^5} = 0 \qquad (101)$$

Integration of Eq 101 gives

$$\frac{a}{F_o} - \frac{a}{F} + \frac{1}{8} \ln \left[\frac{F - a}{F + a} \ \frac{F_o + a}{F_o - a}\right] - \frac{5\sqrt{5}}{8} \ln \left[\frac{F - \frac{a}{\sqrt{5}}}{F + \frac{a}{\sqrt{5}}} \ \frac{F_o + \frac{a}{\sqrt{5}}}{F_o - \frac{a}{\sqrt{5}}}\right]$$

$$+ \frac{a}{F_o} \ \frac{5\eta}{25} = 0 \qquad (102)$$

If the value of F aproaches the value for a, the third term ln ( ) in Eq 102 approaches $-\infty$, and the value of t must tend to $+\infty$. This implies that the limiting value of the depth of scour $Z_{s\infty}$ is reached only after infinite time. For m = 2 Eq 96 can be obtained from Eq 102 by expanding its terms in a logarithmic series.

Since F = a corresponds to $t \longrightarrow \infty$ in Eq 102, it is possible to compute the time necessary to reach the point where F, a function of $Z/B_o$, is equal to 99 per cent of a. Furthermore, it is to be noted that $Z/B_o$ is a function of the flow and sediment characteristics.

For m = 2, Eq 79 becomes

$$\eta = \frac{10}{9}(25)\,\beta\,\frac{A_o}{1-\lambda}(\alpha F_o)^5\left(\frac{V_o}{V_s}\right)^4 R'\,R^4\,\frac{d_s}{B_o}\left(\frac{V_o t}{b}\right) \tag{103}$$

The last term of Eq 102 can be written

$$\left(\frac{a}{F_o}\right)^5\frac{\eta}{25} = \frac{10}{9}\,\beta\,\frac{A_o}{1-\lambda}\,(\alpha\, a)^5\,\frac{V_o}{V_s}^4\,R'\,R^4\,\frac{d_s}{B_o}\,\frac{U_{*c}t}{b} \tag{104}$$

Substituting Eq 80 into Eq 104 yields

$$\left(\frac{a}{F_o}\right)^5\frac{\eta}{25} = \frac{10}{9}\,\beta\,\frac{A_o}{1-\lambda}\left(\frac{U_{*c}}{V_s}\right)^4\frac{R'}{R}\,\frac{d_s}{B_o}\,\frac{U_{*c}t}{b} \tag{105}$$

Define now the quantity

$$\frac{1}{\eta'} = \frac{10}{9}\,\beta\,\frac{A_o}{1-\lambda}\left(\frac{U_{*c}}{V_s}\right)^4\frac{R'}{R}\,\frac{d_s}{B_o} \tag{106}$$

With the notation of Eq 106, Eq 105 substituted into Eq 102 gives

$$\frac{a}{F_o} - \frac{1}{0.99} + \frac{1}{8}\ln\left[\frac{0.01}{1.99}\,\frac{F_o+a}{F_o-a}\right] - \frac{5\sqrt{5}}{8}\ln\left[\frac{0.99\sqrt{5}-1}{0.99\sqrt{5}+1}\right]\left[\frac{\sqrt{5}\,F_o+a}{\sqrt{5}\,F_o-a}\right]$$

$$\frac{1}{\eta'}\left[\frac{U_{*c}t}{b}\right]_{0.99} = 0 \tag{107}$$

or

$$\left[\frac{U_*}{b}\right]_{0.99} = \eta'\left\{0.307 + 3.22 \log \left[\frac{\sqrt{5} + a/F_o}{\sqrt{5} - a/F_o}\right] - 0.29 \log \left[\frac{1 + a/F_o}{1 - a/F_o}\right] - \left(\frac{a}{F_o}\right)\right\} \tag{108}$$

It is to be noted that

$$\frac{a}{F_o} < \frac{a}{F} = \frac{1}{0.99} \simeq 1$$

IMPINGEMENT OF A SUBMERGED, TWO-DIMENSIONAL, VERTICAL JET OF REAL FLUID ON AN ERODIBLE BED

Rate of Sediment Transport - The rate of sediment transport for the submerged jet is the same as for the non-submerged jet.

Integral Expression for Depth of Scour - In a manner similar to the case of the non-submerged jet, the following quantities are now defined.

$$S = [illegible]\ \gamma''^{-\frac{1}{8}} \left[\frac{2.28\ V_o b}{\nu}\ \ \frac{X}{b}\right]^{-\frac{1}{10}} \left[1 - e^{-8.48\ \frac{X}{b}^{2}}\right]^{\frac{9}{20}} \tag{109}$$

$$S' = \frac{dS}{d(X/b)} \tag{110}$$

$$\xi = \frac{9}{20}(2m + 1)\frac{A_o}{1 - \lambda}(2.28)^{2m + 1}\left(\frac{V_o}{V_s}\right)^{2m} S'\ S^{2m}\ \frac{d_s}{B_o}\ \frac{V_o t}{b} \tag{111}$$

$$\beta = \frac{U_{*c}}{2.28\ V_o}\ \frac{1}{S} \tag{112}$$

$$\alpha = \frac{C}{\sqrt{2m + 1}} \tag{113}$$

$$G = \left(\frac{B_o}{b - Z}\right)^{\frac{9}{20}};\ dZ = \frac{20}{9}\ B_o\ \frac{dG}{G^{29/9}} \tag{114}$$

$$G_{[illegible]} = \left(\frac{B_o}{b}\right)^{\frac{9}{20}} \tag{115}$$

Eq 87 with $\alpha = 2.28$, $a = C$, and $a' = C'$ becomes

$$\frac{\partial q_s}{\partial (X/b)} = A_o\, d_s \left(\frac{V_{bm}}{2.28\, V_o}\right)^{\frac{9}{10}} \frac{(2.28\, V_o)^{2m+1}}{V_s^{2m}}\, S'\, S^{2m}\, (2m + 1)$$

$$\cdot \left\{\left[\frac{V_{bm}}{2.28\, V_o}\right]^{\frac{9}{5}} - C^2\right\}^{m-1} \left\{\left[\frac{V_{bm}}{2.28\, V_o}\right]^{\frac{9}{5}} - C'^2\right\} \tag{116}$$

Eq 65 expressing $V_{bm}$ as a function of $Z$ can be written

$$\frac{V_{bm}}{2.28\, V_o} = \sqrt{\frac{B_o}{b - Z}} \tag{117}$$

or with notation of Eq 114

$$\left[\frac{V_{bm}}{2.28\, V_o}\right]^{\frac{9}{10}} = G \tag{118}$$

Substituting Eq 118 into Eq 116 gives

$$\frac{\partial q_s}{\partial (X/b)} = A_o\, d_s \frac{(2.28\, V_o)^{2m+1}}{V_s^{2m}}\, S'\, S^{2m}\, (2m + 1) G (G^2 - C^2)^{m-1}$$

$$\cdot\, (G^2 - C'^2) \tag{119}$$

If Eq 119 is substituted into Eq 85, in which $dZ$ is given by Eq 114, one obtains by integration the expression.

$$\frac{20}{9}\, B_o \int_{G_o}^{G} \frac{dG}{G^{\frac{38}{9}} (G^2 - C^2)^{m-1} (G^2 - C'^2)}$$

$$+ \frac{(2m + 1)\, A_o\, d_s}{(1 - \lambda)\, b} \frac{(2.28\, V_o)^{2m+1}}{V_s^{2m}}\, S'\, S^{2m} \int_o^t dt = 0 \tag{120}$$

Substituting Eq 111 into Eq 120 yields

$$\int_{G_o}^{G} \frac{dG}{G^{\frac{38}{9}} (G^2 - C^2)^{m-1} (G^2 - C'^2)} + \xi = 0 \tag{121}$$

which is the integral expression for depth of scour

Variation of Scour Depth for $U_* \gg U_{*c}$ - For the case of $U_* \gg U_{*c}$, the quantities C and C' are much smaller than G and Eq 121 becomes

$$\int_{G_o}^{G} \frac{dG}{G^{2m + 38/9}} + \xi = 0 \qquad (122)$$

Integrating Eq 122, one obtains

$$\frac{1}{(2m + 29/9)} \left[\frac{1}{G_o^{2m + 29/9}} - \frac{1}{G^{2m + 29/9}}\right] + \xi = 0 \qquad (123)$$

Substituting Eqs 114 and 115 into Eq 123, and denoting the actual depth of scour by $Z_s = -Z$ , gives the equation

$$\frac{b + Z_s}{B_o} = \left\{\left[2m + \frac{29}{9}\right]\xi + \left(\frac{b}{B_o}\right)^{\frac{18m + 29}{20}}\right\}^{\frac{20}{18m + 29}} \qquad (124)$$

in which $\xi$ is a function of the following parameters (noting that m and $A_o$ are functions of $U_*^2/\left[\sigma/\rho - 1\right] g d_s$ ):

$$\left[\frac{V_o^2}{(\frac{\sigma}{\rho} - 1)\, g\, d_s}\; ; \; \frac{X}{b} \; ; \; \frac{d_s}{B_o} \; ; \; \frac{V_o t}{b} \; ; \; \frac{V_o b}{\nu} \; ; \; \frac{b}{B_o}\right]$$

Eq 124 is applicable only for $Z_s \ll b$.

Final Depth of Scour $Z_{s\infty}$ - The final depth of scour $Z_{s\infty}$ will be attained when $U_* = U_{*c}$. For this case $q_s = 0$ and an equation similar to Eq 98 is applicable, or

$$S = \frac{U_{*c}}{(V_{bm})^{9/10}(2.28\ V_o)^{1/10}} \qquad (125)$$

Substituting Eq 117 into Eq 112 gives

$$\left[\frac{b + Z_{s\infty}}{B_o}\right]^{9/20} = S\,\frac{2.28\ V_o}{U_{*c}}$$

or

$$\frac{b + Z_{s\infty}}{B_o} = \left[\frac{S\ 2.28\ V_o}{U_{*c}}\right]^{20/9} \qquad (126)$$

in which

$$S = k^{1/2}\ \gamma'^{\ -1/8} \left[\frac{2.28\ V_o\ X}{\nu}\right]^{-1/10} \left[1 - e^{-8.48\ (X/b)^2}\right]^{9/20}$$

A closer approximation for $Z_{s\infty}$ is obtained if the last term of $S$ is written with the exponent $e^{-8.48\ (X/b\ +\ Z_{s\infty})^2}$ instead of $e^{-8.48(X/b)^2}$, particularly when $b$ and $Z_{s\infty}$ are of the same order of magnitude.

<u>Example of Integration</u> - When $U_*$ is of the same order of magnitude as $U_{*c}$, and $m = 2$, then the integral of Eq 121 is approximately

$$\int_{G_o}^{G} \frac{dG}{G^4\ (G^2 - C^2)(G^2 - C'^2)} + \xi = 0 \tag{122}$$

in which the exponent $38/9 = 4.22$ has been replaced by 4.00 in the denominator of the integral.

Separating the fraction of the first term of Eq 122 into its several integral parts, and with $C'^2 = C^2/5$, one obtains

$$\frac{5}{C^4}\int_{G_o}^{G} \frac{dG}{G^4} + \frac{30}{C^6}\int_{G_o}^{G} \frac{dG}{G^2} + \frac{5}{4C^6}\int_{G_o}^{G} \frac{dG}{G^2 - C^2} - \frac{125}{4C^6}\int_{G_o}^{G} \frac{dG}{G^2 - \frac{C^2}{5}} + \xi = 0 \tag{123}$$

Integration of Eq 123 gives

$$\frac{1}{3}\left[\frac{C^3}{G_o^3} - \frac{C^3}{G^3}\right] + 6\left[\frac{C}{G_o} - \frac{C}{G}\right] + \frac{1}{8}\ln\left[\frac{G - C}{G + C}\ \frac{G_o + C}{G_o - C}\right] - \frac{25\sqrt{5}}{8}$$

$$\ln\left[\frac{G - \frac{C}{\sqrt{5}}}{G + \frac{C}{\sqrt{5}}}\ \frac{G_o + \frac{C}{\sqrt{5}}}{G_o - \frac{C}{\sqrt{5}}}\right] + \frac{C^7}{5}\ \xi = 0 \tag{124}$$

Eq 124 gives the general relation between $F(Z)$ and $\xi(t)$, when $t \to \infty$, $G \to C$, and $\ln\ (G-C) \longrightarrow -\infty$.

To determine the time of scour necessary to attain the condition $G = 0.99C$, it is necessary to rewrite the last term of Eq 124. With $n = 2$ and $\xi$ as defined by Eq 111, one can obtain

$$\xi = \frac{9}{20} (5) \frac{A_o}{1 - \lambda} (2.28)^5 \left(\frac{V_o}{V_s}\right)^4 S' S^4 \frac{d_s}{B_o} \frac{V_o t}{b} \tag{125}$$

which, when substituted into the last term of Eq 124, gives

$$\frac{C^7}{5} \xi = \frac{9}{20} \frac{A_o}{1 - \lambda} (2.28)^5 \left(\frac{V_o}{V_s}\right)^4 S' S^4 \frac{d_s}{B_o} \frac{V_o t}{b} C^7 \tag{126}$$

Substituting Eq 112 into the right hand side of Eq 126 yields

$$\frac{C^7}{5} \xi = \frac{9}{20} \frac{A_o}{1 - \lambda} \left(\frac{U_{*c}}{2.28 V_o}\right)^2 \left(\frac{U_{*c}}{V_s}\right)^4 \frac{S'}{S} \frac{1}{S^2} \cdot \frac{d_s}{B_o} \frac{U_{*c} t}{b} \tag{127}$$

Define the quantity

$$\frac{1}{\xi_1} = \frac{9}{20} \frac{A_o}{1 - \lambda} \left(\frac{U_{*c}}{2.28 V_o}\right)^2 \left(\frac{U_{*c}}{V_s}\right)^4 \frac{S'}{S} \frac{1}{S^2} \frac{d_s}{B_o} \tag{128}$$

so that Eq 127 becomes

$$\frac{C^7}{5} \xi = \frac{1}{\xi_1} \left[\frac{U_* t}{b}\right]_{0.99} \tag{129}$$

Substituting $G = 0.99\,C$ and Eq 129 into Eq 124, one obtains

$$\frac{1}{3}\left[\left(\frac{C}{G_o}\right)^3 - \left(\frac{1}{0.99}\right)^3\right] + 6\left[\frac{C}{G_o} - \frac{1}{0.99}\right] + \frac{2.3}{8}\left\{\log \frac{0.01}{1.99} + \log\left[\frac{G_o + C}{G_o - C}\right]\right\}$$

$$- \frac{25\sqrt{5}}{8}\, 2.3 \left\{\log\left[\frac{0.99\sqrt{5} - 1}{0.99\sqrt{5} + 1}\right] + \log\left[\frac{G_o + C/\sqrt{5}}{G_o - C\sqrt{5}}\right]\right\}$$

$$+ \frac{1}{\xi_1} \left[\frac{U_{*c} t}{b}\right]_{0.99} = 0 \tag{130}$$

or, rearranging and reducing terms yields

$$\left[\frac{U_{*c}t}{b}\right]_{0.99} = \xi_1\left\{0.25 - 0.33\left(\frac{C}{G_o}\right)^3 - 6\left(\frac{C}{G_o}\right) + 16.1 \log\left[\frac{\sqrt{5} + \frac{C}{G_o}}{\sqrt{5} - \frac{C}{G_o}}\right] - 0.29 \log\left[\frac{1 + \frac{C}{G_o}}{1 - \frac{C}{G_o}}\right]\right\} \tag{131}$$

It is to be noted that

$$\frac{C}{G_o} < \frac{C}{G} = \frac{1}{0.99} \simeq 1$$

IMPINGEMENT OF A TWO-DIMENSIONAL INCLINED JET OF REAL FLUID ON AN ERODIBLE BED

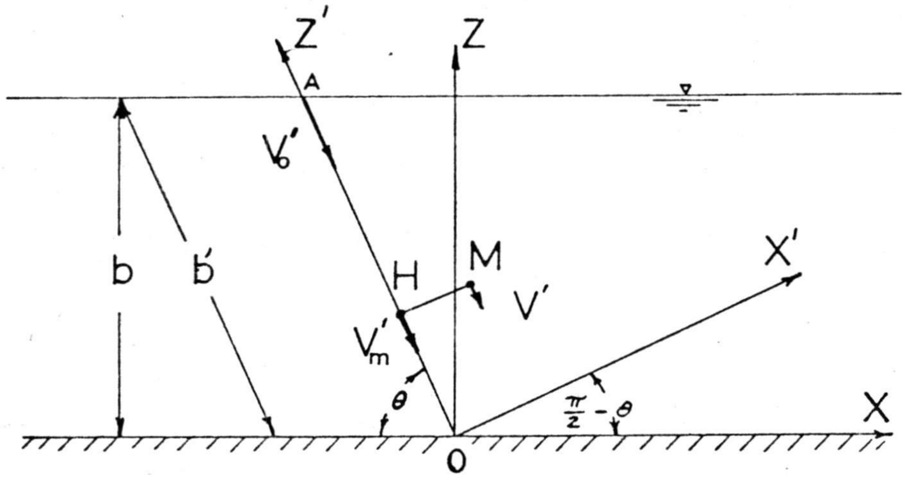


Fig. 16 Coordinate system for an inclined jet

Jet Issuing from a Non-Submerged Outlet - For an inclined jet the Z'-axis and X'- axis are taken along the centerline of the jet and transverse to the jet respectively as shown in Fig. 16. On the vertical plane containing the centerline of the jet, the relationships between the system of coordinates (X, Z) and X',Z') are

$$X' = X \sin\theta + Z\cos\theta \tag{132}$$

$$Z' = -X\cos\theta + Z\sin\theta \tag{133}$$

Since $\overline{AH} = (b'-Z')$ and $\overline{HM} = X'$ in Fig. 16, an expression similar to Eq 44 for the maximum velocity of the jet $V_m'$ along the centerline can be written as follows:

$$\frac{V_m'}{V_o'} = \alpha' \, e^{-\beta\left(\frac{b' - Z'}{B_o}\right)} \tag{134}$$

An expression for the velocity distribution along the transverse section $\overline{HM}$ of the jet can be similarily defined as in Eq 45, by

$$\frac{V'}{V_m'} = \left(\frac{b' - Z'}{B_o}, \ \frac{X'}{b'}, \ \frac{b'}{B_o}\right) \tag{135}$$

Expressions for Eqs 134 and 135 using the relationships of Eqs 132 and 133 as functions of $X$, $Z$, and $\theta$ are as follows:

1. For any point on the vertical plane

$$\frac{V_m'}{V_o'} = \alpha' \exp\left\{-\beta'\left(\frac{b'}{B_o} + \frac{X}{B_o}\cos\theta - \frac{Z}{B_o}\sin\theta\right)\right\} \tag{136}$$

$$\frac{V'}{V_m'} = \Psi\left[\left(\frac{b'}{B_o} + \frac{X_1}{B_o}\cos\theta - \frac{Z}{B_o}\sin\theta\right); \left(\frac{X}{b'}\sin\theta + \frac{Z}{b'}\cos\theta\right); \frac{b'}{B_o}\right] \tag{137}$$

2. For a point along the OX-axis $(Z = 0)$.

$$\frac{V_{bm}'}{V_o'} = \alpha' \exp\left\{-\beta'\left(\frac{b'}{B_o} + \frac{X}{B_o}\cos\theta\right)\right\} \tag{138}$$

$$\frac{V_b'}{V_{bm}'} = \Psi\left[\left(\frac{b'}{B_o} + \frac{X}{B_o}\cos\theta\right); \left(\frac{X}{b'}\sin\theta\right); \frac{b'}{B_o}\right] \tag{139}$$

3. For the stagnation point $(Z = 0$ and $X = 0)$

$$\frac{V_{bmo}'}{V_o'} = \alpha' \exp\left(-\frac{\beta' b'}{B_o}\right) \tag{140}$$

Velocity and Shear Distrubiton Along the Bed - As in the case of a vertical jet, the horizontal velocity component $U_b$ along the X-axis can be expressed by

$$\frac{U_b}{V_{bmo}' \sin \theta} = \left\{ 1 - \left( \frac{V_b'}{V_{bmo}'} \right)^2 \right\}^{1/2} \tag{141}$$

Combining Eqs 138, 139, and 140 , one obtains

$$\frac{V_b'}{V_{bmo}'} = \frac{V_b'}{V_{bm}'} \, \frac{V_{bm}'}{V_o'} \, \frac{V_o'}{V_{bmo}'}$$

$$= \psi \left[ \left( \frac{b'}{B_o} + \frac{X}{B_o} \cos \theta \right) ; \left( \frac{X}{b'} \sin \theta \right) ; \frac{b'}{B_o} \right] \exp \left\{ - \frac{\beta' X}{B_o} \cos \theta \right\} \tag{142}$$

Denoting Eq 142 by

$$\Psi_1 \left( \frac{X}{b} , \theta \right)$$

or

$$\Psi_1 = \Psi \left[ \frac{b'}{B_o} + \frac{X}{B_o} \cos \theta \right) ; \left( \frac{X}{b} \sin \theta \right) ; \frac{b'}{B_o} \right] \exp \left\{ - \frac{\beta' X}{B_o} \cos \theta \right\} \tag{143}$$

Then with the notation of Eqs 142 and 143, Eq 141 becomes

$$\frac{U_b}{V_{bmo}' \sin \theta} = \left\{ 1 - \Psi_1^2 \right\}^{1/2} \tag{144}$$

For the distribution of the shear velocity along the bed, the approach given for the case of a vertical jet is still valid if L in Eq 58 is assumed constant. For this condition Eq 63 can be written

$$\frac{U_*^2}{U_b^2} = k \gamma'^{-n} \left( \frac{U_b X}{\nu} \right)^{-\frac{n}{1+n}} \tag{63}$$

Introducing Eq 144 into Eq 63 and taking n as 1/4 gives

$$\frac{U_*^2}{(V_{bmo}' \sin \theta)^2} = k \gamma'^{-1/4} \left( \frac{V_{bmo}' \sin \theta \, X}{\nu} \right)^{-1/5} (1 - \Psi_1^2)^{9/10} \tag{145}$$

<u>Variation of Scour Depth for $U_* \gg U_{*c}$</u> - Since Eq 75 is an expression for the rate of sediment transport for the case in which $U_* \gg U_{*c}$, Eq 145 may be substituted into Eq 75 to give

$$\frac{q_s}{\alpha' V_o' d_s} = A_o \left[\frac{\alpha' V_o'}{\nu}\right]^{2m} k^{\frac{2m+1}{2}} \gamma'^{-\frac{2m+1}{8}} \left[\frac{V_{bmo}' \sin\theta}{\alpha' V_o'}\right]^{\frac{9(2m+1)}{10}}$$

$$\cdot \left[\frac{\alpha' V_o' X}{\nu}\right]^{-\frac{2m+1}{10}} \left[1 - \Psi_1^{2}\right]^{\frac{9(2m+1)}{20}} \tag{146}$$

Define the quantity $P$ a function of $X/b$ and $\theta$, as

$$P = \alpha' A_o \left[\frac{\alpha' V_o'}{V_s}\right]^{2m} k^{\frac{2m+1}{2}} \gamma'^{-\frac{2m+1}{8}}$$

$$\cdot \left[\frac{\alpha' V_o' b}{\nu} \frac{X}{b}\right]^{-\frac{2m+1}{10}} \left[1 - \psi_1^{2}\right]^{\frac{9(2m+1)}{20}} \tag{147}$$

Eq 146 with the notation of Eq 147, becomes

$$\frac{q_s}{V_o' d_s} = \left[\frac{V_{bmo}' \sin\theta}{\alpha' V_o'}\right]^{\frac{9(2m+1)}{10}} P \tag{148}$$

Let

$$P' = \frac{dP}{d(X/b)} \tag{150}$$

The derivative of $q_s$ with respect to $(X/b)$ is

$$\frac{1}{V_o' d_s} \frac{\partial q_s}{\partial (X/b)} = \left[\frac{V_{bmo}' \sin\theta}{\alpha' V_o'}\right]^{\frac{9(2m+1)}{10}} P' \tag{151}$$

Substituting Eq 151 into Eq 3 gives

$$(1 - \lambda) \frac{\partial Z}{\partial t} + \frac{V_o' d_s}{b} P' \left[\frac{V_{bmo}' \sin\theta}{\alpha' V_o'}\right]^{\frac{9(2m+1)}{10}} = 0 \tag{152}$$

For the case of an erodible bed, $V_{bmo}'/V_o'$ depends on $Z$, and, as given by Eq 134, is equal to $V_m'/V_o'$, or

$$\frac{V_{bmo}'}{V_o'} \cong \frac{V_m'}{V_o'} = \alpha' e^{-\beta'\left(\frac{b' - Z'}{B_o}\right)} \tag{153}$$

Since Eqs 132 and 133 can be written

$$X = X' \sin \theta - Z' \cos \theta \tag{154}$$

$$Z = X' \cos \theta - Z' \sin \theta \tag{155}$$

along the Z'-axis, that is, if X' = 0, Eq 155 becomes

$$Z = Z' \sin \theta \tag{156}$$

Therefore, Eq 153 is written for any point along the Z'-axis as follows:

$$\frac{V_{bmo}'}{V_o'} = \alpha' \, e^{-\beta'\left[\frac{b' - (Z/\sin\theta)}{B_o}\right]} \tag{157}$$

Substitution of Eq 157 into Eq 152 yields

$$\left[e^{\beta'\left[\frac{b' - (Z/\sin\theta)}{B_o}\right]} \frac{1}{\sin \theta}\right]^{\frac{9(2m+1)}{10}} dZ + \frac{1}{1-\lambda} \frac{V_o'}{b} d_s P' \, dt = 0 \tag{158}$$

Define the following quantity

$$\omega' = \frac{9(2m+1)}{10} \frac{\beta'}{1-\lambda} \left\{P' (\sin\theta)^{\frac{18m-1}{10}}\right\} \cdot \exp\left[-\frac{9}{10}(2m+1) \frac{\beta' b'}{B_o}\right] \frac{d_s}{B_o} \frac{V_o' t}{b} \tag{159}$$

in which

$$P' (\sin\theta)^{\frac{18m-1}{10}} = \alpha' A_o \left[\frac{\alpha' V_o' \sin\theta}{V_s}\right]^{2m} \frac{k \frac{2m+1}{2}}{\gamma' \frac{2m+1}{8}} \cdot \frac{d}{d(X/b)} \left\{\frac{(1-\Psi_1^2)^{\frac{9(2m+1)}{20}}}{\left[\frac{\alpha' V_o' \sin\theta}{\nu} \frac{X}{b}\right]^{\frac{2m+1}{10}}}\right\} \tag{160}$$

With the definition given by Eq 159, integration of Eq 159, after dividing

the result by

$$\left(e^{\frac{9(2m+1)}{20} \frac{\beta' b'}{B_o}}\right),$$

yields

$$\left(1 - e^{-\frac{9(2m+1)}{10} \frac{\beta' Z}{B_o \sin\theta}}\right) + \omega' = 0 \tag{161}$$

Denoting the actual depth of scour by $Z_s = -Z$, Eq 161 becomes

$$\frac{Z_s}{B_o \sin\theta} = \frac{10}{9(2m+1)\beta'} \ln(\omega' + 1) \qquad (162)$$

From Eqs 142, 147, 148, and 159, it can be seen that $\omega'$ depends upon the following dimensionless parameters.

$$\left[\frac{d_s}{B_o}\ \frac{V_o' t}{b}\right];\ \frac{(V_o' \sin\theta)^2}{(\sigma/\rho - 1) g\, d_s};\ \Psi_1\left[\frac{X}{b}\ \frac{b}{B_o \sin\theta},\ \theta\right];\ \frac{X}{b};$$

$$\left[\frac{V_o' \sin\theta\ b}{\nu}\right] \text{and } \frac{b}{B_o \sin\theta}$$

Final Depth of Scour $Z_{s\infty}$ - The final value of the scour depth $Z_{s\infty}$ is attained when $U_* = U_{*c}$. From this equality, Eq 145 yields

$$\frac{U_{*c}^2}{(\alpha' V_o')^2} = k\, \gamma'^{-1/4} \left[\frac{V_{bmo}' \sin\theta}{\alpha V_o'}\right]^{9/5} \left[\frac{\alpha' V_o' X}{\nu}\right]^{-1/5} \left[1 - \Psi_1^2\right]^{9/10} \qquad (163)$$

Replacing $V_{bmo}'/\alpha V_o'$ by $V_m'/\alpha V_o'$ and using Eq 153, 163 becomes, for the critical value of scour depth $(Z_{s\infty} = -Z'_\infty)$

$$\frac{Z_{s\infty} + b}{B_o \sin\theta} = \frac{5}{9\beta'} \ln\left\{\left[\frac{\alpha' V_o' \sin\theta}{U_{*c}}\right]^2 \frac{k}{\gamma'^{\frac{1}{4}}} \frac{(1 - \Psi_1^2)^{9/10}}{\left[\frac{\alpha V_o \sin\theta X}{\nu}\right]^{1/5}}\right\} \qquad (164)$$

in which

$$Z'_{s\infty} = Z_{s\infty}/\sin\theta$$

Thus, from Eqs 142 and 164, it can be seen that $(Z_{s\infty} + b)/(d \sin\theta)$ is a function of the following dimensionless parameters

$$\left[\frac{V_o' \sin\theta}{U_{*c}}\right];\ \Psi_1\left[\frac{X}{b}\ \frac{b}{B_o \sin\theta},\ \theta\right];\ \left[\frac{V_o' \sin\theta\ X}{\nu}\right];\ \frac{X}{b} \text{ and } \frac{b}{B_o \sin\theta}$$

Jet Issuing From a Submerged Outlet - In this case, the following expressions for the distribution of the longitudinal velocity component $V_m'$ along the centerline of the jet and the transverse distribution of the velocity V' are applicable:

1. Along the centerline

$$\frac{V_m'}{V_o'} = 2.28 \left[\frac{B_o}{b' - Z'}\right]^{1/2} \tag{165}$$

2. Transverse to the direction of flow

$$\frac{V'}{V_m'} = e^{-4.24 \frac{X'^2}{(b' - Z')^2}} \tag{166}$$

Using the relationship between (X', Z') and (X, Z) given by Eqs 132 and 133, expressions for the velocity distributions are given for the following cases:

1. For a point along the X-axis or bed (Z = 0).

$$\frac{V_{bm}'}{(2.28\ V_o')} = \left[\frac{B_o}{b' + X\cos\theta}\right]^{1/2} \tag{167}$$

$$\frac{V_b'}{V_{bm}'} = \exp\left\{-4.24\left[\frac{X\sin\theta}{b' + X\cos\theta}\right]^2\right\} \tag{168}$$

2. For the stagnation point (Z = 0 and X = 0).

$$\frac{V_{bmo}'}{(2.28\ V_o')} = \left(\frac{B_o}{b'}\right)^{1/2} \tag{169}$$

Combining Eqs 167, 168, and 169, one obtains

$$\frac{V_b'}{V_{bmo}'} = \frac{V_b'}{V_{bm}'}\ \frac{V'_{bm}}{2.28\ V_o'}\ \frac{(2.28\ V_o')}{V_{bmo}'} = \left(1 + \frac{X}{b'}\cos\theta\right)^{-1/2} \exp\left\{-4.24\left[\frac{\frac{X}{b'}\sin\theta}{1 + \frac{X}{b'}\cos\theta}\right]^2\right\} \tag{170}$$

Define now the quantity of $\omega$ as

$$\omega = \frac{V_b'}{V_{bmo}'} = \left(1 + \frac{X}{2b}\sin 2\theta\right)^{-1/2} \exp\left\{-4.24\left[\frac{(X/b)\sin^2\theta}{1 + (X/2b)\sin 2\theta}\right]^2\right\} \tag{171}$$

Velocity and Shear Distribution Along the Bed - For the distribution of the shear velocity Eq 145 is applicable in which $\psi_1$ is replaced by $\omega$. Also, Eq 141 can be written

$$\frac{U_b}{V_{bmo}' \sin\theta} = (1 - \omega^2)^{1/2} \tag{172}$$

However, for the distribution of the horizontal velocity $U_b$, L given by 58 is assumed to be constant. Since

$$L = \frac{(X/b)}{U_b^2} \frac{d\, U_b^2}{d\,(X/b)} \tag{58}$$

Then by applying Eqs 171 and 172, one obtains

$$\frac{d\, U_b^2}{U_b^2} = -\frac{d(\omega^2)}{1-\omega^2} = \left\{ \frac{\sin 2\theta}{2} + 16.96 \sin^4\theta \cdot \frac{X/b}{(1 + \frac{X}{2b} \sin 2\theta)^2} \right\} \frac{d\,(X/b)}{(1 + \frac{X}{2b} \sin 2\theta)\,(\frac{1}{\omega^2} - 1)} \tag{173}$$

The value of L can be obtained from Eq 173 as follows:

Consider the case when (X/b) is very small, then Eqs 171 and 173 become

$$\frac{1}{\omega^2} = 1 + \frac{X}{2b} \sin 2\theta + \cdots$$

$$\frac{d\, U_b^2}{U_b^2} = \left[ \frac{\sin 2\theta}{2} \frac{1}{(1 + \frac{X}{2b} \sin 2\theta) - 1} + \cdots \right] d\,(X/b)$$

and finally

$$L = \frac{(X/b)\, d\, U_b^2}{U_b^2\, d\,(X/b)} \approx 1 \tag{174}$$

Consider the case when (X/b) is very large, then $L \approx 0$. Therefore, L can be considered as a constant either for small values of X/b with $L \approx 1$ or for large values of X/b with $L \approx 0$.

Variation of Scour Depth for $U_* \gg U_{*c}$ - For the case $U_* \gg U_{*c}$ Eq 152, with $P'$ replaced by $Q'$ and $\alpha' = 2.28$, can be written as

$$(1 - \lambda)\ \frac{\partial Z}{\partial t} + \frac{V_o' d_s}{b}\left[Q'\ (\sin\theta)^{\frac{18m-1}{10}}\right]\left[\frac{V_{bmo}}{2.28\ V_o}\right]^{\frac{9}{10}(2m+1)} \sin\theta = 0 \tag{175}$$

in which

$$Q'(\sin\theta)^{\frac{18m-1}{10}} = 2.28\ A_o\left[\frac{2.28\ V_o'\ \sin\theta}{V_s}\right]^{2m}\ \frac{k^{\frac{2m+1}{2}}}{\beta'^{\frac{2m+1}{8}}}$$

$$\cdot\left[\frac{2.28\ V_o'\ \sin\theta\ b}{\nu}\right]^{-\frac{2m+1}{10}}\ \frac{d}{d(X/b)}\left\{\frac{(1-\omega^2)^{\frac{9}{20}(2m+1)}}{(X/b)^{\frac{2m+1}{10}}}\right\} \tag{176}$$

The velocity distribution along the centerline of the jet, as given by Eq 165 for a non-submerged outlet, is obtained similarly, or

$$\frac{V_{bmo}'}{V_o'} \approx \frac{V_m'}{V_o'} = 2.28\left[\frac{B_o}{b' - Z'}\right]^{1/2} \tag{177}$$

in which

$$Z' = \frac{Z}{\sin\theta} \tag{178}$$

Substituting Eq 178 into Eq 175, one obtains

$$\left[\frac{b' - \left(\frac{Z}{\sin\theta}\right)}{B_o}\right]^{\frac{9}{20}(2m+1)} dZ + \frac{1}{1-\lambda}\ \frac{V_o'\ d_s}{b}\left[Q'(\sin\theta)^{\frac{18m-1}{10}}\right]\sin\theta\ dt = 0 \tag{179}$$

Defining the following quantity

$$\xi' = \frac{18m+29}{20}\ \frac{1}{1-\lambda}\left[Q'(\sin\theta)^{\frac{18m-1}{10}}\right]\frac{d_s}{B_o}\ \frac{V_o't}{b} \tag{180}$$

Substituting Eq 180 into Eq 179 and integrating gives

$$\left(\frac{b'}{B_o}\right)^{\frac{18m+29}{20}} - \left[\frac{b' - \frac{Z}{\sin\theta}}{B_o}\right]^{\frac{18m+29}{20}} + \xi' = 0 \tag{181}$$

from which, if $Z_s = -Z$, one obtains

$$\frac{b + Z_s}{B_o\ \sin\theta} = \left\{\xi' + \left(\frac{b'}{B_o}\right)^{\frac{18m+29}{20}}\right\}^{\frac{20}{18m+29}} \tag{182}$$

Eqs 176 and 180 show that $\xi'$ depends, since $m$ and $A_o$ are functions of $U_*^2/(\sigma/\rho) - 1)g\, d_s$, upon the following dimensionless parameters

$$\left(\frac{d_s}{B_o}\ \frac{V_o't}{b}\right);\ \frac{(V_o'\sin\theta)^2}{(\sigma/\rho - 1)\ g\ d_s};\quad \psi_1\ (X/b,\ \theta)\ ;\left[\frac{V_o'\sin\theta\ b}{\nu}\right];\ (X/b)$$

and $\dfrac{b}{B_o \sin\theta}$.

It is to be noted that the foregoing parameters are the same as for the case of the non-submerged outlet, where the variation of $Z_s$ with the parameters follows a power type law. Whereas, for the non-submerged outlet, the variation of $Z_s$ with the parameters follows an exponential type law.

<u>Final Depth of Scour ($U_* = U_{*c}$)</u> - For this case, introducing 2.28 $V_o'$ into Eq 145 and substituting $\omega''$ for $\psi_1$ yields

$$\left[\frac{U_{*c}}{2.28\ V_o'\ \sin\theta}\right]^2 = \frac{k}{\gamma' - 1/4}\left(\frac{V_{bmo}'}{2.28\ V_o'}\right)^{9/5} \cdot \left(\frac{2.28\ V_o'\ \sin\theta\ X}{\nu}\right)^{-1/5}(1 - \omega''^2)^{9/10} \tag{183}$$

Substituting $Z'_{s\infty} = -Z'$ into Eq 177 gives

$$\left[\frac{V_{bmo}'}{2.28\ V_o'}\right]_c \simeq \left[\frac{V_m'}{2.28\ V_o'}\right]_c = \left[\frac{B_o}{b' + Z_{s\infty}'}\right]^{1/2} \tag{184}$$

in which $Z_{s\infty}'$ is defined as

$$Z_{s\infty}' = \frac{Z_{s\infty}}{\sin\theta} \tag{185}$$

Introducing Eq 184 into Eq 183 yields

$$\left[\frac{b' + Z_{s\infty}'}{B_o}\right]^{9/10} = \frac{k}{\gamma'\ 1/4}\left(\frac{2.28\ V_o'\ \sin\theta}{U_{*c}}\right)^2\left(\frac{2.28\ V_o'\ \sin\theta\ X}{\nu}\right)^{-1/5} \cdot (1 - \omega''^2)^{9/10} \tag{186}$$

Replacing $Z_{s\infty}'$ by its value given by Eq 185 gives the following expression for the depth of scour

$$\frac{b + Z_{s\infty}}{B_o \sin\theta} = \left\{ \frac{k}{\gamma'^{\frac{1}{4}}} \left[ \frac{2.28\ V_o' \sin\theta}{U_{*c}} \right] \right\}^{10/9} \left[ \frac{2.28\ V_o' \sin\theta\ X}{\nu} \right]^{-2/9} (1 - \omega^2) \qquad (187)$$

The final depth of scour $Z_{s\infty}$ is a function of the following dimensionless parameters.

$$\left[ \frac{V_o' \sin\theta}{U_{*c}} \ ; \ \frac{V_o' \sin\theta\ X}{\nu}, \ \psi_1 \left( \frac{X}{b}, \theta \right) \text{ and } \frac{b}{B_o \sin\theta} \right]$$

These parameters are similar to those obtained for the case of a non-submerged outlet; however, the depth $Z_{s\infty}$ is function of the logarithmic law, while for the case of the non-submerged outlet $Z_{s\infty}$ is function of the logarithmic law.

## SUMMARY AND CONCLUSIONS

An elementary, theoretical analysis has been attempted herein for the mechanics of scour caused by a two-dimensional jet issuing from a submerged or non-submerged outlet and acting upon an erodible bed. The erodible bed for this analysis is considered to be covered by a tailwater of given depth. In developing the theory, some assumptions have been made and some flow phenomena have been neglected, such as:

1. It has been assumed that the Bernoulli equation is valid in the neighborhood of the stagnation point.
2. The value of L given by Eq 58 has been assumed constant in solving the boundary layer equation.
3. It has been assumed that the depth of scour is small compared with the tailwater depth in integrating the continuity equation of mass sediment transport.
4. A generalized equation of sediment transport for open channel flow has been applied, that is, it has been assumed that the rate of sediment transport depends only upon the shear velocity and the size and specific weight of sediment.
5. The lateral diffusion along the bed, after impingement of the jet on the bed, has been neglected.
6. The bed has been treated as a hydraulically smooth boundary in obtaining the shear distribution from the boundary layer equation.

From an analysis of the flow phenomena of a jet issuing from a non-submerged outlet, it is evident that the jet entrains air and results in jet diffusion, which as given by Eq 44, follows an exponential type of law. Both vertical and transverse distributions of velocity have not been determined experimentally for this case.

For a jet issuing from a submerged outlet, Albertson and others have determined both the vertical and transverse distributions of velocity within the diffusing jet. Equations 65 and 66 express these two types of velocity distribution. In particular, the vertical diffusion of the jet with time is expressed by a power-type law.

For the case of the non-submerged outlet, the transverse distribution of velocity given by Eq 45 was, as a first approximation, assumed to be equal to the one obtained by Albertson and others for the case of the submerged outlet as expressed by Eq 67.

The importance of velocity distribution within the diffusing jet is its effect upon the scour phenomena, namely: (a) the shape of the scour hole depends upon the velocity distribution transverse to the direction of flow, and (b) the development of the scour hole with respect to time depends upon the velocity distribution in the direction of flow.

The angle of impingement has a noticeable influence on the constants giving the depth of scour. This has been demonstrated by the experimental study of Homma for the three-dimensional jet issuing from a non-submerged outlet.

Since the phenomena of scour involving fluid jets is of such a complex nature, all of the variables describing the phenomena cannot be included in an initial theoretical development such as presented herein. Therefore before the theory of this analytical study can be utilized for design purposes, it will be necessary to test the theory and evaluate the various exponents by means of experimental data covering a wide range of flow, fluid and sediment characteristics and boundary geometry. By means of dimensional analysis, a judicious selection of essential hydraulic conditions and boundary geometry can be made, which will expedite the development of the generalized theory.

REFERENCES


1. Rouse, Hunter, "Criteria for similarity in the transportation of sediment." Proceedings Hydraulic Conference, University of Iowa Studies in Engineering, Bulletin 20, 1940, pp. 33-49.

2. Doddiah, Doddiah, "Comparison of scour caused by hollow and solid jets of water." M.S. Thesis, Colorado A and M College, 1950.

3. Thomas, R K., "Scour in a gravel bed." M.S. Thesis, Colorado A and M College, 1953.

4. Hallmark, D. E., "Scour at the base of a free overfall." M.S. Thesis, Colorado A and M College, 1955.

5. Albertson, M. L. and Smith, G.L., "Principles of energy dissipation in erosion control structures." Colorado A and M College, Civil Engineering Department. Report prepared for "Joint ARS-SCS Irrigation - Drainage Conference, 1957. Paper No. 15--For in-service use only."

6. Smith, George L., "An analysis of scour below culvert outlets." M.S. Thesis, Colorado State University, 1957.

7. Schlichting, H. and Truckenbrodt, E., "Die Stromung an einer angeblasenen rotierenden Scheibe." ZAMM Band 32. Heft 4/5. April/Mai, 1952. S. 97-111.

8. Truckenbrodt, E., "Die Trubulent Stromung an einer angeblasenen rotierenden Scheibe," ZAMM, Band 34, Heft 4/5, April/Mai, 1954, S. 150-162.

9. Albertson, M. L., Dai, Y. B., Jensen, R. A., and Rouse, H., "Diffusion of Submerged Jets," Transactions A.S.C.E., Vol. 115, 1950, pp. 639-697.

10. Brown, C. B., and Laursen, E. M., "Sediment transportation," (In Proceedings Fourth Hydraulic Conference, Iowa Institute of Hydraulic Research edited by Hunter Rouse, John Wiley and Sons, 1950. p. 769-857).

11. Shields, A., "Anwendung der aehnlichkeitsmechanik und der turbulenz-forschung auf die geschiebebewegung." Mitteilungen der Preuss. Versuchsanst. f. Wasserbau u. Schiffbau., Berlin, Heft 26, 1936

12. Kalinske, A. A., "Movement of sediment as bed load in rivers." Transactions A.G.U., Vol. 28, No. 4, 1947, pp. 615-620.

13. Exner, F., "Uber die Wechselwirkung zwischen Wasser und Geschiebe in Flussen," Proceedings Vienna Academy of Sciences, Section II A, Vol. 134, 1925, p. 165.

14. Poreh, M., "Flow characteristics of a circular submerged jet impinging normally on a smooth boundary." M.S. Thesis, Colorado State University, 1959.

15. Iwagaki, Y., Smith, George L., and Albertson, M. L., "Analytical Study of the Mechanics of Scour for Three-Dimensional Jet." Colorado State University, Deaprtment of Civil Engineering, Report CER60GLS9. January 1960. Prepared for "U. S. Bureau of Public Roads under contract CPR11-5504."

16. Schlichting, H., "Boundary Layer Theory," McGraw-Hill Book Company New York, 1955, p. 73.

17. Howarth, L., "On the calculation of the steady flow in the boundary layer near the surface of a cylinder in a stream" A.R.C. Report and Memo. 1632, 1935.

18. Homma, M., "An Experimental Study on Water Fall." Proc. of Minnesota International Hydraulic Convention. September 1953, pp. 477-481.