DISSERTATION

A SIMPLICIAL HOMOTOPY GROUP MODEL FOR K_2 OF A RING

Submitted by

JaDon Saeed Whitfield

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Fall 2010

Doctoral Committee:

Department Chair: Gerhard Dangelmayr

Advisor: Jeanne Duflot

Rick Miranda
Jeff Achter
Martin Gelfand

ABSTRACT

A SIMPLICIAL HOMOTOPY GROUP MODEL FOR K_2 OF A RING

We construct an isomorphism between the group $K_2(R)$ from classical, algebraic K-Theory for a ring R and a simplicial homotopy group constructed using simplicial homotopy theory based on that same ring R. First I describe the basic aspects of simplicial homotopy theory. Special attention is paid to the use of category theory, which will be applied to the construction of a simplicial set. K-Theory for $K_0(R), K_1(R)$ and $K_2(R)$ is then described before we set to work describing explicitly the nature of isomorphisms for $K_0(R)$ and $K_1(R)$ based on previous work[11]. After introducing some theory related to K-Theory, some considerations and corrections on previous work motivate more new theory that helps the isomorphism with $K_2(R)$. Such theory is developed, mainly with regards to finitely generated projective modules over R and then elementary matrices with entries from R, culminating in the description of the Steinberg Relations that are central to the understanding of $K_2(R)$ in terms of homotopy classes. We then use new considerations on the previous work to show that a map whose image is constructed through this article is an isomorphism since it is the composition of isomorphisms.

In Chapter 1 we explore Simplicial Homotopy Theory from the "canonical" point of view of [2]. The emphasis of the entire paper will be on the calculations involving this structure and how they give explicit instructions for the isomorphism that is our final result. Accordingly, less attention is given to the fine details and examples from either classical or modern K-Theory, which we give a brief description of in Chapter 2. Our goal is not to describe or work with K-Theory as much as it is to accurately reflect the properties involved through the algebraic structures provided by Simplicial Homotopy Theory, so Chapter 2 only describes what is necessary to see how the later constructions will provide isomorphisms.

Chapter 3 establishes the simplicial sets that we will work with in detail, and provides constructions that lead to the maps we will connect together to form our final result. Those connections are then introduced in Chapter 4, where we describe some work that has already been done [11] with $K_1(R)$ which will be helpful. Chapter 5 establishes the main theory that will allow us to reflect the structure of $K_2(R)$ through the simplicial sets introduced in Chapter 3, and Chapter 6 connects these properties into an explicit isomorphism.

ACKNOWLEDGMENTS

There are too many people whose thoughts, prayers, support and camaraderie have helped make this amazing milestone possible, but I must honor those who believed in me, literally, for the rest of their lives after meeting me: in fond and grateful remembrance of the late Billy Joe "Mr. B." Burleson, Dr. Robert Liebler and Dr. Daniel Rudolph.

Also, I cannot ever write a note of gratitude without thanking God, from Whom all good things come; my parents Donnie and Billetta Whitfield; my wife Eliina and children Jalynn and Jadiina.

Thanks, as always, to the Mathematics Department of Colorado State University, for giving me a shot at this in the first place.

TABLE OF CONTENTS

Chapter 1

Simplicial Homotopy Theory

This chapter is an exposition of simplicial homotopy theory, relying mostly (and heavily) on [2] with some ideas and proofs from [9],[17] and [18], which are also used and expounded upon in [11]. Examples come from [2],[9], [3] and [11].

1 Simplicial Sets

1.1 Combinatorics and Extension Condition[2]

Definition 1.1.1 *A* ***Simplicial Set*** *is a sequence of sets* $\{X_n\}_{n\in\mathbb{Z},n\geq 0}$ *together with two types of maps –* ***face maps*** $d_i : X_n \to X_{n-1}$ *and* ***degeneracy maps*** $s_i : X_n \to X_{n+1}$ *for each* $i \in \{0, 1, \ldots, n\}$ *– which satisfy the following relations with respect to composition:*

i) If $i < j$ *then* $d_i d_j = d_{j-1} d_i$.

ii) If $i < j$ *then* $s_j s_i = s_i s_{j-1}$.

iii) $$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j \\ id_{X_n}, & i = j \text{ or } i = j+1 \\ s_j d_{i-1}, & i > j+1. \end{cases}$$

A simplicial set is often referred to as a **Complex**. The elements of X_n are called n**-simplices**, or the elements in X of **dimension** n. The index of a face map or of a degeneracy map is the **degree** of that map. We omit parentheses and write the images of these maps as simply $d_i x$ and $s_j x$ for $x \in X_n$, $0 \leq i, j \leq n$.

Definition 1.1.2 *Given simplicial sets* X, L, *a* ***simplicial map****, or* ***map of simplicial sets****,* $f : X \to L$, *is a collection of functions* $f_n : X_n \to L_n, n \in \mathbb{N}$ *such that* $d_i \circ f_n = f_{n-1} \circ d_i$ *and* $s_i \circ f_n = f_{n+1} \circ s_i$, $\forall\ 0 \leq i \leq n$.

Definition 1.1.3 *Given a complex X, a* ***subcomplex*** *L of X is a sequence of subsets $\{L_n \subseteq X_n\}_{n\in\mathbb{Z},n\geq 0}$ for which the face maps and degeneracy maps of X have $d_i|_{L_n} : L_n \to L_{n-1}$ and $s_j|_{L_n} : L_n \to L_{n+1}$ $\forall\ 0 \leq i, j \leq n$ for each $n \in \mathbb{N}$.*

The sets L_n establish $L = \{L_n\}$ as a simplicial set in its own right, with (restrictions of) the same face maps and degeneracy maps as defined for X.

Definition 1.1.4 *If $X' \subseteq X$ and $L' \subseteq L$ are subcomplexes of Complexes X and L respectively, a* ***simplicial map of pairs*** *$f : (X, X') \to (L, L')$ is a simplicial map $f : X \to L$ for which $f|_{X'} : X' \to L'$.*

It is easy to see that simplicial sets together with simplicial maps form a category:

Definition 1.1.5 *$\mathcal{SS}$ is the category whose objects are simplicial sets and whose morphisms are simplicial maps.*

Definition 1.1.6 *Given a simplicial set X, a* ***compatible list*** *in X is a list of $n+1$ n-simplices,*

$$C_{(n,k)} = (x_0, x_1, \ldots, \hat{x}_k, \ldots, x_{n+1})$$

(with $n > 0$) such that $d_i x_j = d_{j-1} x_i$ whenever $i < j$, $i \neq k$, $j \neq k$.

Here $\hat{x}_k$ indicates that x_k is omitted from the ordered list.

Definition 1.1.7 *Given a compatible list $C_{(n,k)}$ in a Complex X as above, an* ***extender*** *for $C_{(n,k)}$ is an $(n+1)$-simplex y for which $d_i y = x_i$ whenever $i \neq k$.*

Notice that the image of a compatible list under a simplicial map will also be a compatible list. A simplicial set in which *every* compatible list has an extender satisfies the **Extension Condition**; such a simplicial set is known as a **Kan Complex**.

Example 1.1.8 *$X = \Delta^m$[2]: Given $n \in \mathbb{Z}_{\geq 0}$, define $\boldsymbol{n} = (0 \leq 1 \leq \cdots \leq n)$ as an ordered set. Let*

$$\Delta^m_n = \{c : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\} \mid c(i) \leq c(j) \forall i \leq j\}.$$

(Another notation for this is $\Delta^m_n \doteq Hom_\Delta(\boldsymbol{n}, \boldsymbol{m})$.) Define $s^j \in \Delta^n_{n+1}$ by

$$s^j(k) = \begin{cases} k, & k \leq j \\ k-1, & k > j \end{cases},$$

and $d^i \in \Delta^n_{n-1}$ *by*

$$d^i(k) = \begin{cases} k, & k < i \\ \\ k+1, & k \geq i \end{cases},$$

for $0 \leq i, j \leq n$. *If we fix* $m \in \mathbb{Z}_{\geq 0}$ *and define face maps and degeneracy maps respectively by* $d_i c = c \circ d^i$ *and* $s_j c = c \circ s^j$ *for any* $c \in \Delta^m_n$, *then* $\Delta^m = \{(\Delta^m_n; \{d_i\}; \{s_j\})\}$ *is a simplicial set, called the* ***Standard Simplicial*** m***-simplex****. We call the* d^i *and* s^j ***coface maps*** *and* ***codegeneracy maps****, respectively.*

The following Lemma provides a way to uniquely "factor" elements of Δ^m:

Lemma 1.1.9 *If* $c \in \Delta^m_n, c \neq id_{\mathbf{n}}$ *has image*

$$\boldsymbol{m} - \{i_u < i_{u-1} < \cdots < i_1\}$$

and

$$\{j \mid c(j) = c(j+1)\} = \{j_1 < j_2 < \cdots < j_v\},$$

then $n - v + u = m$ *and* $c = d^{i_1} \circ d^{i_2} \circ \cdots d^{i_u} \circ s^{j_1} \circ s^{j_2} \circ \cdots \circ s^{j_v}$. *Moreover, this factorization is unique when "reduced" using rules (i)-(iii) of Definition 1.1.1.*

□□

1.2 Categorical Description of Simplicial Sets[2, 17, 18]

An alternative construction of simplicial sets begins with the category Δ^{op}, which is the opposite category of the category Δ. The objects of Δ are the ordered sets $\mathbf{n}$ as seen in Example 1.1.8, and the morphisms are the maps $c : \mathbf{n} \to \mathbf{m}$ as discussed in that same example. The definition of Δ^{op} then requires that the objects be the same as those of Δ, and that the morphisms be $Hom_{op}(\mathbf{m}, \mathbf{n}) = \Delta^m_n$ (i.e. maps *over* $\mathbf{n}$ as opposed to maps *into* $\mathbf{n}$).

Definition 1.2.1 *A (category-theoretic)* ***simplicial set*** *is a (covariant) functor* $X : \Delta^{op} \to \mathcal{S}$, *where* $\mathcal{S}$ *is the category of sets.*

Now, given a simplicial set X (by the original definition), identify $X_n := X(\mathbf{n})$ for every $n \in \mathbb{Z}_{\geq 0}$ and $d_i := X(d^i), s_j := X(s^j)$, $0 \leq i, j \leq n$ for coface maps d^i and codegeneracy maps s^j. More generally, to every

$\mu \in Hom_\Delta(\mathbf{n}, \mathbf{m}) = \Delta_n^m$ represented according to Lemma 1.1.9 by

$$\mu = d^{i_1} \circ d^{i_2} \circ \cdots d^{i_u} \circ s^{j_1} \circ s^{j_2} \circ \cdots \circ s^{j_v},$$

there corresponds a map $\mu^* = X(\mu) : X_m \to X_n$ uniquely defined by

$$\mu^*(x) = s_{j_v} s_{j_{v-1}} \cdots s_{j_1} d_{i_u} d_{i_{u-1}} \cdots d_{i_1} x.$$

From here on, we will use both definitions of a simplicial set interchangeably to perform various calculations, depending on which provides the most advantage. This chapter shall rely mostly on the combinatorial description.

The combinatorial data for a simplicial map $f : X \to L$ is a collection of maps, one defined for each dimension n, but by Definition 1.2.1 the simplicial set is itself a set map sending objects $\mathbf{n} \in \Delta^{op}$ to sets $X_n \in \mathcal{S}$. So the simplicial map assigns to each $\mathbf{n}$ a map (i.e. morphism of sets) $f(\mathbf{n}) = f_n : X(\mathbf{n}) \to L(\mathbf{n})$. Furthermore, with this assignment we see that the required degree-preserving behavior toward face maps and degeneracy maps(Definition 1.1.2) implies

$$f(\mathbf{n\text{-}1}) \circ X(d^i) = L(d^i) \circ f(\mathbf{n})$$

and

$$f(\mathbf{n+1}) \circ X(s^j) = L(s^j) \circ f(\mathbf{n})$$

for each $0 \leq i, j \leq n$. Since any morphism α in Δ^{op} can be written as a unique combination of coface maps and codegeneracy maps (Lemma 1.1.9) and X and L must be covariant functors, it follows that

$$f(\mathbf{m}) \circ X(\alpha) = L(\alpha) \circ f(\mathbf{n}) \; \forall \; \alpha \in Hom_{op}(\mathbf{n}, \mathbf{m}),$$

in which case a simplicial map $f : X \to L$ is a *natural transformation*[2, 9] from the functor X to the functor L.

Given a simplicial set X and a fixed $\phi_0 \in X_0$, define the **one-point simplicial set**

$$\Phi = \{\phi_n = s_{n-1} \circ s_{n-2} \circ \cdots \circ s_0 \phi_0, n \in \mathbb{N}\}$$

(i.e. Φ contains only one simplex in each dimension). Then Φ is a subcomplex of X. The pair (X, Φ) is called a **Pointed Complex** with **basepoint** Φ, and use of this definition motivates us to define the elements of

X_0 as the **vertices** of the simplicial set X. When X is a Kan Complex, (X, Φ) is called a **Kan Pair**. When $X' \subseteq X$ is a Kan subcomplex (and X is a Kan Complex) for which $\Phi \subseteq X'$ as a subcomplex, we call the data (X, X', Φ) a **Kan Triple**.

Definition 1.2.2 *Given any subset $S \subseteq X_m$, $m \in \mathbb{N}$, the* ***subcomplex generated by*** *S is the simplicial set $X(S)$ with n-simplices*

$$X(S)_n = \{\mu^*(s) \mid s \in S, \mu* : X_m \to X_n\}.$$

Definition 1.2.2 can be easily extended to general subsets of the simplicial set X (i.e. $S = S_0 \cup S_1 \cup \cdots \cup S_m$, $S_i \subset X_i$). Also, note that $\Phi = X(\phi_0)$.

Definition 1.2.3 *$\mathcal{SS}_*$ is the category whose objects are Pointed Complexes (X, Φ_X) and whose morphisms are simplicial maps of pairs $f : (X, \Phi_X) \to (Y, \Phi_Y)$ for appropriate basepoints Φ_X, Φ_Y.*

Definition 1.2.4 *A simplicial set X is* ***reduced*** *if it has only one vertex: $X_0 = \{\phi_0\}$.*

1.3 Homotopy in Kan Complexes

Definition 1.3.1 *Let X be a simplicial set. For $n \geq 1$, n-simplices x and y are* ***homotopic*** *(in the simplicial set), denoted $x \sim y$, if $\forall\ 0 \leq i \leq n, d_i x = d_i y \in X_{n-1}$ and for some $(n+1)$-simplex z,*

$$d_i z = \begin{cases} y, & i = n+1, \\ x, & i = n \\ s_{n-1} d_i x = s_{n-1} d_i y, & 0 \leq i \leq n-1. \end{cases}$$

Such z is a ***homotopy*** *(in the simplicial set) from x to y.*

Theorem 1.3.2 *([2], Proposition 3.2) If X is a Kan Complex then the relation $x \sim y$ for $x, y \in X_n$ is an equivalence relation on X_n for any given $n \in \mathbb{N}$.*

The purpose of many of the constructions we perform is to *ensure* that the simplicial set we are working with is a Kan Complex, and so has the equivalence relation (and hence equivalence classes) given by homotopy.

The relation above only applies when $n \geq 1$. We have the following relation on the 0-simplices X_0 of a simplicial set X:

Definition 1.3.3 *In a simplicial set X, two 0-simplices $x, y \in X_0$ are in the same* ***path component*** *of X if there is a list of 1-simplices $D_k = (z_1, \ldots z_k) \subset X_1$ so that $x = d_0 z_1$ or $x = d_1 z_1$, $y = d_0 z_k$ or $y = d_1 z_k$, and for each $1 \leq i < k$, one of the following is true: $d_0 z_i = d_0 z_{i+1}, d_1 z_i = d_0 z_{i+1}, d_0 z_i = d_1 z_{i+1}$ or $d_1 z_i = d_1 z_{i+1}$.*

We consider the above definition to be synonymous with homotopy for 0-simplices: given $x, y \in X_0$, $x \sim y$ if and only if x and y are in the same path component of X. This is clearly an equivalence relation on X_0.

Definition 1.3.4 *Given a Pointed Complex (X, Φ), define*

$$\widetilde{X}_n := \{x \in X_n \mid d_i x = \phi_{n-1} \ \forall\ 0 \leq i \leq n\}$$

for $n > 0$. In case $n = 0$ define $\widetilde{X}_0 = X_0$.

Note that the equivalence relation on X_n restricts to an equivalence relation on $\widetilde{X}_n$ for each n when X is a Kan Complex.

Definition 1.3.5 *When (X, Φ) is a Kan Pair with homotopy $x \sim y$ between n-simplices as an equivalence relation, define $\pi_n(X, \Phi) = \widetilde{X}_n/_\sim$, for $n > 0$, with elements $[x]$. When $n = 0$ use Definition 1.3.3 and set $\pi_0(X, \Phi) = \widetilde{X}_0/\sim$.*

Note that by definition, $\pi_0(X) := \pi_0(X, \Phi)$ is independent of the choice of 0-simplex ϕ_0. Some constructions later will be made for the purpose of producing a reduced Kan Complex, so that these homotopy sets for $n > 0$ are unambiguous in terms of the choice of Φ, and can be denoted as simply $\pi_n(X)$.

Definition 1.3.6 *Let $(X, \Phi) \in \mathcal{SS}_*$ be a Kan Pair. Given $n > 0$, $[x], [y] \in \pi_n(X, \Phi)$, define the specific, compatible list $C_{xy} = C_{(n,n)} = (x_i)_{i \neq n}$ where*

$$x_i = \begin{cases} \phi_n, & 0 \leq i \leq n-2, \\ x, & i = n-1, \\ y, & i = n+1 \end{cases}$$

(i.e. $k = n$ in the usual compatible list notation). Then

$$[x] \bullet [y] = [d_n z]$$

where $z \in X_{n+1}$ is an Extender of C_{xy}.

One can show that the above multiplication $\bullet$ above is well-defined, and we have

Theorem 1.3.7 *[2]Let $(X, \Phi) \in \mathcal{SS}_*$ be a Kan Pair. With respect to the multiplication $\bullet$ defined above, $\pi_n(X, \Phi)$ is a group if $n \geq 1$. Moreover,if $n \geq 2$, $\pi_n(X, \Phi)$ is an abelian group.*

When Theorem 1.3.7 holds, we call $\pi_n(X, \Phi)$ the n^{th} **simplicial homotopy group of** X (with respect to Φ). We have $\pi_0(X, \Phi)$ as a pointed set, with basepoint the class of ϕ_0, but this is not necessarily a group.

Definition 1.3.8 *If $f : (X, \Phi_X) \to (L, \Phi_L)$ is a simplicial map of Kan Pairs, then an* ***induced map*** *$f_* : \pi_n(X, \Phi_X) \to \pi_n(Y, \Phi_Y)$ is defined by $f_*([x]) = [f_n(x)]$.*

It is straightforward to see that

Lemma 1.3.9 *If $f : (X, \Phi_X) \to (L, \Phi_L)$ is a simplicial map of Kan Pairs, then the* ***induced map*** *$f_* : \pi_n(X, \Phi_X) \to \pi_n(Y, \Phi_Y)$ is a homomorphism of groups, if $n \geq 1$, and is a map of pointed sets if $n = 0$.*

We will construct a long exact sequence of homotopy groups; in order to do this, we will need a more general theory on homotopy.

Definition 1.3.10 *Given $X \in \mathcal{SS}$ with a subcomplex $X' \subseteq X$ and $n \geq 1$, two n-simplices $x, y \in X_n$ have $x \sim y(rel\ X')$ (i.e. x and y are* ***homotopic relative to*** *X') if*

1) $d_0x \sim d_0y$ as elements in X'_{n-1}.

2) $\forall\ 1 \leq i \leq n, d_ix = d_iy$.

3) There is some homotopy in the simplicial set, $w \in X'_n$, between d_0x and d_0y and there is an $(n+1)$-simplex $z \in X_{n+1}$ such that

$$d_iz = \begin{cases} y, & i = n+1, \\ x, & i = n, \\ s_{n-1}d_ix = s_{n-1}d_iy, & 1 \leq i \leq n-1, \\ w, & i = 0 \end{cases}$$

(such a z is a ***relative homotopy*** *(in the simplicial set) from x to y).*

Definition 1.3.11 *Given a Kan Triple* (X, X', Φ), *and* $n \geq 1$,

$$\widetilde{X}(X')_n = \left\{x \in X_n : d_0 x \in X'_{n-1}, d_i x = \phi_{n-1} \ \forall\ 1 \leq i \leq n\right\}$$

.

Definition 1.3.12 *Relative Homotopy Groups as Sets: Given Kan Triple* (X, X', Φ) *and* $n \geq 1$,

$$\pi_n(X, X', \Phi) = \widetilde{X}(X')_n/_{\sim_{X'}},$$

with elements $[x]_{X'}$.

Similar to Definition 1.3.8, given a simplicial map $f : (X, X', \Phi_X) \to (L, L', \Phi_L)$ between Kan Triples, define the **induced map** $f_* : \pi_n(X, X', \Phi_X) \to \pi_n(L, L', \Phi_L)$ by $f_*([x]_{X'}) = [f(x)]_{L'}$

In light of Definitions 1.3.11 and the rules of Definition 1.1.1, notice that $x \in \widetilde{X}(X')_n$ implies

$$d_i d_0 x = d_0 d_{i+1} x = \phi_{n-2}$$

$\forall\ 0 \leq i \leq n-1$, so that (since $d_0 x \in X'_{n-1}$) $d_0 x \in \widetilde{X'}_{n-1}$. Thus for $x, y \in \widetilde{X}(X')_n$ we have

$$[d_0 x] \bullet [d_0 y] = [d_{n-1} u] \in \pi_{n-1}(X', \Phi)$$

for some $u \in X'_n$ that extends the compatible list $C_{d_0(x) d_0 y} \subseteq X'_{n-1}$. This in turn gives a compatible list

$$C'_{xy} = C_{(n,n)} = (x_i)_{i \neq n},$$

where

$$x_i = \begin{cases} u, & i = 0, \\ \phi_n, & 1 \leq i \leq n-2, \\ x, & i = n-1, \\ y, & i = n+1. \end{cases}$$

Since X' is a Kan subcomplex by definition, there is an extender $v \in X'_{n+1}$ for C'_{xy}. The result is a group product on $\pi_n(X, X', \Phi)$ similar to Definition 1.3.6:

Definition 1.3.13 *Given Kan Triple* (X, X', Φ), $n \geq 2$ *and corresponding set* $\widetilde{X}(X')_n$, *define* $[x]_{X'} \bullet_{X'} [y]_{X'} = [d_n v]_{X'}$ *where* $v \in X'_{n+1}$ *extends the compatible set* C'_{xy} *described above.*

Remark 1.3.14 *It is easy to see that* $\pi_n(X, \Phi, \Phi) = \pi_n(X, \Phi)$ *when* $n \geq 1$.

Similar to Theorem 1.3.7 and Lemma 1.3.9, we have

Theorem 1.3.15 *Given a Kan Triple* (X, X', Φ)*and* $n \geq 2$, $\pi_n(X, X', \Phi)$ *is a group with respect to the multiplication* $\bullet$, *and is an abelian group if* $n \geq 3$. *If* $f : (X, X', \Phi_X) \to (Y, Y', \Phi_Y)$ *is a map of Kan Triples, then the induced map* $f_* : \pi_n(X, X', \Phi) \to \pi_n(Y, Y', \Phi)$ *is a homomorphism of groups.*

Note that while $\pi_1(X, X', \Phi)$ is not necessarily a group, it is a pointed set.

We may now write down the "long exact sequence for a Kan Triple"; first of course we define the connecting homomorphism.

Definition 1.3.16 *Define*

$$d : \pi_n(X, X', \Phi) \to \pi_{n-1}(X', \Phi)$$

by $[x]_{X'} \mapsto [d_0 x]$, *which is a* ***connecting homomorphism*** *for* $n \geq 2$ *and a (connecting) set map when* $n = 1$.

Theorem 1.3.17 *([2], Theorem 3.7) Let* (X, X', Φ) *be a Kan Triple, with inclusion (simplicial) maps*

$$i : (X', \Phi) \to (X, \Phi) \text{ and } j : (X, \Phi, \Phi) \to (X, X', \Phi)$$

(see Remark 1.3.14). Then there exists a ***long exact sequence of Homotopy Groups****,*

$$\cdots \longrightarrow \pi_{n+1}(X, X', \Phi) \xrightarrow{d} \pi_n(X', \Phi) \xrightarrow{i_*} \pi_n(X, \Phi) \xrightarrow{j_*} \pi_n(X, X', \Phi) \longrightarrow \cdots.$$

Remark 1.3.18 *The maps at the end of this long exact sequence are not necessarily group homomorphism, but are maps of pointed sets, and "exact" here means exact as a sequence of pointed sets.*

1.4 Dimension-wise Map Homotopy

Definition 1.4.1 *Let* X *and* L *be simplicial sets. Simplicial maps* $f, g : X \to L$ *are* ***homotopic*** *via a* ***dimension-wise homotopy*** $h : f \simeq g$ *if given* $n \in \mathbb{Z}_{\geq 0}$ *there is a sequence of maps*

$$\left\{ h_i^{(n)} : X_n \to L_{n+1} \mid 0 \leq i \leq n \right\}$$

for which the following relations hold with respect to composition for any $x \in X_n$:

i) $d_0 h_0^{(n)}(x) = f_n(x)$ *and* $d_{n+1} h_n^{(n)}(x) = g_n(x)$.

$$ii)\ d_i h_j^{(n)}(x) = \begin{cases} h_{j-1}^{(n-1)} d_i(x), & i < j \\ \\ d_{j+1} h_{j+1}^{(n)}(x) = d_{j+1} h_j^{(n)}(x), & i = j \text{ or } i = j+1 \\ \\ h_j^{(n-1)} d_{i-1}(x), & i > j+1. \end{cases}$$

$$iii)\ s_i h_j^{(n)}(x) = \begin{cases} h_{j+1}^{(n+1)} s_i(x), & i \leq j \\ \\ h_j^{(n+1)} s_{i-1}(x), & i > j. \end{cases}$$

If $f, g : (X, X') \to (L, L')$ are simplicial maps of Pairs and as a homotopy of simplicial maps the homotopy $h : f \simeq g$ has $h|_{X'} : X' \to L'$ and $h|_{X'} : f|_{X'} \simeq g|_{X'}$, we say that f and g are homotopic ***relative to*** *X' via* ***relative homotopy*** *$h : f \simeq g\ rel(X')$. In case $X' = \Phi_X \subseteq X$ and $L' = \Phi_L \subset L$ for appropriate one-point simplicial sets, we say f and g are homotopic relative to the* ***basepoint*** *Φ.*

Theorem 1.4.2 *Let $(X, \Phi_X), (L, \Phi_L) \in \mathcal{SS}_*$ be Kan Pairs. If $f, g : (X, \Phi_X) \to (L, \Phi_L)$ are simplicial maps of Pairs with $f \simeq g\ rel(\Phi_X)$, then for each $n \in \mathbb{N}$, $f_*([x]) = g_*([x])\ \forall\ [x] \in \pi_n(X, \Phi_X)$.*

Definition 1.4.3 *Simplicial Sets X and L are of the same* ***homotopy type*** *or are* ***homotopy equivalent*** *if there are simplicial maps $f : X \to L$ and $f' : L \to X$ for which $f \circ f' \simeq id_L$ and $f' \circ f \simeq id_X$. Such f and f' are then called* ***homotopy equivalences****[17]. X is* ***of the homotopy type of a point*** *if and only if it is contractible.*

As a consequence of Theorem 1.4.2, we note that if K and L are of the same homotopy type, then $\pi_n(K, \Phi)$ is isomorphic to $\pi_n(L, f(\Phi))\ \forall\ n \geq 0$ given homotopy equivalence f between them.

2 Simplicial Groups

2.1 Definition ([2], Chapter 17)

Definition 2.1.1 *A* ***simplicial group*** *is a simplicial set $G = \{(G_n; \{d_i\}; \{s_i\})\}$ for which each G_n is a group and each of the corresponding collections $\{d_i\}$ and $\{s_i\}$ consists of group homomorphisms. Denote the identity element of each such group by e_n. A* ***map of simplicial groups*** *is a simplicial map between simplicial groups whose dimension-wise maps are group homomorphisms.*

The category-theoretic version of this definition is that a simplicial group is a (covariant) functor $G : \Delta^{op} \to \mathscr{G}$ where $\mathscr{G} \subset \mathcal{S}$ is the (sub)category of groups.

Theorem 2.1.2 *([2], Theorem 17.1) Every Simplicial Group is a Kan Complex.*

Corollary 2.1.3 *Suppose G is a simplicial group. Let e be the one-point simplicial set consisting of identity elements $e_n \in G_n$. Then the homotopy groups $\pi_n(G, e)$ exist for each $n > 0$.*

Recall that when G is a simplicial group, $[x] \bullet [y]$ denotes the group operation in $\pi_n(G, e)$ (well-defined since G is a Kan Complex from Theorem 2.1.2), and let concatenation xy denote the group operation in each group G_n from here on.

Proposition 2.1.4 *([2], Proposition 17.2) If G is a Simplicial Group then*

$$[x] \bullet [y] = [xy] \in \pi_n(G, e) \ \forall \ x, y \in \widetilde{G}_n.$$

Consequently, $[x]^{-1} = [x^{-1}] \in \pi_n(G, e)$ and $[e]$ is the identity of the group $\pi_n(G, e)$.

As a corollary we have a stronger property than what we already have in Theorem 1.3.7

Proposition 2.1.5 *([2],Proposition 17.3) If G is a simplicial group then $\pi_n(G, e)$ is abelian $\forall \ n > 0$.*

2.2 Chain Complex Construction

Definition 2.2.1 *Given simplicial group G, define $\bar{G}_n = G_n \cap ker(d_0) \cap ker(d_1) \cap \cdots \cap ker(d_{n-1})$ and $\widetilde{G}_n = G_n \cap ker(d_1) \cap ker(d_2) \cap \cdots \cap ker(d_n)$ for each $n \in \mathbb{N}$.*

Lemma 2.2.2 *([2], Proposition 17.3.iii) If G is a simplicial group with $\bar{G}_n$ as defined above, then*

$$d_{n+1}(\bar{G}_{n+1}) \triangleleft \bar{G}_n \text{ and } d_{n+1}(\bar{G}_{n+1}) \triangleleft G_n.$$

The above lemma allows definition of a **Chain Complex** [19], $\bar{G}$, by

$$\cdots \xrightarrow{d_{n+2}} \bar{G}_{n+1} \xrightarrow{d_{n+1}} \bar{G}_n \xrightarrow{d_n} \bar{G}_{n-1} \xrightarrow{d_{n-1}} \cdots .$$

We denote the restriction $d_i|_{\bar{G}} = \bar{d}_i$. Now we can define n**-cycles** $Z_n(\bar{G}) = ker(\bar{d}_n) \leq \bar{G}_n$, n**-boundaries** $B_n := B_n(\bar{G}) = im(\bar{d}_{n+1})$, and we define

$$\pi'_n(G) := Z_n(\bar{G})/B_n(\bar{G}).$$

This allows an alternative to the canonical construction for the homotopy group given in Definition 1.3.6 using these Chain Complexes rather than the sometimes-cumbersome homotopy of Definition 1.3.1:

Proposition 2.2.3 *([2], Proposition 17.4)* $\pi_n(G) \approx \pi'_n(G)\ \forall\ n \geq 0$ *by the natural identification* $[x] \mapsto [x]$.

Now we see that when X happens to be a simplicial group we are able to use the group operation inherited from the bijection between $\pi_0(X, \Phi)$ and $\pi'_0(X)$ to define $\pi_0(X, \Phi)$ as a group in a natural way.

3 Kan Fibrations([2], Ch.7 and Ch.18)

3.1 Definition

Definition 3.1.1 *Let* X, L *be simplicial sets,* $f : X \to L$ *a simplicial map, and*

$$C_{(n,k)} = (x_0, \ldots, \hat{x}_k, \ldots, x_{n+1})$$

be a compatible list in X. *Suppose that* f *has the property that some preimage* x *of* y *(i.e. some* x *with* $f(x) = y$*) is an extender for* $C_{(n,k)}$ *whenever* y *extends the corresponding compatible set* $f(C_{(n,k)}) = (f(x_q))_{q \neq k} \subseteq L$. *Then* f *satisfies the* ***Image-Extension Condition*** *on* $C_{(n,k)}$.

A simplicial map f satisfying the Image-Extension Condition on every "extensible," compatible list in its domain is known as a **Kan Fibration**, or a **fibration of simplicial sets**; in this case, X is the **total complex**, L is the **base complex** and the collection of data (X, f, L) is the **fiber space** defined by the Kan fibration f. Notice that this structure requires neither X nor L to be Kan Complexes, but only for any "extensible" sets that exist within these simplicial sets to satisfy the Image-Extension Condition.

Lemma 3.1.2 *Given a simplicial set* X *and any one-point simplicial set* $\Phi \subseteq X$, X *is a Kan Complex if and only if the unique simplicial map* $p : X \to \Phi$ *is a Kan fibration.*

Proof: Since $\Phi_n = \{\phi_n = s_{n-1}s_{n-2}\cdots s_1 s_0 \phi_0\}$ for given $\phi_0 \in X_0$, any simplicial map $p : X \to \Phi$ must have $x \mapsto \phi_n$ for every $x \in X_n$. So p is uniquely defined by $p_n(x) = \phi_n\ \forall\ x \in X_n$.

Suppose X is a Kan Complex and let $C_{(n,k)} = (x_0, \ldots, \hat{x}_k, \ldots, x_{n+1})$, be a compatible list in X_n. Since X is a Kan Complex, there is an extender $x \in X_{n+1}$ for $C_{(n,k)}$. But $p(x) = \phi_{n+1}$ and $p(C_{(n,k)}) = (\phi_n, \ldots, \phi_n)$, by definition, so that $p(x)$ extends $p(C_{(n,k)})$. Therefore the preimage, x, of the extender ϕ_{n+1} of $p(C_{(n,k)})$ extends $C_{(n,k)}$. Such ϕ_{n+1} is the only possible extender of $p(C_{(n,k)})$ by definition of the one-point simplicial set Φ, so every extender of $p(C_{(n,k)})$ has a preimage that extends $C_{(n,k)}$. Therefore p is a Kan fibration.

Conversely, if the simplicial map $p : X \to \Phi$ by $p(x) = \phi_n$ for every $x \in X_n$ is a Kan fibration, then consider the compatible list $C_{(n,k)}$ once more. By definition we have $p(C_{(n,k)}) = (\phi_n, \dots, \phi_n) \subseteq \Phi_n$. Since p is a Kan fibration, any extender of $p(C_{(n,k)})$ has a preimage that extends $C_{(n,k)}$. But $\phi_{n+1} \in \Phi_{n+1}$ exists and extends $p(C_{(n,k)})$. Therefore there must be an extender $x \in X_{n+1}$ for $C_{(n,k)}$ which has $p(x) = \phi_{n+1}$. Thus every compatible list in X has an extender, in which case X is a Kan Complex.

□

3.2 Long Exact Sequence of Kan Fibrations

Given simplicial map $f : (X, \Phi_X) \to (L, \Phi_L)$, set $F = f^{-1}(\Phi_L)$.

Proposition 3.2.1 *([2], Proposition 7.3) Let $(X, \Phi_X) \in \mathcal{SS}_*$ be a Pointed Complex and (X, f, L) be a fiber space, with $\Phi_L = f(\Phi_X)$. Then (F, Φ_X) is a Kan Pair.*

From now on, whenever we write a homotopy group $\pi_n(X, \Phi_X)$, we assume that (X, Φ_X) is a Kan Pair, and similarly for relative homotopy groups.

Consider the compatible list $C_{(n-1,0)} = (\phi^{(X)}_{n-1}, \phi^{(X)}_{n-1}, \dots, \phi^{(X)}_{n-1})$. Any given $y \in \widetilde{L}_n$ extends the list $f(C_{n-1,0}) = (\phi^{(L)}_{n-1}, \dots, \phi^{(L)}_{n-1})$, and if f is a Kan Fibration this implies $\exists\, x \in X_n$ with $d_i x = \phi^{(X)}_{n-1}$ $\forall\, 1 \le i \le n$ (so that x extends $C_{(n-1,0)}$) and $f(x) = y$. But $d_0 y = \phi^{(L)}_{n-1}$ since $y \in \widetilde{L}_n$ and since f is a simplicial map we have $d_0 y = d_0 f(x) = f(d_0 x) = \phi_{n-1}$, so $d_0 x \in F_{n-1}$. Now we have class $[d_0 x] \in \pi_{n-1}(F, \Phi_X)$ and we can define a **connecting homomorphism** $d_\sharp : \pi_n(L, \Phi_L) \to \pi_{n-1}(F, \Phi_X)$ by $[y] \mapsto [d_0 x]$.

Lemma 3.2.2 *The induced map $f_* : \pi_n(X, F, \Phi_X) \to \pi_n(L, \Phi_L)$ is an isomorphism* $\forall\, n \ge 2$.

Recall that $\pi_n(L, \Phi_L, \Phi_L) = \pi_n(L, \Phi_L)$ when $n \ge 2$. The inverse isomorphism to f_* is the map q defined by $q[y] = [x]$ for such x as used to define the connecting homomorphism $d_\sharp$ above. We also notice that $d_\sharp f_*[x] = d_\sharp[y] = [d_0 x] = d[x]$ for each $[x] \in \pi_n(X, F, \Phi_X), n \ge 2$ (using extender $y = f(x)$ for $f(C_{(n-1,0)})$ in the construction above and Definition 1.3.16). It follows that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(X, F, \Phi_X) \arrow[r, "d"] \arrow[d, "f_*"'] & \pi_n(F, \Phi_X) \arrow[r, "i"] \arrow[d, "="] & \pi_n(X, \Phi_X) \arrow[r, "j"] \arrow[d, "="] & \pi_n(X, F, \Phi_X) \arrow[r] \arrow[d, "f_*"'] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}(L, \Phi_L) \arrow[r, "d_\sharp"] & \pi_n(F, \Phi_X) \arrow[r, "i"] & \pi_n(X, \Phi_X) \arrow[r, "f_*"] & \pi_n(L, \Phi_L) \arrow[r] & \cdots
\end{tikzcd}.$$

In this diagram, exactness at $\pi_n(F, \Phi_X)$ via $i \circ d_\sharp$, exactness at $\pi_n(X, \Phi_X)$ via $f_* \circ i$, and exactness at $\pi_n(L, \Phi_L)$ via $d_\sharp \circ f_*$ all follow from the long exact sequence of Homotopy Groups on X(Theorem 1.3.17). The result is another exact sequence:

Definition 3.2.3 *The sequence*

$$\cdots \longrightarrow \pi_{n+1}(L,\Phi_L) \xrightarrow{d_\sharp} \pi_n(F,\Phi_X) \xrightarrow{i} \pi_n(X,\Phi_X) \xrightarrow{f_*} \pi_n(L,\Phi_L) \longrightarrow \cdots$$

is the ***long exact sequence of Kan Fibrations****. For Kan fibration f the sequence $(F,\Phi_X) \subseteq (X,\Phi_X) \xrightarrow{f} (L,\Phi_L)$ is called a* ***fiber sequence****.*

3.3 Simplicial Group Action and Twisted Cartesian Products

Definition 3.3.1 *Given simplicial sets X (with maps d_i^X and s_j^X) and L (with maps d_i^L and s_j^L), the* ***Cartesian Product*** *of X and L is the simplicial set $P = X \times L$ with n-simplices*

$$P_n := X_n \times L_n,$$

face maps and degeneracy maps

$$d_i := d_i^X \times d_i^L$$

and

$$s_j := s_j^X \times s_j^L$$

respectively.

Definition 3.3.2 *A simplicial group G* ***acts*** *(from the left) on a simplicial set X if each group G_n acts (from the left) on the corresponding set X_n, and these actions commute through face maps and degeneracy maps: $d_i(gx) = (d_i g)(d_i x)$ and $s_j(gx) = (s_j g)(s_j x)$.*

Put another way, G acts on X if the map $\psi : G \times X \to X$ defined for each dimension n by $\psi_n(e_n, x) = x \;\forall\, x \in X_n$ and $\psi_n(g_1 g_2, x) = \psi_n(g_1, \psi_n(g_2, x))$ (giving the group action) is actually a *simplicial* map on the Cartesian Product $G \times X$.

Let simplicial group G act on simplicial set F. Consider another simplicial set B and a map t, defined for each dimension $n > 0$ by $t_n : B_{n+1} \to G_n$ having the following relationships with face maps and degeneracy maps ($b \in B_{n+1}$):

a) $d_n t_n(b) = (t_{n-1}(d_{n+1}b))^{-1} t_{n-1}(d_n b)$.

b) $d_i t_n(b) = t_{n-1}(d_i b) \;\forall\, 0 \le i \le n-1$.

c) $s_j t_n(b) = t_{n+1}(s_j b) \;\forall\, 0 \le j \le n$.

d) $t_{n+1}(s_{n+1}b) = e_{n+1}$.

We can define a simplicial set structure with n-simplices $F_n \times B_n$ by letting face maps be

$$d_i(f,b) = \begin{cases} (d_i f, d_i b) & 0 \leq i < n, \\ \\ (t_{n-1}(b) d_n f, d_n b) & i = n \end{cases}$$

and

$$s_j(f,b) = (s_j f, s_j b) \ \forall \ 0 \leq j \leq n$$

for degeneracy maps. This simplicial set is the **Twisted Cartesian Product** with **fiber** F, **base** B, **twisting function** t and **group** G, denoted $F \times_t B$.

Theorem 3.3.3 *([2], Proposition 18.4.i) The natural projection map $p : F \times_t B \to B$ is a Kan fibration with total space $F \times_t B$, fiber F and base B.*

Definition 3.3.4 *If G acts on X such that for every $n \in \mathbb{N}$ the only $g \in G_n$ for which any one $x \in X_n$ has $gx = x$ is $g = e_n$, then G acts **principally** on X. Thus if F in a Twisted Cartesian Product $F \times_t B$ is a simplicial group then we call $F \times_t B$ a **Principal Twisted Cartesian Product**.*

When G acts principally on X we have equivalence classes $[x] = \{gx | g \in G_n \subseteq X_n\}$, which form a "quotient subcomplex," B, of X. The projection $p : X \to B$ by $x \mapsto [x]$ is the **principal fibration** of X with group G and base B.

Lemma 3.3.5 *([2], Lemma 18.2) Every principal fibration is a Kan fibration.*

Theorem 3.3.6 *([2], Proposition 18.4.iii) If $F = G$ then the projection $p : F \times_t B \to B$ on the principal Twisted Cartesian Product is a principal fibration.*

4 Loop Groups([17, 18] and [2] Ch.18)

The canonical definition (Definitions 1.3.5 and 1.3.6) of the homotopy groups requires the simplicial set under consideration to be a Kan Complex. By virtue of Theorem 2.1.2, Proposition 2.1.4 and Proposition 2.2.3, simplicial groups are among the most convenient and transparent Kan Complexes to work with. Thus methods have been developed to construct a simplicial group from a simplicial set. An important result of Kan's ([17, 18]) is that the construction we describe in this section canonically describes the homotopy of a simplicial set whether that simplicial set is a Kan Complex or not, by construction of a particular simplicial group.

4.1 Kan's Loop Construction

Definition 4.1.1 *A simplicial group G is a **free simplicial group** if each group $G_n, n \in \mathbb{Z}_{\geq 0}$ is a free group, and each group G_n has a basis B_n so that the collection $\{B_n\}$ of bases is preserved by degeneracy maps: for every $b \in B_n$, $s_j b \in B_{n+1}$ $\forall$ $0 \leq j \leq n$.*

We use a construction by Kan[17, 18] that results in a free simplicial group starting with a simplicial set X. We start with the requirement not that X be a Kan Complex, but just that X be a *reduced* simplicial set (i.e. $X_0 = \{\phi_0\}$). Later on even this requirement will be relaxed. Take the set X_{n+1} of $(n+1)$-simplices, and define a basis element σ_x for each $x \in X_{n+1}$:

Definition 4.1.2 *Given $n \in \mathbb{Z}_{\geq 0}$ and a reduced simplicial set X (with maps d_i^X and s_j^X), the **Loop Group** GX of X is a simplicial group wherein the set GX_n of n-simplices is a group with one generator σ_x for each $x \in X_{n+1}$ and a relation*

$$\sigma_{s_n^X y} = e_n$$

defining the identity of GX_n, for each $y \in X_n$. Let face maps $d_i : GX_n \to GX_{n-1}$ be defined by setting

$$d_i \sigma_x = \begin{cases} \sigma_{d_i^X x}, & 0 \leq i \leq n-1 \\ \\ (\sigma_{d_{n+1}^X x})^{-1} \sigma_{d_n^X x}, & i = n \end{cases}$$

and extending linearly. Similarly, extend

$$s_j \sigma_x = \sigma_{s_j^X x} \; \forall \; 0 \leq j \leq n$$

linearly to define degeneracy maps $s_j : GX_n \to GX_{n+1}$.

Theorem 4.1.3 *Given a reduced simplicial set X, GX is a free simplicial group.*

Proof: Recall that the identity element is a required generator for a group unless other relations are specified. The relation $\sigma_{s_n^X(y)} = e_n, y \in X_n$ merely assigns the generator e_n to each "n-degenerate" element of X_{n+1}. Otherwise, there are no nontrivial relations among the generators $\sigma_x, x \in X_n$ since each distinct generator corresponds to a distinct $(n+1)$-simplex in X. Therefore each GX_n is a free group.

Since $(n+1)$-simplices of X correspond directly to generators of GX_n, the face and degeneracy relationships of Definition 1.1.1 for simplicial set X imply the same relationships on the face maps and degeneracy maps acting on generators. For instance, note that for any $i < n$ and any $x \in X_{n+1}$ with corresponding generator $\sigma_x \in GX_n$,

$$d_i d_n \sigma_x = d_i((\sigma_{d^X_{n+1}x})^{-1}\sigma_{d^X_n x}) = (d_i\sigma_{d^X_{n+1}x})^{-1} d_i\sigma_{d^X_n x} = (\sigma_{d^X_i d^X_{n+1}x})^{-1}\sigma_{d^X_i d^X_n x} = (\sigma_{d^X_n d^X_i x})^{-1}\sigma_{d^X_{n-1} d^X_i x}.$$

But $\sigma_{d^X_i x} \in GX_{n-1}$, so

$$(\sigma_{d^X_n d^X_i x})^{-1}\sigma_{d^X_{n-1} d^X_i x} = d_{n-1}\sigma_{d^X_i x} = d_{n-1} d_i \sigma_x.$$

It follows that GX is a simplicial group with free groups GX_n as sets of n-simplices. Furthermore, we see that $s^X_j x \in X_{n+2}\ \forall\ 0 \leq j \leq n+1$ implies $\sigma_{s^X_j x} = s_j \sigma_x$ is a generator for GX_{n+1} for each $0 \leq j \leq n$. So if we identify $\{\sigma_x \mid x \in X_{n+1}\} = B_n$ as the basis for GX_n, we see that

$$s_j b \in B_{n+1}\ \forall\ b \in B_n\ \forall\ 0 \leq j \leq n.$$

Therefore GX is a free simplicial group.

□

The identification $x \mapsto \sigma_x$ gives functions $t_n : X_{n+1} \to GX_n$ for which

$$d_i t_n(x) = d_i \sigma_x = \sigma_{d^X_i x} = t_{n-1}(d^X_i x)$$

$\forall\ 0 \leq i \leq n-1$,

$$d_n t_n(x) = d_n \sigma_x = (\sigma_{d^X_{n+1}x})^{-1}\sigma_{d^X_n x} = (t_{n-1}(d^X_{n+1}x))^{-1} t_{n-1}(d^X_n x),$$

and

$$s_j t_n(x) = s_j(\sigma_x) = \sigma_{s^X_j x} = t_{n+1}(s^X_j x)\ \forall\ 0 \leq j \leq n.$$

So the map $t : X \to GX$ defined for each dimension by these t_n is a twisting function from which we can form a Twisted Cartesian Product:

Definition 4.1.4 *Given a reduced simplicial set X with corresponding, constructed free simplicial group GX, $EX = GX \times_t X$ is the* ***loop complex of*** *X, where $t_n : X_{n+1} \to GX_n$, is defined by $t_n(x) = \sigma_x$.*

We denote the generators of GX by $t(x) := t_n(x) \in GX_n$ for $x \in X_{n+1}$ from here on. Since any group acts naturally on itself (hence any simplicial group acts naturally on itself), EX is a Principal Twisted Cartesian Product with base X, group GX and fiber GX. Kan shows[18] that EX is a contractible simplicial set when X is a reduced simplicial set.

4.2 Functoriality of Kan's Loop Group Construction

At this point, we are mostly concerned with applying the loop group construction to reduced simplicial sets.

Definition 4.2.1 *A map of reduced simplicial sets $f : X \to Y$ is a* ***weak homotopy equivalence*** *if and only if f_* induces isomorphisms of all homotopy groups. A homomorphism of simplicial groups $f : A \to B$ is a weak homotopy equivalence if and only if $f_* : \pi_i(A, e) \to \pi_i(B, e)$ is an isomorphism, for e the identity element of the groups A or B as appropriate, and for every $i \geq 0$.*

Denote the category of reduced simplicial sets by $\mathcal{SS}_{red}$ and the category of simplicial groups by $\mathcal{SG}$. Some features of the loop group construction from the previous section are:

Lemma 4.2.2 *If $f : X \to Y$ is a simplicial map on reduced simplicial sets, then there is an induced homomorphism of simplicial groups $Gf : GX \to GY$, defined on generators by $t(x) \mapsto \tilde{t}(f(x))$ for $x \in X$ and generators $t(x)$ of GX and $\tilde{t}(y)$ for GY. This admits a functor G from $\mathcal{SS}_{red}$ to $\mathcal{SG}$.*

Lemma 4.2.3 *The loop group construction fits GX into a fibration*

$$GX \to EX \to X$$

of simplicial sets, with EX of the homotopy type of a point. This fibration is also functorial: a map $f : X \to Y$ in $\mathcal{SS}$ gives a map of fibrations

$$\begin{array}{ccccc} GX & \to & EX & \to & X \\ \downarrow Gf & & \downarrow Ef & & \downarrow f \\ GY & \to & EY & \to & Y. \end{array}$$

Using the homotopy long exact sequence for a fibration, we have

Lemma 4.2.4 *If $f : X \to Y$ is a simplicial map that is a weak homotopy equivalence, then the homomorphism of free simplicial groups $Gf : GX \to GY$ is also a weak homotopy equivalence.*

Kan [18] defines a relation of "loop homotopy" between two homomorphisms of simplicial groups $f, g : A \to B$ and then proves that if A is free, then this relation is an equivalence relation. The definition of loop homotopy implies that the homotopy leaves the basepoint (the identity element) of the simplicial group "fixed"; i.e., it is a homotopy relative to the basepoint so that loop homotopic maps are always simplicially homotopic as in Definition 1.4.1. A "loop homotopy equivalence" of free simplicial groups A and B is defined to be a homomorphism $f : A \to B$ of simplicial groups such that there exists a homomorphism

$g : B \to A$ of simplicial groups such that both $g \circ f$ and $f \circ g$ are loop homotopic to the appropriate identity homomorphisms.

It is clear then, that any loop homotopic equivalence is a weak homotopy equivalence, and Kan proves the converse in special case:

Theorem 4.2.5 *(Proposition 6.5 of [17]) Let $f : A \to B$ be a homomorphism of free simplicial groups that is also a weak homotopy equivalence. Then f is a loop homotopy equivalence.*

So, we have

Corollary 4.2.6 *If $f : X \to Y$ is a simplicial map (with X and Y reduced simplicial sets) that is a weak homotopy equivalence, then $Gf : GX \to GY$ is a loop homotopy equivalence.*

In addition, using Kan's work in [18] one can prove

Theorem 4.2.7 *If X and Y are reduced simplicial sets (neither necessarily Kan complexes) and $f, g : X \to Y$ are maps of simplicial sets that are simplicially homotopic, relative to the basepoint, then the induced homomorphisms $Gf, Gg : GX \to GY$, are loop homotopic.*

4.3 Loop Groups on Nonreduced Simplicial Sets

We will also need to construct Loop Groups on nonreduced simplicial sets in a functorial way. Kan [18] constructs such loop groups using maximal trees; another construction is obtained by Berger as described by Duflot, and functoriality may be obtained by incorporating the choice of maximal tree into the category of definition.

One way of doing this is exposited in Duflot[11]and briefly summarized below.

Definition 4.3.1 *A simplicial set X is **star-connected at basepoint** $\phi_0 \in X_0$ if and only if for any $z \in X_0$ $\exists$ $y(z) \in X_1$ for which $d_1(y(z)) = \phi_0$ and $d_0(y(z)) = z$. Call such $y(z)$ a **ray** at z.*

We see easily that any star-connected simplicial set is connected, since by definition $z \sim \phi_0$ for every $z \in X_0$ (i.e. the required list of 1-simplices from Definition 1.3.3 is $D_1 = (y(z))$).

Definition 4.3.2 *Given $(X, \Phi) \in \mathcal{SS}_*$ with X star-connected at ϕ_0, a **ray function** $\omega : X_0 \to X_1$ is any function such that $\omega(x) = y$ where y is a ray at x.*

Definition 4.3.3 *[18]Let $(X, \Phi) \in \mathcal{SS}_*$ with X (star-)connected at ϕ_0. An n-**loop of** X is a sequence*

$$(x_1, x_2, \ldots, x_{2k}) \subset X_{n+1},$$

$k > 0$, *wherein*

$$d_{n+1}x_{2j-1} = d_{n+1}x_{2j} \;\forall\; 1 \leq j \leq k,$$

$$d_0 d_1 \cdots d_n x_{2j} = d_0 d_1 \cdots d_{2j+1} \;\forall\; 1 \leq j \leq k-1,$$

and

$$d_0 d_1 \cdots d_n x_1 = d_0 d_1 \cdots d_n x_{2k}.$$

Such loop is ***reduced*** *if* $x_j \neq x_{j+1} \;\forall\; 1 \leq j \leq 2k$.

Definition 4.3.4 *Let* $(X, \Phi) \in \mathcal{SS}_*$ *with* X *(star-)connected at* ϕ_0*. A* ***tree*** *in* X *is a connected subcomplex* $T \subset X$ *such that* $\phi_0 \in T_0$ *and* T *contains no reduced loops.* T *is a* ***maximal tree*** *if* $T_0 = X_0$.

Proposition 4.3.5 *Given* $(X, \Phi) \in \mathcal{SS}_*$ *with* X *star-connected at* ϕ_0 *and corresponding ray function* $\omega :$ $X_0 \to X_1$*, let* $T_\omega = X(X_0, \omega(X_0))$*. Then* T_ω *is a maximal tree in* X.

Recall Definition 1.2.2 for T_ω and note that Φ is always a subcomplex of T_ω. Using either Kan [18] or Berger (see the variation of Berger's construction discussed in [11]), given a star-connected simplicial set X with ray function ω, and maximal tree $T_\omega(X)$, one may construct a loop group $G(X, \omega)$:

Definition 4.3.6 *For every* $n \geq 0$*,* $G(X, \omega)_n$ *is constructed by taking the free group on the set* X_{n+1} *and imposing the following relations:*

1) $s_n x \mapsto 1$*, for every* $x \in X_n$.

2) $y \mapsto 1$*, for every* $y \in (T_\omega(X))_{n+1}$.

One sees that $G(X, \omega)_n$ is a free group on the set $X_{n+1} - s_n(X_n) - (T_\omega(X))_{n+1}$.

As in the reduced case, we denote the generator of $G(X, \omega)$ corresponding to $x \in X_{n+1}$ by $t(x)$.

4.3.1 Functoriality for the Nonreduced Case

The domain category of the functors we consider is the category whose objects are the triples (X, Φ, ω) where (X, Φ) is a pointed star-connected simplicial set, and ω is a ray function. A morphism $f : (X, \Phi_X, \omega_X) \to (Y, \Phi_Y, \omega_Y)$in this category is a map of pointed simplicial sets $f : X \to Y$ such that $\omega_Y \circ f = f \circ \omega_X$.

Using the construction details (for either Kan's or Berger's construction), we have the analogs of the theorems in the previous section:

Lemma 4.3.7 *([11], Lemma 4.0.22) If $f : (X, \Phi_X, \omega_X) \to (Y, \Phi_Y, \omega_Y)$ is a morphism as defined above, then there is a functorial induced homomorphism of simplicial groups $Gf : G(X, \omega_X) \to G(Y, \omega_Y)$ defined by*

$$Gf(t(x)) = t(f(x)).$$

Corollary 4.3.8 *Given a triple (X, Φ_X, ω) in our domain category, there is a fibration*

$$G(X, \omega) \to E(X, \omega) \to X$$

of simplicial sets, with $E(X, \omega)$ of the homotopy type of a point. This fibration is also functorial: a map of triples $f : (X, \phi_X, \omega_X) \to (Y, \phi_Y, \omega_Y)$ in our domain category gives a map of fibrations

$$\begin{array}{ccccc} G(X,\omega_X) & \to & E(X,\omega_X) & \to & X \\ \downarrow Gf & & \downarrow Ef & & \downarrow f \\ G(Y,\omega_Y) & \to & E(Y,\omega_Y) & \to & Y. \end{array}$$

Since our simplicial groups $G(X, \omega)$ are always free simplicial groups, we also have

Corollary 4.3.9 *If $f : (X, \Phi_X, \omega_X) \to (Y, \Phi_Y, \omega_Y)$ is a morphism that is a weak homotopy equivalence, then $Gf : G(X, \omega_X) \to G(Y, \omega_Y)$ is a loop homotopy equivalence.*

and

Corollary 4.3.10 *If (X, Φ_X, ω_X) and (Y, Φ_Y, ω_Y) are objects in our category and*

$$f, g : (X, \Phi_X, \omega_X) \to (Y, \Phi_Y, \omega_Y)$$

are morphisms that are simplicially homotopic, relative to the basepoint, then the induced homomorphisms $Gf, Gg : G(X, \omega_X) \to G(Y, \omega_Y)$, are loop homotopic.

5 More Examples of Simplicial Sets and Groups

5.1 Nerve Constructions

Example 5.1.1 *Nerve of a Small Category[3, 9]: Given a category A which we can think of as "small" (i.e. the objects form a set) and any $n \in \mathbb{Z}_{\geq 0}$, define*

$$NA_n = \left\{ x = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_n} a_n := (\alpha_1|\alpha_2|\cdots|\alpha_n) \right\},$$

the set of n-tuples of composeable morphisms, as the set of n-simplices; let

$$d_i x = a_0 \overset{\alpha_1}{\to} \cdots \overset{\alpha_{i-1}}{\to} a_{i-1} \overset{\alpha_{i+1}\circ\alpha_i}{\to} a_{i+1} \overset{\alpha_{i+2}}{\to} \cdots \overset{\alpha_n}{\to} a_n$$

for each $0 < i < n$(with d_0 and d_n by deleting a_0 and a_n, respectively, from the n-tuple), and

$$s_j x = a_0 \overset{\alpha_1}{\to} \cdots \overset{\alpha_j}{\to} a_j \overset{id_{a_j}}{\to} a_j \overset{\alpha_{j+1}}{\to} a_{j+1} \overset{\alpha_{i+2}}{\to} \cdots \overset{\alpha_n}{\to} a_n$$

*for each $0 \leq j \leq n$. Then $NA = \{NA_n\}$ with face maps and degeneracy maps as defined above constitutes a simplicial set, called the **nerve** of the category A. When the morphisms involved are of more concern to us, we will denote the nerve elements by $\boldsymbol{\alpha} := (\alpha_1|\alpha_2|\cdots|\alpha_n)$. Note that with this notation we write $s_0\boldsymbol{\alpha} = (id_{a_0}|\alpha_1|\alpha_2|\cdots|\alpha_n)$ and $s_n\boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_n|id_{a_n})$.*

Notice that NA_0 is just the set of objects of A, while NA_1 is the set of morphisms of A. We will not need to discuss the *geometric realization* of a simplicial set for our purposes (see [2],[9], etc. for descriptions), but it suffices to say that the geometric realization of the nerve of a category, denoted the *Classifying Space* of that category, has widespread applications. The nerve itself will be the centerpiece of an important construction later on, and with this application we will note that geometric realization is a functorial operation.

Example 5.1.2 *As a category, a group G has one object, $*$, and a morphism $g : * \to *$ corresponding to each group element g such that each morphism has an inverse. With this viewpoint, the 1-simplices of the nerve $N(G)$ from Example 5.1.1 would correspond to the elements of G, but NG_0, defined to consist of the objects the category G, would just be the object $*$. Therefore NG is a reduced simplicial set whenever G is a group.*

Example 5.1.3 *Recall from Example 5.1.2 that when G is a group (viewed as a category) with identity element e, the nerve NG is a reduced simplicial set. So there is a single 0-simplex which we denote $* := \phi_0$. The one-point simplicial set is constructed as*

$$\phi_1 = \phi_0 \overset{id}{\to} \phi_0 := e;$$

$$\phi_2 = \phi_0 \overset{id}{\to} \phi_0 \overset{id}{\to} \phi_0 := (e|e) \in G \times G$$

$$\vdots$$

$$\phi_n = \overbrace{\phi_0 \overset{id}{\to} \phi_0 \overset{id}{\to} \cdots \overset{id}{\to} \phi_0}^{n\ times} := (e|e|\cdots|e) = 1 \in G^n.$$

So if we identify $NG_n := G^n$ *we see that an element* $\boldsymbol{g} = (g_1|g_2|\cdots|g_n) \in NG_n$ *has*

$$d_i\boldsymbol{g} = \begin{cases} (g_2|g_3|\cdots|g_n), & i = 0 \\ (g_1|g_2|\cdots|g_{i-1}|g_{i+1}g_i|g_{i+2}|\cdots|g_n), & 1 \leq i \leq n-1 \\ (g_1|g_2|\cdots|g_{n-1}), & i = n. \end{cases}$$

Now $\boldsymbol{x} \in \widetilde{NG_n}$ *with* $n > 1$ *implies* $d_i\boldsymbol{x} = 1 \in G^{n-1}$*, so that* $\boldsymbol{x} = 1 \in G^n$*. Therefore* $\pi_0(NG) = [\phi_0] := 1$ *and* $\pi_n(NG) = [1] = 1\ \forall\ n > 1$*. Again,* $\boldsymbol{x} \in NG_1$ *is identified with* $x \in G$ *via morphism* $\phi_0 \xrightarrow{x} \phi_0$*, so clearly* $d_0x = d_1(x) = \phi_0$*. Therefore* $\widetilde{NG_1} = NG_1 = G$*. But homotopy in* NG_1 *dictates that if* $g_1 \sim g_2 \in NG_1$ *then there is a homotopy* $\boldsymbol{y} = (a|b) \in NG_2$ *such that*

1) $d_0\boldsymbol{y} = s_0d_0(g_1) = s_0(\phi_0) = \phi_1 = e$*, which implies* $b = e$*.*

2) $d_1\boldsymbol{y} = g_1 = ba = ea$*, so* $\boldsymbol{y} = (g_1|e)$*.*

3) $d_2\boldsymbol{y} = g_2 = a = g_1$*, so* $g_1 = g_2$*.*

We conclude that each $x \in \widetilde{NG_1} := G$ *represents a distinct homotopy class in* $\pi_1(NG)$*, in which case* $\pi_1(NG) := G$*. Therefore*

$$\pi_n(NG) = \begin{cases} G, & n = 1 \\ 1, & else. \end{cases}$$

□□

5.2 Functorial Constructions

Example 5.2.1 *Reverse of a Simplicial Set: Given a simplicial set* X*, we can define another simplicial set by keeping simplices as they are but "reversing" the degrees of face maps and degeneracy maps:*

$$X_n^{rev} = X_n; d_i^{rev} = d_{n-i}; s_j^{rev} = s_{n-j}.$$

The degree of the face maps and degeneracy maps in X^{rev} depends on the dimension on which they act, and calculations must reflect this: for example, when $0 \leq i < j \leq n$ and $x \in X_n = X_n^{rev}$ we have

$$d_i^{rev}d_j^{rev}x = d_{n-1-i}d_{n-j}x = d_{n-j}d_{n-1-i+1}x = d_{n-1-(j-1)}d_{n-i}x = d_{j-1}^{rev}d_i^{rev}x,$$

and

$$s_j^{rev}s_i^{rev}x = s_{n+1-j}s_{n-i}x = s_{n+1-i}s_{n+1-j}x = s_{n+1-i}s_{n-(j-1)}x = s_i^{rev}s_{j-1}^{rev}x.$$

Note that the second face map operates on the $n-1$-simplex $d_j^{rev}x$ and the second degeneracy map operates on the $n+1$-simplex $s_i^{rev}x$. The simplicial set whose n-simplices are X_n and whose face maps and degeneracy maps are the d_i^{rev} and s_j^{rev} as above is called the **reverse** of the simplicial set X, denoted X^{rev}.

Now we formulate the definition for X^{rev} as a functor from Δ^{op} to $\mathcal{S}$. Given a functor $X : \Delta^{op} \to \mathcal{S}$, X^{rev} is the functor $X^{rev} : \Delta^{op} \to \mathcal{S}$ defined by

$$X^{rev}(\mathbf{n}) = X(\mathbf{n}),$$

on objects of Δ^{op}, and on morphisms as follows.

If $\alpha : \mathbf{n} \to \mathbf{m}$ is a morphism in Δ, define $\alpha^{rev} : \mathbf{n} \to \mathbf{m}$ by

$$\alpha^{rev}(u) = m - \alpha(n-u).$$

Proposition 5.2.2 *If α is a morphism in Δ (hence in Δ^{op}) then α^{rev} is a morphism in Δ (hence in Δ^{op}).*

Now, given a morphism $\alpha \in Hom_{op}(\mathbf{m}, \mathbf{n})$, let

$$X^{rev}(\alpha) := X(\alpha^{rev}) : X(\mathbf{m}) \to X(\mathbf{n}).$$

Theorem 5.2.3 *Given a functor $X : \Delta^{op} \to \mathcal{S}$, X^{rev} as defined above is a functor from Δ^{op} to $\mathcal{S}$.*

Proposition 5.2.4 *$(X^{rev})^{rev} = X$, for every $X \in \mathcal{SS}$.*

Proposition 5.2.5 *There is a functor ${}^{rev} : \mathcal{SS} \to \mathcal{SS}$ defined on objects by ${}^{rev}(X) = X^{rev}$ for functor $X : \Delta^{op} \to \mathcal{S}$, and on morphisms by ${}^{rev}(f) = f$ for appropriate morphism f. Furthermore, ^{rev} is an isomorphism of categories.*

□□

Example 5.2.6 *Segal Subdivision[10]: Given simplicial set $X = \{(X_n; \{d_i\}; \{s_j\})\}$, set*

$$Sd(X)_n = X_{2n+1}; d_i^{Sd}x = d_i d_{2n+1-i}x; s_j^{Sd}x = s_j s_{2n+1-j}x$$

for any $x \in X_{2n+1}$.

A change of dimension by 1 in $Sd(X)$ amounts to a change of dimension in X by 2, and the dependence of the degrees of face maps and degeneracy maps on dimension warrants care in the arithmetic: $0 \leq i < j \leq$

$n, x \in X_{2n+1} = Sd(X)_n$ implies

$$\begin{aligned}
d_i^{Sd} d_j^{Sd}(x) &= d_i d_{2(n-1)+1-i} d_j d_{2n+1-j} x \\
&= d_i d_{2n-1-i} d_j d_{2n+1-j} x \\
&= d_{j-1} d_i d_{2n-i} d_{2n+1-j} x \\
&= d_{j-1} d_{2n-j} d_i d_{2n+1-i} x \\
&= d_{j-1} d_{2n-1-(j-1)} d_i d_{2n+1-i} x \\
&= d_{j-1} d_{2(n-1)+1-(j-1)} d_i d_{2n+1-i} x \\
&= d_{j-1}^{Sd} d_i^{Sd} x,
\end{aligned}$$

and

$$s_j^{Sd} s_i^{Sd} x = s_j s_{2(n+1)+1-j} s_i s_{2n+1-i} x$$

$$= s_j s_{2n+2+1-j} s_i s_{2n+1-i} x = s_i s_{j-1} s_{2n+2-j} s_{2n+1-i} x$$

$$= s_i s_{2n+2+1-i} s_{j-1} s_{2n+1-(j-1)} x = s_i^{Sd} s_{j-1}^{Sd} x.$$

It can be shown, similar to the case for functors G and ^{rev}, that this construction admits a covariant functor, $Sd : \mathcal{SS} \to \mathcal{SS}$. See[10] for a good description of this and other properties of the Subdivision.

□□

Chapter 2

Algebraic K-Theory

From here on, let R be a ring with identity 1, commutative where necessary, and consider (subcategories of) the category of R-modules with R-module homomorphisms $Hom_R(P, Q)$ for modules P and Q.

1 Projective Modules

1.1 Definitions[8][1]

Definition 1.1.1 *We adopt the following, equivalent definitions for our objects of interest – **finitely generated projective** R**-modules**:*

a) *"Diagram Completion Property": P is a projective R-module if and only if given any R-modules N, M, surjective homomorphism $\psi : M \to N$ and any homomorphism $\phi : P \to N$, $\exists\ \theta \in Hom_R(P, M)\ \ni \phi = \psi \circ \theta$.*

b) *"Section Property": P is projective if and only if given any R-module M, any surjective homomorphism $\psi \in Hom_R(M, P)$ has a right inverse (i.e. there is a section $s : P \to M\ \ni\ \psi \circ s = id_P$ and $s \in Hom_R(P, M)$).*

c) *"Splitting Property": P is projective if and only if any short exact sequence*

$$0 \to N \overset{\phi}{\to} M \overset{\psi}{\to} P \to 0$$

*of R-modules ending at P **splits**: $M \approx N \oplus P \approx im(\phi) \oplus im(s)$ where $s \in Hom_R(P, M)$ is a section for ψ as described above.*

d) *"Summand Property": P is a finitely generated projective R-module if and only if $\exists$ R-module Q and $n \in \mathbb{N}$ for which $P \oplus Q \approx R^n$.*

Example 1.1.2 *Projective But Not free: Suppose* $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*. Then the* R*-module* $P_1 = \langle(1,0)\rangle$ *is projective, but not free. Indeed, if* $P_2 = \langle(0,1)\rangle$ *then* $P_1 \oplus P_2 = R = R^1$ *so we have* P_1 *as a direct summand of a free module. But* R *has order 4 as an additive group, so that any free* R*-module of finite rank must have an order that is a multiple of 4. Since both* P_1 *and* P_2 *have order 2, neither module can be free.*

1.2 The Category $\mathcal{P}R$

Definition 1.2.1 *For finitely generated projective modules* P *and* Q*, define morphisms to be* R*-module homomorphisms* $Hom_R(P,Q)$*, and let* $\mathcal{P}R$ *be the resulting category of finitely generated projective* R*-modules. Among morphisms are* ***admissible injections****, which are injective homomorphisms* $P \overset{\phi}{\rightarrowtail} Q$ *for which* $0 \to P \overset{\phi}{\rightarrowtail} Q \twoheadrightarrow Q/P \to 0$ *is a short exact sequence in* $\mathcal{P}R$*, and* ***admissible surjections****, which are surjective homomorphisms* $P \overset{\psi}{\twoheadrightarrow} Q$ *for which there is a short exact sequence* $0 \to N \rightarrowtail P \overset{\psi}{\twoheadrightarrow} Q \to 0$ *in* $\mathcal{P}R$*.*

Note that by Definition 1.1.1.b,c that all surjective homomorphisms in $\mathcal{P}R$ are admissible surjections.

Definition 1.2.2 *For a finitely generated projective* R*-module* P *define the dual of* P *as*

$$P^* := Hom_R(P, R).$$

Lemma 1.2.3 *Suppose* $P, Q \in \mathcal{P}R$*.*

a) P^* *is a finitely generated projective* R*-module, and given an* R*-module homomorphism* $f : P \to Q$*, there is an* R*-module homomorphism* $f^* : Q^* \to P^*$ *defined by* $f^*(\alpha) = \alpha \circ f$ *for any* $\alpha \in Q^*$*. When* f *is injective* f^* *is surjective and when* f *is surjective* f^* *is injective.*

b) $\forall\ P, Q \in \mathcal{P}R, (P \oplus Q)^* \approx P^* \oplus Q^*$*.*

c) There is an exact contravariant functor, $^* : \mathcal{P}R \to \mathcal{P}R$*, defined on objects by* $P \mapsto P^*$ *and on morphisms by* $f \mapsto f^*$*. In particular,* $f^* : Q^* \to P^*$ *is admissible whenever* $f : P \to Q$ *is.*

d) The composite functor $^*\circ^* :=^{**}$ *is a covariant functor, equivalent to the identity functor; in fact, there is a natural transformation* $\eta : id \to\ ^{**}$ *of functors on* $\mathcal{P}R$ *such that for every object* P*,* $\eta(P) : P \to P^{**}$ *is an isomorphism.*

As a brief note, we define the isomorphism $\eta : id_{\mathcal{P}R} \to^{**}$ by assigning $[\eta(P)(p)](\psi) = \psi(p) \in R$ for $p \in P, \psi \in P^*$, so that $\eta(P) \in Hom_R(P, P^{**})$.

2 Classical $K_0(R)$ ([8], Chapter 1)

2.1 Generators and Relations

Definition 2.1.1 *Given the isomorphism classes $[P]$ of finitely generated projective modules over ring R, let F be the free abelian group on these classes and $S = \langle [P] + [Q] - [P \oplus Q] \rangle$ as a subgroup. Then $K_0(R) = F/S$ (i.e. $K_0(R)$ is the* ***Grothendieck Group,*** *or* ***Group Completion of the Semigroup*** *of isomorphism classes of finitely generated projective R-modules).*

Theorem 2.1.2 *([8], Lemma 1.1) Every element $A \in K_0(R)$ can be represented by a difference $A = [P] - [Q]$ of two isomorphism classes, and $[P_1] - [Q_1] = [P_2] - [Q_2] \in K_0(R)$ if and only if $\exists\ r \in \mathbb{N}\ \ni\ P_1 \oplus Q_2 \oplus R^r \approx P_2 \oplus Q_1 \oplus R^r$.*

Corollary 2.1.3 *Two generators $[P]$ and $[Q]$ of $K_0(R)$ are equal if and only if $\exists\ r \in \mathbb{N}\ \ni\ P \oplus R^r \approx Q \oplus R^r$.*

We refer these generators $[P]$ of $K_0(R)$ as **stable isomorphism classes** of finitely generated projective modules over R.

Example 2.1.4 *Grothendieck Group of a field: Let $R = F$ be a field. It can be shown through basic linear algebra principles that if R is a field then any (finitely generated) R-module P is a free R-module: from any generating set for P a basis B can be selected for which given any $p \in P$ there is a unique sum $p = \sum_{i \in I} r_i b_i$, over R for p[20]. Since any free module is a projective module by default, we know that the objects of $\mathcal{P}R$ are free R-modules of finite rank, i.e. finite-dimensional vector spaces over R. Since two vector spaces of the same (finite) dimension are isomorphic, the canonical isomorphism classes can be represented by their dimension. We can show that this same representation works for stable isomorphism classes as well.*

Indeed, let $P_1 \approx R^{n_1}, Q_1 \approx R^{m_1}$, $P_2 \approx R^{n_2}, Q_2 \approx R^{m_2}$. Suppose that the corresponding representatives have $n_1 - m_1 = n_2 - m_2$. Then $n_1 + m_2 = n_2 + m_1 \in \mathbb{N}$ and $R^{n_1} \oplus R^{m_2} \approx R^{n_2} \oplus R^{m_1}$, so $P_1 \oplus Q_2 \approx P_2 \oplus Q_1$. Since all of these modules are free modules, it follows that for any $r \in \mathbb{N}$,

$$R^{n_1} \oplus R^{m_2} \oplus R^r \approx P_1 \oplus Q_2 \oplus R^r \approx R^{n_2} \oplus R^{m_1} \oplus R^r \approx P_2 \oplus Q_1 \oplus R^r\ \forall\ r \in \mathbb{N}.$$

Therefore differences $n_1 - m_1 = n_2 - m_2$ represent differences of stable isomorphism classes in $K_0(R)$. We conclude that $K_0(R) \approx \mathbb{Z}$ whenever $R = F$ is a field.

□□

Example 2.1.5 *$K_0(\mathbb{Z})$[8]: Example 2.1.4 has a more general case, in that $K_0(R) = \mathbb{Z}$ whenever R is a principal ideal domain. This follows from the fact that any finitely generated projective module over R is a* *free module, in which case we apply a similar method to the above example, mapping a difference of ranks for these modules to a difference of isomorphism classes. But the property that every finitely generated projective module over a principal ideal domain is free follows from the Direct Summand Property and the Fundamental Theorem for Finitely Generated Modules over a Principal Ideal Domain (also known as the Structure Theorem[20]). From that theorem we have that if an R-module P is a direct summand of a free module* $R^n, n \in \mathbb{N}$, *then it must be torsion-free (i.e. the kernel of the map* $a \mapsto ap$ *from* $R \to P$ *is trivial), so that* P *itself is free.*

□□

3 Classical $K_1(R)$

3.1 $GL(R)$ and Elementary Matrices[1, 8, 9]

Definition 3.1.1 *The* ***infinite general linear group*** $GL(R)$ *is the direct limit of the general linear groups* $GL(n, R), n \in \mathbb{N}$, *or the union of the sequence*

$$R^* = GL(1, R) \subseteq GL(2, R) \subseteq \cdots$$

under the inclusion $GL(n, R) \hookrightarrow GL(n + 1, R)$ *via* $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 3.1.2 *Given* $a \in R, i, j \in \mathbb{N}, i \neq j$ *the* ***elementary matrix*** $e_{ij}(a)$ *is the matrix* $e_{ij}(a) \in GL(R)$ *having* a *as the* (i, j)*-entry, 1 on the diagonal and 0 everywhere else.*

Definition 3.1.3 $E(n, R) \leq GL(n, R)$ *is the subgroup generated by the elementary* $n \times n$ *matrices.* $E(R)$ *is the direct limit of such groups as in the definition of* $GL(R)$.

Theorem 3.1.4 *([1], Proposition 2.1.4)*$E(R)$ *is the commutator subgroup of* $GL(R)$.

3.1.1 $K_1(R)$ as a quotient group

Definition 3.1.5 $K_1(R) = GL(R)/E(R)$, *the* ***abelianization*** *of the infinite general linear group.*

From this definition, we think of elements of $K_1(R)$ as (classes of) matrices $A \in GL(R)$.

Example 3.1.6 $K_1(F)$ *where* F *is a field, local ring, or Euclidean Domain [1]: In the case of rings* R *where multiplicative inverses (i.e. fields and local rings), or at least where a quotient-remainder analog exists (i.e. Euclidean Domains), we may think of elementary matrices as representing the elementary row* *or column operations that work on invertible matrices as elements of $GL(R)$. Of course invertible matrices must have unit "determinant" (i.e. a corresponding element of $R^\times$, the units in R), and a sequence of elementary operations serves to change that determinant. Therefore equivalence classes of matrices in $K_1(R)$ are matrices with the same determinant, and we have that $K_1(R) = R^\times$ in the special case of R being a field, local ring or Euclidean Domain.*

Example 3.1.7 *As a corollary to the previous example, we have the well-known result $K_1(\mathbb{Z}) = \{1, -1\}$ (i.e. the cyclic group of order 2) since $\mathbb{Z}$ is a Euclidean Domain whose only units are 1 and -1.*

4 Classical $K_2(R)$

4.1 Steinberg Group

Note the following relations[1] between generators $e^n_{ij}(a) \in E(n, R)$ when $n \geq 3$, whether considered as elements in a particular $E(n, R)$ or in $E(R)$; also note the commutators

$$[e^n_{ij}(a), e^n_{kl}(b)] = e^n_{ij}(a)e^n_{kl}(b)(e^n_{ij}(a))^{-1}(e^n_{kl}(b))^{-1} :$$

$$1)\ e^n_{ij}(a)e^n_{kl}(b) = \begin{cases} e^n_{ij}(a+b), & i = k, j = l \\ e^n_{kl}(b)e^n_{ij}(a), & j \neq k, i \neq l. \end{cases}$$

$$2)\ [e^n_{ij}(a), e^n_{kl}(b)] = \begin{cases} e^n_{il}(ab), & i \neq l, j = k \\ e^n_{kj}(-ba), & j \neq k, i = l \\ 1 & j \neq k, i \neq l. \end{cases}$$

Definition 4.1.1 *Fix $n \in \mathbb{N}$, $n \geq 3$. Assign to each $e^n_{ij}(a), 1 \leq i, j \leq n, i \neq j$ and $a \in R$ a generator $x^n_{ij}(a)$, let F be the free group on these generators and S the subgroup generated by the relations (for every i, j as above, similar k, l, and every $a, b \in R$)*

$$1)\ x^n_{ij}(a)x^n_{kl}(b) = \begin{cases} x^n_{ij}(a+b), & i = k, j = l \\ x^n_{kl}(b)x^n_{ij}(a), & j \neq k, i \neq l. \end{cases}$$

$$2)\ [x^n_{ij}(a), x^n_{kl}(b)] = \begin{cases} x^n_{il}(ab), & i \neq l, j = k \\ x^n_{kj}(-ba), & j \neq k, i = l \\ 1 & j \neq k, i \neq l. \end{cases}$$

*Then the **nth Steinberg Group over** R is a quotient $St(n, R) = F/S$.*

From now on when we speak of the Steinberg group $St(n, R)$, or its elements, we assume that $n \geq 3$.

By definition there is a unique surjective homomorphism $\phi_n : St(n, R) \to E(n, R) \subseteq GL(n, R)$ by $x^n_{ij}(a) \mapsto e^n_{ij}(a)$. However, there may be other relations between the $e^n_{ij}(a)$ depending on the specific structure of R that are ignored by the subgroup S; that is, $S \subset ker(\phi_n)$ but we may not have $S = ker(\phi_n)$. We can define homomorphisms of groups $\iota_{n,n+1} : St(n, R) \to St(n + 1, R)$ (not inclusions) that match generators $x^n_{ij}(a)$ of $St(n, R)$ to generators $x^{n+1}_{ij}(a)$ of $St(n+1, R)$. So we define the **infinite Steinberg Group** $St(R)$ as the direct limit of this sequence of groups and homomorphisms.

Note that the direct limit construction[20] gives a canonical homomorphism

$$\iota_n : St(n, R) \to St(R)$$

such that $\iota_{n+1} \circ \iota_{n,n+1} = \iota_n$, and a homomorphism

$$\phi : St(R) \to E(R)$$

such that

$$\iota_n \circ \phi = \phi_n \circ i_n,$$

where $i_n : E(n, R) \to E(R)$ is the canonical inclusion defining $E(R)$ as a direct limit of the $E(n, R)$ (in this case i_n is an inclusion).

Definition 4.1.2 *Given $\phi : St(R) \to E(R)$ as above, $K_2(R) = ker(\phi)$.*

From this definition, we will assume an element of $K_2(R)$ to be (represented by) simply a word over generators of $St(n, R)$ for some $n \in \mathbb{N}$:

$$w = [x_{i_1 j_1}(a_1)][x_{i_2 j_2}(a_2)] \cdots [x_{i_k j_k}(a_k)].$$

Also, we have an exact sequence:

$$1 \to K_2 \hookrightarrow St(R) \xrightarrow{\phi} GL(R) \xrightarrow{\pi} K_1(R) \to 1$$

from Definition 3.1.5.

An important fact about the Steinberg group is

Theorem 4.1.3 *([8], Theorem 5.1)$K_2(R)$ is precisely the center of $St(R)$.*

Example 4.1.4 *It is an interesting result that*

$$K_2(\mathbb{Z}) = \{-1, 1\}\,;$$

it is generated by the element $(x_{12}(1)x_{21}(-1)x_{12}(1))^4$. *This amazing result is worthy of a chapter in and of itself, as in [8] Chapter 10, and is therefore not fully described here.*

5 Higher K-Theory from Quillen [16, 11]

5.1 $N(Q\mathcal{P}R)$

Let R be a commutative ring with identity 1, and recall that $\mathcal{P}R$ is the category of finitely generated projective R-modules. In fact $\mathcal{P}R$ is an *exact* category, with admissible injections (indicated by arrows $\rightarrowtail$) and admissible surjections (indicated by $\twoheadrightarrow$).

Note that by Definition 1.2.1 an admissible injection of finitely generated projective R-modules, $P \rightarrowtail Q$, is an injection such that the quotient of Q modulo the image of P is also projective, and all surjections in the category $\mathcal{P}R$ are admissible as note earlier. From this category we make another:

Definition 5.1.1 *Quillen's Category: In the category* $Q\mathcal{P}R$, *the objects are the objects of* $\mathcal{P}R$. *Given* $P, Q \in \mathcal{P}R$, *a morphism* $f : P \cdots \to Q$ *in* $Q\mathcal{P}R$ *is a diagram*

$$\begin{tikzcd} f : P \cdots \arrow[rr] & & Q \\ & U \arrow[ul, twoheadrightarrow, "f_1"] \arrow[ur, rightarrowtail, "f_2"'] & \end{tikzcd}$$

where U, f_1 *and* f_2 *allow* P *and* Q *to be part of short exact sequences (i.e.* f_1 *and* f_2 *are admissible maps). Composition of these morphisms, when appropriate, is given by forming another diagram: given* $f : P \cdots \to Q$ *with admissibles* U, f_1 *and* f_2, *and* $g : Q \cdots \to S$ *with admissibles* V, g_1 *and* g_2, *the composition* $g \circ f : P \cdots \to S$ *is the diagram*

$$\begin{tikzcd} g \circ f : P \cdots \arrow[rr] & & S \\ & U \times_Q V \arrow[ul, twoheadrightarrow, "\bar{f_1}"] \arrow[ur, rightarrowtail, "\bar{g_2}"'] & \end{tikzcd}$$

where $U \times_Q V = \{(u, v) \in U \times V \mid g_1(v) = f_2(u) \in Q\}$, $\bar{f_1}(u, v) = f_1(u)$ *and* $\bar{g_2}(u, v) = g_2(v)$.

Definition 5.1.2 *Given the category $Q\mathcal{P}R$, two morphisms*

$$\begin{tikzcd} f : P \arrow[rr, dotted] & & Q \\ & U \arrow[ul, two heads, "f_1"'] \arrow[ur, tail, "f_2"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} f' : P \arrow[rr, dotted] & & Q \\ & U' \arrow[ul, two heads, "f_1'"'] \arrow[ur, tail, "f_2'"'] & \end{tikzcd}$$

are ***equivalent*** *if there is an R-module isomorphism $F : U \to U'$ for which $f_1' \circ F = f_1$ and $f_2' \circ F = f_2$ in $\mathcal{P}R$.*

As seen in Example 5.1.1, we can construct the nerve of this category if we think of an appropriate "small" category corresponding to $Q\mathcal{P}R$. This nerve $N(Q\mathcal{P}R)$ is a simplicial set, with 0-simplices corresponding the the objects of $Q\mathcal{P}R$ (which consequently are the objects of $\mathcal{P}R$ by definition). The 1-simplices are (isomorphism classes of) the morphisms of $Q\mathcal{P}R$ as described above, and n-simplices are n-tuples of composeable morphisms. For instance, a 2-simplex would be

$$\left(\begin{tikzcd} P_0 \arrow[rr, dotted] & & P_1 \\ & U_1 \arrow[ul, two heads, "f_1^{(1)}"'] \arrow[ur, tail, "f_2^{(1)}"'] & \end{tikzcd}, \begin{tikzcd} P_1 \arrow[rr, dotted] & & P_2 \\ & U_2 \arrow[ul, two heads, "f_1^{(2)}"'] \arrow[ur, tail, "f_2^{(2)}"'] & \end{tikzcd} \right),$$

and a 3-simplex would look like

$$\left(\begin{tikzcd} P_0 \arrow[rr, dotted] & & P_1 \\ & U_1 \arrow[ul, two heads, "f_1^{(1)}"'] \arrow[ur, tail, "f_2^{(1)}"'] & \end{tikzcd}, \begin{tikzcd} P_1 \arrow[rr, dotted] & & P_2 \\ & U_2 \arrow[ul, two heads, "f_1^{(2)}"'] \arrow[ur, tail, "f_2^{(2)}"'] & \end{tikzcd}, \begin{tikzcd} P_2 \arrow[rr, dotted] & & P_3 \\ & U_3 \arrow[ul, two heads, "f_1^{(3)}"'] \arrow[ur, tail, "f_2^{(3)}"'] & \end{tikzcd} \right).$$

Since $N(Q\mathcal{P}R)_0 = Ob(\mathcal{P}R)$, this nerve cannot be a reduced simplicial set. However we can use it due to Kan's work on star-connectedness as described in Chapter 1:

Theorem 5.1.3 *$N(Q\mathcal{P}R)$ is star-connected at the basepoint $0 \in \mathcal{P}R$, with ray function defined on finitely generated projective R-modules P by*

$$\omega(P) = \quad \begin{tikzcd} q_P : 0 \arrow[rr, dotted] & & P \\ & 0 \arrow[ul, two heads, "="'] \arrow[ur, tail, "incl."'] & \end{tikzcd}$$

5.2 $G(N(Q\mathcal{P}R))$

As we have seen, $N(Q\mathcal{P}R)$ is not a reduced simplicial set, although it is star-connected. We will see later (i.e. Lemma 4.0.14 in Chapter 3) that more than one ray function can accomplish this, but for now we use the function $\omega(P) = q_P$ from Theorem 5.1.3 and apply Definition 4.3.6 from Chapter 1:

Definition 5.2.1 *For any $n \geq 0$, the set $G(N(Q\mathcal{P}R))_n$ of n-simplices of the loop group $G(N(Q\mathcal{P}R))$ is the free group on the set*

$$\mathcal{B}_n = \{t(x) \mid x \in N(Q\mathcal{P}R_{n+1} - s_n(N(Q\mathcal{P}R))_n) - (T_\omega(N(Q\mathcal{P}R)))_{n+1}\}.$$

5.3 Quillen's K_0, K_1, K_2 versus Classical K-Theory

Chapter 4(IV) of [10] gives a good account of Quillen's results for higher Algebraic K-Theory:

Quillen constructs higher K-Theory as

$$K_i(R) := \pi_i(\Omega|N(Q\mathcal{P}R)|),$$

where $\Omega|N(Q\mathcal{P}R|$ denotes the combinatorial loop space of the geometric realization of $N(Q\mathcal{P}R)$ (see [10, 11]). It is then possible to show that for $i \in \{0, 1, 2\}$ the groups K_i constructed this way are isomorphic in a natural sense to the classical K-Theory groups $K_0(R)$, $K_1(R)$ and $K_2(R)$ as described earlier in this chapter (in fact, these groups are constructed specifically so that this is true). One of Quillen's constructions which affords this definition is known as **Quillen's +-construction**, and results in a space $|N(Q\mathcal{P}R)|^+$, which we will refer to in Chapter 6. Although the theory tells us that, for example,

$$K_2(R) \approx \pi_2(\Omega|N(Q\mathcal{P}R)|),$$

explicit isomorphisms are not constructed. Such is the inspiration for this dissertation.

Homotopy theory on topological spaces then tells us that

$$\pi_i(G(N(\mathcal{P}R))) \approx \pi_i(\Omega|N(Q\mathcal{P}R)|)$$

as described in [2]. Thus we call $G(N(Q\mathcal{P}R))$ a **simplicial model for K-Theory**. Later exposition in this dissertation will show other simplicial models for K-Theory under the same definition: the Gillet-Grayson simplicial set $\mathcal{G}.\mathcal{P}R$, the loop group on Waldhausen's simplicial set $\mathfrak{s}.\mathcal{P}R$, and some other simplicial sets derived from these via techniques described in Chapter 1.

Chapter 3

More Constructions for $\mathcal{P}R$

1 Waldhausen's $\mathfrak{s}.\mathcal{P}R$

1.1 The Simplicial Set $\mathfrak{s}.\mathcal{C}$

This section explains some definitions from Waldhausen's paper [7] and exposits a notion of duality.

Define a poset $Ar[\mathbf{n}]$ as the set

$$Ar[\mathbf{n}] = \{(i,j) \in \mathbf{n} \times \mathbf{n} \mid 0 \leq i \leq j \leq n\},$$

with order defined by

$$(i,j) \leq (k,l) \Leftrightarrow i \leq k \text{ and } j \leq l.$$

Let $\mathcal{C}$ be a "category with cofibrations" as defined by Waldhausen [7], with initial (and final) object $0 := 0_{\mathcal{C}}$. Then the idea of a short exact sequence (a "cofibration" sequence) is defined in $\mathcal{C}$. In particular, while we do not give the formal definition of a category with cofibrations (as in [7], for example) here, we do notice that the category $\mathcal{P}R$ introduced in Chapter 2 is a category with cofibrations. The cofibrations in $\mathcal{P}R$ are admissible injections as in Definition 1.2.1 of Chapter 2, while admissible surjections are the quotient maps.

Considering $Ar[\mathbf{n}]$ as a category in the usual way (see [9]), define

Definition 1.1.1 *A functor $A : Ar[\mathbf{n}] \to \mathcal{C}$ is a* ***normalized exact functor*** *if*

a) for every $(i,i) \in Ar[\mathbf{n}]$, $A(i,i) = 0$.

b) for every (i,j) and (i,k) in $Ar[\mathbf{n}]$, such that $i \leq j \leq k$,

$$0 \to A(i,j) \to A(i,k) \to A(j,k) \to 0$$

is a short exact sequence in $\mathcal{C}$.

For any n, define

$$\mathfrak{s}_n\mathcal{C} = \{A : Ar[\mathbf{n}] \to \mathcal{C} \mid A \text{ is a normalized exact functor}\}.$$

Proposition 1.1.2 *An element A of $\mathfrak{s}_n\mathcal{C}$ is a triangular, commutative diagram in $\mathcal{C}$, where each*

$$A(0,i) \xrightarrow{A(\leq)} A(0,j) \xrightarrow{A(\leq)} A(i,j)$$

is a short exact sequence and each vertical row is a quotient map.

When we refer to the objects in these triangles without reference to the underlying functor, we will use the notation A_{ij} instead of $A(i,j)$ for the objects, so that such elements have the form

$$\left(\begin{tikzcd}
 & & & A_{n-1,n} \\
 & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\
 & A_{12} \arrow[r, tail] & \vdots \arrow[u, two heads] \arrow[r, tail] & A_{1,n} \arrow[u, two heads] \\
A_1 \arrow[r, tail] & A_2 \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[u, two heads] \arrow[r, tail] & A_n \arrow[u, two heads]
\end{tikzcd}\right).$$

Waldhausen also defines a category

$$\mathcal{S}_n\mathcal{C},$$

which has objects $\mathfrak{s}_n\mathcal{C}$, and in which a morphism $F : A \to B$ is a natural transformation from A to B, but we do not use this in the present paper. Instead, we assemble the sets $\mathfrak{s}_n\mathcal{C}$, as n varies, into a simplicial set $\mathfrak{s}.\mathcal{C}$ with n-simplices $\mathfrak{s}.\mathcal{C}(\boldsymbol{n}) = \mathfrak{s}.\mathcal{C}_n := \mathfrak{s}_n\mathcal{C}$ by defining, for each morphism $\alpha : \mathbf{n} \to \mathbf{m}$ in Δ, a function

$$\mathfrak{s}.\mathcal{C}(\alpha) : \mathfrak{s}_m\mathcal{C} \to \mathfrak{s}_n\mathcal{C}$$

given by

$$[\mathfrak{s}.\mathcal{C}(\alpha)(A)](k,l) = A(\alpha(k), \alpha(l)),$$

and

$$[\mathfrak{s}.\mathcal{C}(\alpha)(A)]((k,l) \leq (k_1,l_1)) = A((\alpha(k), \alpha(l)) \leq (\alpha(k_1), \alpha(l_1))).$$

This allows us to define face and degeneracy maps and compositions thereof in $\mathfrak{s}.\mathcal{C}$ according to Definition 1.2.1 and Lemma 1.1.9 of Chapter 1.

Recall that an **exact covariant functor** is a functor that converts short exact sequences (or cofibration sequences) into short exact sequences (or cofibration sequences). An **exact contravariant functor** also converts short exact sequences into short exact sequences, and thus turns cofibrations into quotient maps and quotient maps into cofibrations, using Waldhausen's language [7].

Theorem 1.1.3 *Suppose $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories with cofibrations.*

a) $\mathfrak{s}.\mathcal{C}$ is a simplicial set; moreover, $\mathfrak{s}_0\mathcal{C}$ consists of a single element so that $\mathfrak{s}.\mathcal{C}$ is a reduced simplicial set.

b) A (covariant) exact functor $F : \mathcal{C} \to \mathcal{D}$ induces a map of simplicial sets $\mathfrak{s}.F : \mathfrak{s}.\mathcal{C} \to \mathfrak{s}.\mathcal{D}$; an exact contravariant functor $F : \mathcal{C} \to \mathcal{D}$ induces maps of simplicial sets $\mathfrak{s}.F : \mathfrak{s}.\mathcal{C} \to \mathfrak{s}.\mathcal{D}^{rev}$ and $\mathfrak{s}.F : \mathfrak{s}.\mathcal{C}^{rev} \to \mathfrak{s}.\mathcal{D}$.

In the case of contravariant functor F we use the following definition for $\mathfrak{s}.F : \mathfrak{s}.\mathcal{C} \to \mathfrak{s}.\mathcal{D}^{rev}$: for $A \in \mathfrak{s}_n\mathcal{C}$,

$$\mathfrak{s}.F(A)(i,j) := F(A(n-j, n-i)),$$

and

$$\mathfrak{s}.F(A)((i,j) \leq (k,l)) := F(A((n-l, n-k) \leq (n-j, n-i))).$$

c) If $F : \mathcal{C} \to \mathcal{D}$, and $G : \mathcal{D} \to \mathcal{E}$ are covariant exact functors, then the composite functor GF is a covariant exact functor, and as maps of simplicial sets $\mathfrak{s}.\mathcal{C} \to \mathfrak{s}.\mathcal{E}$,

$$\mathfrak{s}.(GF) = \mathfrak{s}.(G)\mathfrak{s}.(F).$$

Also, if F_1 and G_1 are contravariant exact functors between the same categories as F and G above, then G_1F_1 is an exact covariant functor, and as maps of simplicial sets, either $\mathfrak{s}.\mathcal{C} \to \mathfrak{s}.\mathcal{E}$ or $\mathfrak{s}.\mathcal{C}^{rev} \to \mathfrak{s}.\mathcal{E}^{rev}$, then

$$\mathfrak{s}.(G_1F_1) = \mathfrak{s}.(G_1)\mathfrak{s}.(F_1).$$

d) For the identity functor $id : \mathcal{C} \to \mathcal{C}$, s.id is the identity map on the simplicial sets $\mathfrak{s}.\mathcal{C}$ and $\mathfrak{s}.\mathcal{C}^{rev}$.

Proof: We prove (b) and leave the rest as an exercise.

For the case of covariant functor $F : \mathcal{C} \to \mathcal{D}$, we must show that the natural transformation relation holds on $\mathfrak{s}.F(\boldsymbol{n})(A) = F \circ A$ for any $A \in \mathfrak{s}.\mathcal{C}(\boldsymbol{n})$ and any $\alpha : \mathbf{m} \to \mathbf{n}$ (i.e. any $\alpha \in Hom_{op}(\mathbf{n}, \mathbf{m})$). Given

$(a,b) \in Ar[\mathbf{m}]$ we have

$$\begin{aligned}([\mathfrak{s}.\mathcal{D}(\alpha) \circ \mathfrak{s}.F(\mathbf{n})](A))(a,b) &= \mathfrak{s}.\mathcal{D}(\alpha)([\mathfrak{s}.F(\mathbf{n})(A)])(a,b) \\ &= [\mathfrak{s}.F(\mathbf{n})(A)](\alpha(a),\alpha(b)) \\ &= F(A(\alpha(a),\alpha(b))),\end{aligned}$$

and

$$\begin{aligned}([\mathfrak{s}.F(\mathbf{m}) \circ \mathfrak{s}.\mathcal{C}(\alpha)](A))(a,b) &= [\mathfrak{s}.F(\mathbf{m}) \circ [\mathfrak{s}.\mathcal{C}(\alpha)(A)]](a,b) \\ &= F([\mathfrak{s}.\mathcal{C}(\alpha)(A)](a,b)) \\ &= F(A(\alpha(a),\alpha(b))) \\ &= ([\mathfrak{s}.\mathcal{D}(\alpha) \circ \mathfrak{s}.F(\mathbf{n})](A))(a,b).\end{aligned}$$

Therefore $\mathfrak{s}.\mathcal{D}(\alpha) \circ \mathfrak{s}.F(\mathbf{n}) = \mathfrak{s}.F(\mathbf{m}) \circ \mathfrak{s}.\mathcal{C}(\alpha)$, so that $\mathfrak{s}.F$ is a natural transformation of functors in $\mathcal{SS}$, hence a simplicial map.

Using the recommended definition for $\mathfrak{s}.F$ on contravariant functors, either to or from the reverses of these simplicial sets, we see

$$\begin{aligned}([\mathfrak{s}.\mathcal{D}^{rev}(\alpha) \circ \mathfrak{s}.F(\mathbf{n})](A))(a,b) &= ([\mathfrak{s}.\mathcal{D}(\alpha^{rev}) \circ \mathfrak{s}.F(\mathbf{n})](A))(a,b) \\ &= [\mathfrak{s}.F(\mathbf{n})(A)](\alpha^{rev}(a),\alpha^{rev}(b)) \\ &= [\mathfrak{s}.F(\mathbf{n})(A)](n-\alpha(m-a), n-\alpha(m-b)) \\ &= F(A(n-(n-\alpha(m-b)), n-(n-\alpha(m-a)))) \\ &= F(A(\alpha(m-b),\alpha(m-a))) \\ &= F([\mathfrak{s}.\mathcal{C}(\alpha)(A)](a,b)) \\ &= ([\mathfrak{s}.F(\mathbf{m}) \circ \mathfrak{s}.\mathcal{C}(\alpha)](A))(a,b)\end{aligned}$$

when $\mathfrak{s}.F : \mathfrak{s}.\mathcal{C} \to \mathfrak{s}.\mathcal{D}^{rev}$, and

$$\begin{aligned}
([\mathfrak{s}.\mathcal{D}(\alpha) \circ \mathfrak{s}.F(\mathbf{n})](A))(a,b) &= \mathfrak{s}.\mathcal{D}(\alpha)([\mathfrak{s}.F(\mathbf{n})(A)])(a,b) \\
&= [\mathfrak{s}.F(\mathbf{n})(A)](\alpha(a), \alpha(b)) \\
&= F(A(n-\alpha(b), n-\alpha(a))) \\
&= F(A(n-(\alpha^{rev})^{rev}(b), n-(\alpha^{rev})^{rev}(a))) \\
&= F(A(n-(n-\alpha^{rev}(m-b)), n-(n-\alpha^{rev}(m-a)))) \\
&= F(A(\alpha^{rev}(m-b), \alpha^{rev}(m-a))) \\
&= F([\mathfrak{s}.\mathcal{C}(\alpha^{rev})(A)](a,b)) \\
&= F([\mathfrak{s}.\mathcal{C}^{rev}(\alpha)(A)](a,b)) \\
&= \mathfrak{s}.F(\mathbf{m})([\mathfrak{s}.\mathcal{C}^{rev}(\alpha)](A))(a,b) \\
&= ([\mathfrak{s}.F(\mathbf{m}) \circ \mathfrak{s}.\mathcal{C}^{rev}(\alpha)](A))(a,b)
\end{aligned}$$

when $\mathfrak{s}.F : \mathfrak{s}.\mathcal{C}^{rev} \to \mathfrak{s}.\mathcal{D}$. Therefore the natural transformation relation holds and these two forms of $\mathfrak{s}.F$ are simplicial maps. This gives conclusion (b).

□

1.2 Duality for $\mathfrak{s}.\mathcal{P}R$

Using Theorem 1.1.3 and Example 5.2.1 from Chapter 1, we have

Theorem 1.2.1 *Given the exact category $\mathcal{P}R$ of finitely generated, projective R-modules,*

a) There is a map of simplicial sets $\mathfrak{s}.^* : \mathfrak{s}.\mathcal{P}R \to (\mathfrak{s}.\mathcal{P}R)^{rev}$ defined, for $A \in \mathfrak{s}.\mathcal{P}R_n$, by

$$(\mathfrak{s}.^*A)(i,j) := A(n-j, n-i)^*,$$

and

$$(\mathfrak{s}.^*A)((i,j) \le (k,l)) := A((n-l, n-k) \le (n-j, n-i))^*.$$

b) There is a map of simplicial sets $\mathfrak{s}.^* : (\mathfrak{s}.\mathcal{P}R)^{rev} \to \mathfrak{s}.\mathcal{P}R$ defined, for $A \in \mathfrak{s}.\mathcal{P}R_n^{rev}$, by

$$(\mathfrak{s}.^*A)(i,j) := A(n-j, n-i)^*,$$

and

$$(\mathfrak{s}.^*A)((i,j) \le (k,l)) := A((n-l, n-k) \le (n-j, n-i))^*.$$

c) As maps of simplicial sets (from $\mathfrak{s}.\mathcal{P}R$ to itself, and from $\mathfrak{s}.\mathcal{P}R^{rev}$ to itself)

$$(\mathfrak{s}.^*)(\mathfrak{s}.^*) = \mathfrak{s}.^{**}.$$

Proof: First, note that so far there is no structure on

$$\mathfrak{s}.\mathcal{P}R_n = \{A : Ar[\mathbf{n}] \to \mathcal{P}R \ \mid \text{A is a normalized exact functor}\}$$

beyond it being just a set of functors. But $\mathfrak{s}.^*(\mathbf{n})(A)$ is a functor in $\mathfrak{s}.\mathcal{P}R_n^{rev}$ when $A \in \mathfrak{s}.\mathcal{P}R_n$. Thus $\mathfrak{s}.^*(\mathbf{n})(A) := A^*$ must be defined on objects and morphisms of $Ar[\mathbf{n}]$, so that $\mathfrak{s}.^*(\mathbf{n})$ sends $\mathbf{n} \in \Delta^{op}$ to a *set map* (i.e. a morphism in $\mathcal{S}$) in order to have a natural transformation (i.e. simplicial map). By definition,

$$\begin{aligned} A^*(i,j) &= (A(n-j, n-i))^* \\ &= {}^*(A(n-j, n-i)) \\ &= \mathfrak{s}.({}^*)(\mathbf{n})(A)(i,j)) \end{aligned}$$

(parentheses used for emphasis in the notation of Lemma 1.2.3 of Chapter 2), and

$$\begin{aligned} A^*((i,j) \le (k,l)) &= (A((n-l, n-k) \le (n-j, n-i)))^* \\ &= {}^*(A((n-l, n-k) \le (n-j, n-i))) \\ &= \mathfrak{s}.({}^*)(\mathbf{n})(A)((i,j) \le (k,l)). \end{aligned}$$

Thus Theorem 1.1.3.b applies to the contravariant functor $^* : \mathcal{P}R \to \mathcal{P}R$, so that * induces the simplicial map $\mathfrak{s}.({}^*) : \mathfrak{s}.\mathcal{P}R \to (\mathfrak{s}.\mathcal{P}R)^{rev}$, which proves (a). Also, * must induce the simplicial map $\mathfrak{s}.({}^*) : (\mathfrak{s}.\mathcal{P}R)^{rev} \to \mathfrak{s}.\mathcal{P}R$, which is (b). Finally, $^{**} ={}^* \circ {}^*$ is a covariant functor, in which case Theorem 1.1.3.c and Lemma 1.2.3.b imply that, as simplicial maps either from $\mathfrak{s}.\mathcal{P}R$ to itself or from $(\mathfrak{s}.\mathcal{P}R)^{rev}$ to itself, $\mathfrak{s}.({}^{**}) = \mathfrak{s}.({}^*) \circ \mathfrak{s}.({}^*)$, which proves (c).

□

Remark 1.2.2 *Note that * preserves "weak equivalences" in $\mathcal{P}R$, if these are defined to be the isomorphisms in $\mathcal{P}R$ (although the direction of the isomorphism is reversed of course).*

Now, we note a theorem of Waldhausen:

Theorem 1.2.3 *([7], Lemma 1.4.1 b)) If $\mathcal{C}$ and $\mathcal{D}$ are two categories (with cofibrations), and F_1, F_2 are two exact covariant functors from $\mathcal{C}$ and $\mathcal{D}$ with an isomorphism $\eta : F_1 \to F_2$, then there is a simplicial homotopy equivalence, relative to the basepoint, which we will call $\mathfrak{s}.\eta$, between $\mathfrak{s}.F_1$ and $\mathfrak{s}.F_2$.*

Corollary 1.2.4 *$\mathfrak{s}.^{**}$ and id are homotopic simplicial maps (relative to the basepoint), whether considered as maps from $\mathfrak{s}.\mathcal{P}R$ to itself, or as maps from $\mathfrak{s}.\mathcal{P}R^{rev}$ to itself.*

The general theory of simplicial sets (i.e. Theorem 1.4.2 of Chapter 1) then tells us that

Corollary 1.2.5 *The map on homotopy groups induced by $\mathfrak{s}.^{**}$ is equal to the identity homomorphism, and the simplicial maps*

$$\mathfrak{s}.^{*} : \mathfrak{s}.\mathcal{P}R \to \mathfrak{s}.\mathcal{P}R^{rev}$$

and

$$\mathfrak{s}.^{*} : \mathfrak{s}.\mathcal{P}R^{rev} \to \mathfrak{s}.\mathcal{P}R$$

induce isomorphisms on homotopy groups that are inverse to each other.

Thus we have the following from Theorem 4.2.5 of Chapter 1 and its Corollary.

Theorem 1.2.6 *$G\mathfrak{s}.^{*} : G(\mathfrak{s}.\mathcal{P}R) \to G(\mathfrak{s}.\mathcal{P}R^{rev})$ and $G\mathfrak{s}.^{*} : G(\mathfrak{s}.\mathcal{P}R^{rev}) \to G(\mathfrak{s}.\mathcal{P}R)$ are loop homotopy equivalences. $G\mathfrak{s}.^{**} : G(\mathfrak{s}.\mathcal{P}R) \to G(\mathfrak{s}.\mathcal{P}R)$ is loop homotopic to the identity homomorphism, and thus simplicially homotopic to the identity homomorphism.*

2 The Gillet-Grayson Simplicial Set[11, 12]

2.1

Definition 2.1.1 *The **Gillet-Grayson simplicial set**, $\mathcal{G}.\mathcal{C}$ on a category $\mathcal{C}$ with cofibrations has n-simplices that are pairs of lower-triangular commutative diagrams built from exact sequences, where the* *elements in the pair have all co-kernels in common:*

$$
\mathcal{G}.\mathcal{C}_n = \{ \left(\begin{tikzcd}
 & & & A_{n-1,n} \\
 & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\
 & A_{01} \arrow[r, tail] & \vdots \arrow[r, tail] \arrow[u, two heads] & A_{0n} \arrow[u, two heads] \\
A_0 \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] \arrow[u, two heads] & A_n \arrow[u, two heads]
\end{tikzcd} , \begin{tikzcd}
 & & & A_{n-1,n} \\
 & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\
 & A_{01} \arrow[r, tail] & \vdots \arrow[r, tail] \arrow[u, two heads] & A_{0n} \arrow[u, two heads] \\
B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] \arrow[u, two heads] & B_n \arrow[u, two heads]
\end{tikzcd} \right)
$$

$$
\mid A_i \rightarrowtail A_j \twoheadrightarrow A_{ij}, B_i \rightarrowtail B_j \twoheadrightarrow A_{ij} \text{ are short exact sequences } \forall\ i < j\}.
$$

The face map $d_i, 0 \le i \le n$ is defined by deleting all objects with i in their subscript and composing morphisms accordingly, while degeneracy maps s_i is defined by repeating all such objects and inserting the appropriate identity morphisms.

Looking at Definition 2.1.1, we have 0-simplices of the Gillet-Grayson simplicial set, $\mathcal{G}.\mathcal{P}R$, on the category $\mathcal{P}R$ of finitely generated projective modules as pairs of such modules:

$$
\mathcal{G}.\mathcal{P}R_0 = \{(A, B) \mid A, B \in PR\}.
$$

1-simplices are pairs of short exact sequences

$$
x = \left(\begin{tikzcd}
 & A_{01} \\
A_0 \arrow[r, tail] & A_1 \arrow[u, two heads]
\end{tikzcd} , \begin{tikzcd}
 & A_{01} \\
B_0 \arrow[r, tail] & B_1 \arrow[u, two heads]
\end{tikzcd} \right),
$$

2-simplices are pairs

$$
t = \left(\begin{tikzcd}
 & & A_{12} \\
 & A_{01} \arrow[r, tail] & A_{02} \arrow[u, two heads] \\
A_0 \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[u, two heads] & A_2 \arrow[u, two heads]
\end{tikzcd} , \begin{tikzcd}
 & & A_{12} \\
 & A_{01} \arrow[r, tail] & A_{02} \arrow[u, two heads] \\
B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[u, two heads] & B_2 \arrow[u, two heads]
\end{tikzcd} \right)
$$

where the squares are commutative and the sequences

$$0 \to A_i \rightarrowtail A_j \twoheadrightarrow A_{ij} \to 0$$

and

$$0 \to B_i \rightarrowtail B_j \twoheadrightarrow A_{ij} \to 0$$

are short exact sequences. 3-simplices are pairs

$$q = \left(\begin{tikzcd} & & & A_{23} \\ & & A_{12} \arrow[r, tail] & A_{13} \arrow[u, two heads] \\ & A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[u, two heads] & A_{03} \arrow[u, two heads] \\ A_0 \arrow[r, tail] & A_1 \arrow[r, tail] \arrow[u, two heads] & A_2 \arrow[r, tail] \arrow[u, two heads] & A_3 \arrow[u, two heads] \end{tikzcd} , \begin{tikzcd} & & & A_{23} \\ & & A_{12} \arrow[r, tail] & A_{13} \arrow[u, two heads] \\ & A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[u, two heads] & A_{03} \arrow[u, two heads] \\ B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[u, two heads] & B_2 \arrow[r, tail] \arrow[u, two heads] & B_3 \arrow[u, two heads] \end{tikzcd} \right)$$

where squares are commutative and exact sequences are as described for $\mathcal{G}.\mathcal{P}R_2$ above.

Of course $\mathcal{G}.\mathcal{P}R_0$ has no face maps operating on it, but the degeneracy s_0 is defined by "duplicating" the modules in the pair $x = (A, B) \in \mathcal{G}.\mathcal{P}R_0$ via identity maps:

$$s_0(x) = \left(\begin{tikzcd} & 0 \\ A \arrow[r, tail, "="] & A \arrow[u, two heads] \end{tikzcd} , \begin{tikzcd} & 0 \\ B \arrow[r, tail, "="] & B \arrow[u, two heads] \end{tikzcd} \right) .$$

In higher dimensions, the degeneracy s_j is computed by "duplicating" any module with j in its index (i.e. A_j in the bottom row and the j^{th} column), inserting the identity map and the zero module where appropriate. For instance with $t \in \mathcal{G}.\mathcal{P}R_2$ as above and $j = 1$ we have

$$s_1(t) = \left(\begin{tikzcd} & & & A_{12} \\ & & 0 \arrow[r, tail] & A_{12} \arrow[u, two heads, "="'] \\ & A_{01} \arrow[r, tail, "="] & A_{01} \arrow[r, tail] \arrow[u, two heads] & A_{02} \arrow[u, two heads] \\ A_0 \arrow[r, tail] & A_1 \arrow[r, tail, "="] \arrow[u, two heads] & A_1 \arrow[r, tail] \arrow[u, two heads] & A_2 \arrow[u, two heads] \end{tikzcd} , \begin{tikzcd} & & & A_{12} \\ & & 0 \arrow[r, tail] & A_{12} \arrow[u, two heads, "="'] \\ & A_{01} \arrow[r, tail, "="] & A_{01} \arrow[r, tail] \arrow[u, two heads] & A_{02} \arrow[u, two heads] \\ B_0 \arrow[r, tail] & B_1 \arrow[r, tail, "="] \arrow[u, two heads] & B_1 \arrow[r, tail] \arrow[u, two heads] & B_2 \arrow[u, two heads] \end{tikzcd} \right) .$$

A face d_i is computed by deleting all modules with i in the index and composing homomorphisms and including the zero module where appropriate. For example:

$$d_2(q) = \left(\begin{tikzcd} & & A_{13} \\ & A_{01} \arrow[r, tail, "\circ"] & A_{03} \arrow[u, two heads] \\ A_0 \arrow[r, tail] & A_1 \arrow[u, two heads] \arrow[r, tail, "\circ"] & A_3 \arrow[u, two heads] \end{tikzcd} , \begin{tikzcd} & & A_{13} \\ & A_{01} \arrow[r, tail, "\circ"] & A_{03} \arrow[u, two heads] \\ B_0 \arrow[r, tail] & B_1 \arrow[u, two heads] \arrow[r, tail, "\circ"] & B_3 \arrow[u, two heads] \end{tikzcd} \right),$$

the $\circ$ indicating where a composition occurred.

Notice that for

$$x = \left(\begin{tikzcd} & & & A_{n-1,n} \\ & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\ & A_{01} \arrow[r, tail] & \vdots \arrow[u, two heads] \arrow[r, tail] & A_{0n} \arrow[u, two heads] \\ A_0 \arrow[r, tail] & A_1 \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[u, two heads] \arrow[r, tail] & A_n \arrow[u, two heads] \end{tikzcd} , \begin{tikzcd} & & & A_{n-1,n} \\ & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\ & A_{01} \arrow[r, tail] & \vdots \arrow[u, two heads] \arrow[r, tail] & A_{0n} \arrow[u, two heads] \\ B_0 \arrow[r, tail] & B_1 \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[u, two heads] \arrow[r, tail] & B_n \arrow[u, two heads] \end{tikzcd} \right) \in \mathcal{G}.\mathcal{P}R_n$$

we have

$$D_1 = \left(\begin{tikzcd} & & & A_{n-1,n} \\ & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\ & A_{01} \arrow[r, tail] & \vdots \arrow[u, two heads] \arrow[r, tail] & A_{0n} \arrow[u, two heads] \\ A_0 \arrow[r, tail] & A_1 \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[u, two heads] \arrow[r, tail] & A_n \arrow[u, two heads] \end{tikzcd} \right) = \left(\begin{tikzcd} & & & A'_{n,n+1} \\ & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\ & A'_{12} \arrow[r, tail] & \vdots \arrow[u, two heads] \arrow[r, tail] & A'_{1,n+1} \arrow[u, two heads] \\ A'_{01} \arrow[r, tail] & A'_{02} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[u, two heads] \arrow[r, tail] & A'_{0n+1} \arrow[u, two heads] \end{tikzcd} \right) \in \mathfrak{s}.\mathcal{P}R_{n+1}$$

and

$$
D_2 = \left(\begin{tikzcd}
 & & & A_{n-1,n} \\
 & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\
 & A_{01} \arrow[r, tail] & \vdots \arrow[r, tail] \arrow[u, two heads] & A_{0n} \arrow[u, two heads] \\
B_0 \arrow[r, tail] & B_1 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] \arrow[u, two heads] & B_n \arrow[u, two heads]
\end{tikzcd} \right) = \left(\begin{tikzcd}
 & & & A'_{n,n+1} \\
 & & \cdots \arrow[r, tail] & \vdots \arrow[u, two heads] \\
 & A'_{12} \arrow[r, tail] & \vdots \arrow[r, tail] \arrow[u, two heads] & A'_{1,n+1} \arrow[u, two heads] \\
B'_{01} \arrow[r, tail] & B'_{02} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] \arrow[u, two heads] & B'_{0n+1} \arrow[u, two heads]
\end{tikzcd} \right) \in \mathfrak{s}.\mathcal{P}R_{n+1},
$$

by Proposition 1.1.2, so that elements of $\mathcal{G}.\mathcal{P}R_n$ can be identified with *pairs* of elements in $\mathfrak{s}.\mathcal{P}R_{n+1}$. In fact, it is easily seen that if $x = (D_1, D_2) \in \mathcal{G}.\mathcal{P}R_n$, $D_{1,2} \in \mathfrak{s}.\mathcal{P}R_{n+1}$ then

$$d_i x = (d_{i+1}D_1, d_{i+1}D_2)$$

and

$$s_j x = (s_{j+1}D_1, s_{j+1}D_2).$$

3 Duality on $N(Q\mathcal{P}R)$ and $Sd(\mathfrak{s}.\mathcal{P}R)^{rev}$

3.1

Just as we defined the simplicial maps $\mathfrak{s}.* : \mathfrak{s}.\mathcal{P}R \leftrightarrow \mathfrak{s}.\mathcal{P}R^{rev}$, we want to see the effect of duality on the nerve of Quillen's category. We will use the descriptions of exact categories by Quillen and Waldhausen ([16],[7]), noting that $\mathcal{P}R$ is an exact category.

Since the objects of $Q\mathcal{P}R$ are those of $\mathcal{P}R$ itself, there is no problem with starting the definition of $*$ as a covariant functor on $Q\mathcal{P}R$ by $*(P) = P^*$, but we must define the morphisms carefully. Given a morphism α in $Q\mathcal{P}R$ represented by the diagram

$$
\begin{tikzcd}
P \cdots \arrow[rr, "\alpha"] & & Q \\
 & U_\alpha \arrow[ul, two heads, "p_\alpha"] \arrow[ur, tail, "i_\alpha"'] &
\end{tikzcd}
$$

we have by Definition 5.1.1 of Chapter 2 that p_α is an admissible surjection and i_α is an admissible injection[16]. Taking duals as before for the (exact) category $\mathcal{P}R$ gives a diagram

$$\begin{tikzcd} & U_\alpha^* & \\ P^* \arrow[ur, "p_\alpha^*"] & & Q^* \arrow[ul, two heads, "i_\alpha^*"'] \end{tikzcd},$$

for which there exists a pullback:

$$U_{\alpha^*} := \{(\tau_1, \tau_2) \in P^* \times Q^* \ \mid \ p_\alpha^*(\tau_1) = i_\alpha^*(\tau_2)\} \in \mathcal{P}R.$$

Define morphisms $p_{\alpha^*} : U_{\alpha^*} \to P^*$ and $i_{\alpha^*} : U_{\alpha^*} \to Q^*$ by the appropriate coordinate projections.

Theorem 3.1.1 *The diagram*

$$\begin{tikzcd} P^* \cdots \arrow[rr, "\alpha^*"] & & Q^* \\ & U_{\alpha^*} \arrow[ul, two heads, "p_{\alpha^*}"] \arrow[ur, tail, "i_{\alpha^*}"'] & \end{tikzcd}$$

defines a morphism α^* in $Q\mathcal{P}R$.

This fact comes from the following Lemmas:

Lemma 3.1.2 *p_{α^*} is an admissible surjection.*

Proof: Given $\tau_1 \in P^*$, $p_\alpha^*(\tau_1) \in (U_\alpha)^*$ by definition. But i_α^* is surjective, so there is some $\tau_2 \in Q^*$ such that $p_\alpha^*(\tau_1) = i_\alpha^*(\tau_2)$. So we have $(\tau_1, \tau_2) \in U_{\alpha^*}$ with $p_{\alpha^*}(\tau_1, \tau_2) = \tau_1$, so that p_{α^*} is surjective.

We also see that

$$ker(p_{\alpha^*}) = \{(\tau_1, \tau_2) \in U_{\alpha^*} \ \mid \ (\tau_1, \tau_2) = (0, \tau_2)\}$$

$$= \{(0, \tau_2) \in P^* \times Q^* \ \mid \ i_\alpha^*(\tau_2) = p_\alpha^*(0) = 0\} = \{0\} \times ker(i_\alpha^*).$$

Lemma 1.2.3 tells us that $Q^* \in \mathcal{P}R$ and since i_α is admissible, i_α^* is admissible (and surjective) so that $Q^* \approx ker(i_\alpha^*) \oplus U_\alpha^*$. Since $U_\alpha^* \in \mathcal{P}R$ as well, it follows that $ker(i_\alpha^*) \in \mathcal{P}R$ and $\tilde{Q} := \{0\} \times ker(i_\alpha^*) \in \mathcal{P}R$. Now we have an exact sequence (with the appropriate inclusion on the left)

$$\tilde{Q} \rightarrowtail U_{\alpha^*} \overset{p_{\alpha^*}}{\twoheadrightarrow} P^*$$

in $\mathcal{P}R$, so p_{α^*} is an admissible surjection.

□

Lemma 3.1.3 *i_{α^*} is an admissible injection.*

Proof: Suppose $i_{\alpha^*}(\tau_1,\tau_2)=0\in Q^*$. Then by definition $\tau_2=0$ so $(\tau_1,\tau_2)=(\tau_1,0)\in U_{\alpha^*}$. Thus $p_\alpha^*(\tau_1)=i_\alpha^*(0)=0$, in which case $\tau_1=0$ since p_α^* is injective. It follows that i_{α^*} is injective. Now with $\tilde{Q}:=Q^*/im(i_{\alpha^*})$ and the natural, surjective homomorphism $\pi:Q^*\twoheadrightarrow\tilde{Q}$, we see that

$$U_{\alpha^*}\overset{i_{\alpha^*}}{\rightarrowtail}Q^*\overset{\pi}{\twoheadrightarrow}\tilde{Q}$$

is a short exact sequence. Therefore i_{α^*} is an admissible injection.

□

Lemma 3.1.4 *The assignments $P\mapsto P^*,\alpha\mapsto\alpha^*$ for objects P and morphisms α in $Q\mathcal{P}R$ define a covariant functor from $Q\mathcal{P}R$ to itself.*

To prove this, we must show that the diagrams corresponding to $\beta^*\circ\alpha^*$ and $(\beta\circ\alpha)^*$ are in the same isomorphism class defining a morphism in $Q\mathcal{P}R$ (denoted $(\beta\circ\alpha)^*=\beta^*\circ\alpha^*$ in $Q\mathcal{P}R$) for any composeable (classes of) diagrams $\alpha:P\cdots\to Q,\beta:Q\cdots\to S$ in $Q\mathcal{P}R$ as in Definition 5.1.1 of Chapter 2[11]. By this definition([11],[16]), compositions $\beta\circ\alpha$ in $Q\mathcal{P}R$ are given via pullbacks

$$U_{\beta\circ\alpha}:=U_\alpha\times_Q U_\beta=\{(z,w)\in U_\alpha\times U_\beta\ \mid\ i_\alpha(z)=p_\beta(w)\in Q\}$$

and admissible morphisms $p_{\beta\circ\alpha}(z,w)=p_\alpha(z),i_{\beta\circ\alpha}(z,w)=i_\beta(w)$ for diagram

$$\begin{tikzcd}
P\cdots \arrow[rr, "\beta\circ\alpha"] & & S \\
& U_{\beta\circ\alpha} \arrow[ul, "p_{\beta\circ\alpha}"] \arrow[ur, tail, "i_{\beta\circ\alpha}"'] &
\end{tikzcd}.$$

Taking the dual directly for such diagram yields

$$\begin{tikzcd}
P^*\cdots \arrow[rr, "(\beta\circ\alpha)^*"] & & S^* \\
& U_{(\beta\circ\alpha)^*} \arrow[ul, "p_{(\beta\circ\alpha)^*}"] \arrow[ur, tail, "i_{(\beta\circ\alpha)^*}"'] &
\end{tikzcd}$$

wherein

$$U_{(\beta\circ\alpha)^*}=\{(\chi,\nu)\in P^*\times S^*\ \mid\ (p_{\beta\circ\alpha})^*(\chi)=(i_{\beta\circ\alpha})^*(\nu)\},$$

$p_{(\beta\circ\alpha)^*}(\chi,\nu)=\chi$, and $i_{(\beta\circ\alpha)^*}(\chi,\nu)=\nu$. On the other hand, taking duals first and composing β^* with α^* gives the diagram

$$\begin{tikzcd}
P^* \cdots \arrow[rr, "\beta^* \circ \alpha^*"] & & S^* \\
& U_{\beta^* \circ \alpha^*} \arrow[ul, "p_{\beta^* \circ \alpha^*}"] \arrow[ur, "i_{\beta^* \circ \alpha^*}"'] &
\end{tikzcd}$$

where by definition of α^* and β^*

$$U_{\beta^* \circ \alpha^*} = U_{\alpha^*} \times_{Q^*} U_{\beta^*}$$

$$= \{((\chi, \hat{\chi}), (\hat{\nu}, \nu)) \in U_{\alpha^*} \times U_{\beta^*} \ \mid \ i_{\alpha^*}(\chi, \hat{\chi}) = p_{\beta^*}(\hat{\nu}, \nu)\}$$

$$= \{((\chi, \hat{\chi}), (\hat{\nu}, \nu)) \in U_{\alpha^*} \times U_{\beta^*} \ \mid \ \hat{\chi} = \hat{\nu}\}.$$

Therefore the proof of Lemma 3.1.4 reduces to proving the following theorem.

Theorem 3.1.5 *There is an isomorphism $T : U_{(\beta \circ \alpha)^*} \to U_{\beta^* \circ \alpha^*}$ for which $i_{\beta^* \circ \alpha^*} \circ T = i_{(\beta \circ \alpha)^*}$ and $p_{\beta^* \circ \alpha^*} \circ T = p_{(\beta \circ \alpha)^*}$.*

The following two lemmas give the proof.

Lemma 3.1.6 *Given $(\chi, \nu) \in U_{(\beta \circ \alpha)^*} \subseteq P^* \times S^*$, there is a unique $\hat{\nu} \in Q^*$ for which $\nu \circ i_\beta = \hat{\nu} \circ p_\beta : U_\beta \to R$.*

Proof: Given $y \in Q$, identify a $w \in U_\beta$ for which $y = p_\beta(w)$ and set $\hat{\nu}(y) = \nu(i_\beta(w))$. Since p_β is surjective, given any $y \in Q$ there is such a $w \in U_\beta$, in which case $\nu(i_\beta(w)) \in R$ is defined whenever $\nu \in S^*$ for every $y \in Q$. Set $\hat{\nu}(y) := \nu(i_\beta(w))$ for such w.

Suppose $y = y' \in Q$ with $p_\beta(w) = y$ for some $w \in U_\beta$; then $p_\beta(w) = y'$ as well so $\hat{\nu}(y) = \nu(i_\beta(w)) = \hat{\nu}(y')$, hence $\hat{\nu}$ is well-defined on Q. On the other hand, if $w, w' \in U_\beta$ have $p_\beta(w) = p_\beta(w') = y$ then

$$p_\beta(w) - p_\beta(w') = p_\beta(w - w') = 0 = i_\alpha(0)$$

since i_α is injective. By definition it follows that $(0, w - w') \in U_{\beta \circ \alpha}$. Since $(\chi, \nu) \in U_{(\beta \circ \alpha)^*}$ we now have

$$[(p_{\beta \circ \alpha})^*(\chi)](0, w - w') = [(i_{\beta \circ \alpha})^*(\nu)](0, w - w').$$

Now

$$[(p_{\beta \circ \alpha})^*(\chi)](0, w - w') = \chi(p_{\beta \circ \alpha}(0, w - w')) = \chi(p_\alpha(0)) = \chi(0) = 0$$

and

$$[(i_{\beta \circ \alpha})^*(\nu)](0, w - w') = \nu(i_{\beta \circ \alpha}(0, w - w')) = \nu(i_\beta(w - w')) = \nu(i_\beta(w) - i_\beta(w')) = \nu(i_\beta(w)) - \nu(i_\beta(w')).$$

Therefore $\nu(i_\beta(w)) - \nu(i_\beta(w')) = 0$ so that $\nu(i_\beta(w)) = \nu(i_\beta(w')) = \hat{\nu}(y)$. Thus $\hat{\nu}$ does not depend on the preimage w chosen for y.

Given $r \in R, y_1, y_2 \in Q$, there are w_1, w_2 with $p_\beta(w_1) = y_1$ and $p_\beta(w_2) = y_2$. Since p_β is a homomorphism we have $p_\beta(rw_1 + w_2) = ry_1 + y_2$, so

$$\hat{\nu}(ry_1 + y_2) = \nu(i_\beta(rw_1 + w_2)) = \nu(ri_\beta(w_1) + i_\beta(w_2)) = r\nu(i_\beta(w_1)) + \nu(i_\beta(w_2)) = r\hat{\nu}(y_1) + \hat{\nu}(y_2).$$

Therefore $\hat{\nu} \in Q^*$. Toward uniqueness, suppose that $\nu' \in Q^*$ has $\nu' \circ p_\beta = \nu \circ i_\beta$. Then for any $y \in Q$ with $p_\beta(w) = y$ for $w \in U_\beta$,

$$\nu'(y) = \nu'(p_\beta(w)) = \nu(i_\beta(w)) = \hat{\nu}(y)$$

by definition. Therefore $\nu'(y) = \hat{\nu}(y)\ \forall\ y \in Q$ and $\hat{\nu}$ is unique.

□

Lemma 3.1.7 *Let $(\chi, \nu) \in U_{(\beta\circ\alpha)^*}$ with corresponding $\hat{\nu}$ from Lemma 3.1.6.*

a) $(\chi, \hat{\nu}) \in U_{\alpha^*}$.

b) $(\hat{\nu}, \nu) \in U_{\beta^*}$.

Proof: Let $(\chi, \nu) \in U_{(\beta\circ\alpha)^*}$. Then given any $(z, w) \in U_{\beta\circ\alpha}$,

$$[(p_{\beta\circ\alpha})^*(\chi)](z, w) = [(i_{\beta\circ\alpha})^*(\nu)](z, w),$$

so that $\chi(p_\alpha(z)) = \nu(i_\beta(w))$. Take $u \in U_\alpha$. Since $i_\alpha(u) \in Q$ and p_β is surjective, $\exists\ v \in U_\beta$ with $p_\beta(v) = i_\alpha(u)$. For such v we now have $(u, v) \in U_{\beta\circ\alpha}$, in which case

$$\hat{\nu}(i_\alpha(u)) = \nu(i_\beta(v)) = \chi(p_\alpha(u)) = [(p_\alpha)^*(\chi)](u).$$

It follows that $(p_\alpha)^*(\chi) = (i_\alpha)^*(\hat{\nu})$, hence $(\chi, \hat{\nu}) \in U_{\alpha^*}$ for (a).

Again by definition of $\hat{\nu}$, $y = p_\beta(w)$ for some $w \in U_\beta$ for each $y \in Q$, implies

$$\hat{\nu}(p_\beta(w)) = \nu(i_\beta(w))\ \forall\ w \in U_\beta.$$

Therefore $(p_\beta)^*(\hat{\nu}) = (i_\beta)^*(\nu)$, hence $(\hat{\nu}, \nu) \in U_{\beta^*}$. This proves (b).

□

Now given such $\hat{\nu}$ corresponding to $(\chi, \nu) \in U_{(\beta\circ\alpha)^*}$, define $T : U_{(\beta\circ\alpha)^*} \to P^* \times Q^* \times Q^* \times S^*$ by $(\chi, \nu) \mapsto (\chi, \hat{\nu}, \hat{\nu}, \nu)$. Since $\hat{\nu} = \hat{\nu}$ we see that $T(\chi, \nu) \in U_{\beta^*\circ\alpha^*}$. But $\nu \equiv 0$ if and only if $\hat{\nu} \equiv 0$, and if $T(\chi, \nu) = (0, 0, 0, 0) = (\chi, \hat{\nu}, \hat{\nu}, \nu)$ then clearly $\chi \equiv 0$ and $\nu \equiv 0$. Therefore T is injective.

By definition of U_{β^*}, if $(\chi, \gamma, \gamma, \nu) \in U_{\beta^*\circ\alpha^*}$ then $(\gamma, \nu) \in U_{\beta^*}$, so that $\gamma(p_\beta(u)) = \nu(i_\beta(u))\ \forall\ u \in U_\beta$. Therefore $\gamma \circ p_\beta = \nu \circ i_\beta$, so by uniqueness of $\hat{\nu}$ we have $\gamma = \hat{\nu}$ hence

$$(\chi, \gamma, \gamma, \nu) = (\chi, \hat{\nu}, \hat{\nu}, \nu) = T(\chi, \nu)$$

in which case T is surjective. By construction of this isomorphism T we now have

$$p_{\beta^*\circ\alpha^*}(T(\chi, \nu)) = p_{\beta^*\circ\alpha^*}(\chi, \hat{\nu}, \hat{\nu}, \nu) = p_{\alpha^*}(\chi, \hat{\nu}) = \chi = p_{(\beta\circ\alpha)^*}(\chi, \nu)$$

and

$$i_{\beta^*\circ\alpha^*}(T(\chi, \nu)) = i_{\beta^*\circ\alpha^*}(\chi, \hat{\nu}, \hat{\nu}, \nu) = i_{\beta^*}(\hat{\nu}, \nu) = \nu = i_{(\beta\circ\alpha)^*}(\chi, \nu).$$

Thus Theorem 3.1.5 is proven and such T is sufficient to have $(\beta \circ \alpha)^* \equiv \beta^* \circ \alpha^*$ as morphisms in $Q\mathcal{P}R$.

Consider the identity morphism in $Q\mathcal{P}R$,

$$\begin{tikzcd} P \cdots \arrow[rr, "I_P"] & & P \\ & P \arrow[ul, "p_I = id_P"] \arrow[ur, tail, "i_I = id_P"'] & \end{tikzcd},$$

on an object P of $Q\mathcal{P}R$. Taking the dual results in

$$\begin{tikzcd} P^* \cdots \arrow[rr, "I_P^*"] & & P^* \\ & U_{I^*} \arrow[ul, "p_{I^*}"] \arrow[ur, tail, "i_{I^*}"'] & \end{tikzcd}$$

where

$$U_{I^*} = \{(\chi, \hat{\chi}) \in P^* \times P^* \ \mid\ (p_I)^*(\chi) = (i_I)^*(\hat{\chi})\},$$

$p_{I^*}(\chi, \hat{\chi}) = \chi$ and $i_{I^*}(\chi, \hat{\chi}) = \hat{\chi}$. But since $p_I = id_P = i_I$, we have $U_{I^*} = \{(\chi, \hat{\chi}) \ \mid\ \chi = \hat{\chi}\}$ and $p_{I^*}(\chi, \hat{\chi}) = \chi = id_{P^*}(\chi) = i_{I^*}(\chi, \hat{\chi})$. It follows that $(I_P)^* \equiv I_{P^*}$ as morphisms in $Q\mathcal{P}R$. The result is now the main result of this section:

Theorem 3.1.8 *$* : Q\mathcal{P}R \to Q\mathcal{P}R$ by $P \mapsto P^*$ and $(\alpha : P \cdots \to Q) \mapsto (\alpha^* : P^* \cdots \to Q^*)$, with α^* as described herein, is a covariant functor from $Q\mathcal{P}R$ to itself.*

Since the 1-simplices of the nerve $N(Q\mathcal{P}R)$ are precisely the morphisms of $Q\mathcal{P}R$, notice the effect of duality on two of the important simplices described in [11]: the morphisms

$$\begin{tikzcd} 0\cdots \arrow[rr, "q_P"] & & P \\ & U_q = P \arrow[ul, "p_q\equiv 0"] \arrow[ur, tail, "i_q=id_P"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} 0\cdots \arrow[rr, "\iota_P"] & & P \\ & U_\iota = 0 \arrow[ul, "p_\iota\equiv 0"] \arrow[ur, tail, "i_\iota\equiv 0"'] & \end{tikzcd}.$$

Corollary 3.1.9 *There exists a simplicial map $N.* : N(Q\mathcal{P}R) \to N(Q\mathcal{P}R)$ with $q_P \mapsto \iota_{P^*}$ and $\iota_P \mapsto q_{P^*}$ $\forall\ P \in N(Q\mathcal{P}R)_0$.*

Proof: We calculate $U_{q^*} = \{(\chi,\nu) \in 0\times P^* \ \mid\ (p_q)^*(\chi) = (i_q)^*(\nu)\}$; clearly $\chi \equiv 0$ and $\forall\ u \in P$,

$$[(p_q)^*(\chi)](u) = [(p_q)^*(0)](u) = 0 = [(i_q)^*(\nu)](u) = \nu(i_q(u)) = \nu(u),$$

so $\nu(u) = 0\ \forall\ u \in P$. It follows that $U_{q^*} = \{0\}$. Therefore $(q_P)^* \equiv \iota_{P^*} \in N(Q\mathcal{P}R)_1$. Similarly $U_{\iota^*} = \{(\chi,\nu) \in 0\times P^* \ \mid\ (p_\iota)^*(\chi) = (i_\iota)^*(\nu)\}$. For any $\nu \in P^*$ and any $u \in P$,

$$[(i_\iota)^*(\nu)](u) = \nu(i_\iota(u)) = \nu(0) = 0 = [(p_\iota)^*(0)](u) = [(p_\iota)^*(\chi)](u),$$

in which case $(p_\iota)^*(\chi) = (i_\iota)^*(\nu)\ \forall\ \nu \in P^*$. Therefore $U_{\iota^*} = 0\times P^* \approx P^*$. Also, by definition $p_{\iota^*}(0,\nu) = 0$ so that $p_{\iota^*} \equiv 0$, and $i_{\iota^*}(0,\nu) = \nu$ so $i_{\iota^*} = id_{P^*}$. It follows that $(\iota_P)^* = q_{P^*} \in N(Q\mathcal{P}R)_1$.

We apply this construction to the simplicial set $N(Q\mathcal{P}R)$. Given $x = (\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_n) \in N(Q\mathcal{P}R)_n$, define $N.*(x) = x^* = (\alpha_1^* \mid \alpha_2^* \mid \cdots \mid \alpha_n^*) \in N(Q\mathcal{P}R)_n$. This operation clearly commutes with face maps and degeneracy maps in $N(Q\mathcal{P}R)$ and our conclusion follows.

□

The application of duality to the Segal subdivision $Sd(\mathfrak{s}.\mathcal{P}R)$ of the Waldhausen simplicial set is an easy extension from what we have already calculated for the Waldhausen case. For the subdivision, we have functoriality as mentioned in Example 5.2.6 of Chapter 1(see [10]), so that if $f : X \to Y$ is a simplicial map, then so is $Sd(f) : Sd(X) \to Sd(Y)$ defined by $Sd(f)_n(x) = f_{2n+1}(x) \in Y_{2n+1} = Sd(Y)_n$. From our earlier construction of the simplicial maps $s.* : \mathfrak{s}.\mathcal{P}R \longleftrightarrow \mathfrak{s}.\mathcal{P}R^{rev}$, we now have simplicial maps $Sd(s.*) : Sd(\mathfrak{s}.\mathcal{P}R) \longleftrightarrow Sd(\mathfrak{s}.\mathcal{P}R^{rev})$. Consequently:

Corollary 3.1.10 *There exist simplicial maps*

$$Sd(s.*) : Sd(\mathfrak{s}.\mathcal{P}R)^{rev} \longleftrightarrow Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}.$$

4 Star-Connectedness and $Sd(\mathfrak{s}.\mathcal{P}R)$

First, recall the definition of a map of star-connected simplicial sets from [11] and Example 4.3.1 of Chapter 1, also referred to as a map of triples $f : (X, 0, \omega) \to (\tilde{X}, \tilde{0}, \tilde{\omega})$. These are simplicial maps of pairs (not necessarily Kan) $f : (X, 0) \to (\tilde{X}, \tilde{0})$ where $(X, 0)$ and $(\tilde{X}, \tilde{0})$ are both pointed, star-connected simplicial sets and $\tilde{\omega} \circ f = f \circ \omega$ for ray functions $\omega, \widetilde{\omega}$ on $X_0, \tilde{X}_0$ respectively.

Lemma 4.0.11 *$Sd(\mathfrak{s}.\mathcal{P}R)^{rev}$ is star-connected at* 0.

Proof: Let d_i be face maps on $\mathfrak{s}.\mathcal{P}R$, d_i^{Sd} denote face maps on the Segal subdivision, and $\hat{d}_i^{Sd}$ on its reverse. By definition $Sd(\mathfrak{s}.\mathcal{P}R)_0^{rev} = \mathfrak{s}.\mathcal{P}R_1 = Ob(\mathcal{P}R)$, so consider $P \in \mathcal{P}R$ and define $\omega_1 : Sd(\mathfrak{s}.\mathcal{P}R)_0^{rev} \to Sd(\mathfrak{s}.\mathcal{P}R)_1^{rev}$ by

$$\omega_1(P) = \begin{pmatrix} \begin{tikzcd} & & 0 \\ & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\ P \arrow[r, tail, "id_P"'] & P \arrow[u, two heads] \arrow[r, tail, "id_P"'] & P \arrow[u, two heads] \end{tikzcd} \end{pmatrix} \in \mathfrak{s}.\mathcal{P}R_3 = Sd(\mathfrak{s}.\mathcal{P}R)_1^{rev}.$$

Then

$$\hat{d}_1^{Sd}(\omega_1(P)) = d_0^{Sd}(\omega_1(P)) = d_0 d_3(\omega_1(P))$$
$$= d_0 \begin{pmatrix} \begin{tikzcd} & 0 \\ P \arrow[r, tail] & P \arrow[u, two heads] \end{tikzcd} \end{pmatrix} = 0$$

and

$$\hat{d}_0^{Sd}(\omega_1(P)) = d_1^{Sd}(\omega_1(P)) = d_1 d_2(\omega_1(P))$$
$$= d_1 \begin{pmatrix} \begin{tikzcd} & 0 \\ P \arrow[r, tail] & P \arrow[u, two heads] \end{tikzcd} \end{pmatrix} = P$$

These calculations hold for any $P \in Sd(\mathfrak{s}.\mathcal{P}R)_0^{rev}$, so $Sd(\mathfrak{s}.\mathcal{P}R)^{rev}$ is star-connected at 0 with ray function ω_1

□

Lemma 4.0.12 *$Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}$ is star-connected at 0 with ray function given by*

$$\omega_2(P) = (\omega_1(P^*))^*,$$

where ω_1 is the ray function for star-connected $Sd(\mathfrak{s}.\mathcal{P}R)^{rev}$.

Proof: By definition and what we have seen before for duality,

$$\omega_2(P) = (\omega_1(P^*))^* = (s.*)_3 \left(\begin{tikzcd} & & 0 \\ & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\ P^* \arrow[r, tail, "id_{P^*}"'] & P^* \arrow[u, two heads] \arrow[r, tail, "id_{P^*}"'] & P^* \arrow[u, two heads] \end{tikzcd} \right) = \left(\begin{tikzcd} & & P \\ & 0 \arrow[r, tail] & P \arrow[u, two heads] \\ 0 \arrow[r, tail] & 0 \arrow[u, two heads] \arrow[r, tail] & P \arrow[u, two heads] \end{tikzcd} \right)$$

(up to isomorphism $(P^*)^* \approx P$ via Lemma 1.2.3.d of Chapter 2). Using the face maps d_i, d_i^{Sd}, from Lemma 4.0.11 along with the face maps $\tilde{d}_i^{Sd}$ on $Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}$ and d_i^r on $\mathfrak{s}.\mathcal{P}R^{rev}$, we calculate

$$\tilde{d}_1^{Sd}(\omega_2(P)) = d_0^{Sd}(\omega_2(P)) = d_0^r d_3^r(\omega_2(P))$$

$$= d_2 d_0(\omega_2(P) = d_2 \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads] \end{tikzcd} \right) = 0$$

and

$$\tilde{d}_0^{Sd}(\omega_2(P)) = d_1^{Sd}(\omega_2(P)) = d_1^r d_2^r(\omega_2(P))$$

$$= d_1 d_1(\omega_2(P)) = d_1 \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads] \end{tikzcd} \right) = P$$

These calculations hold for any $P \in Ob(\mathcal{P}R) = \mathfrak{s}.\mathcal{P}R_1 = \mathfrak{s}.\mathcal{P}R_1^{rev} = Sd(\mathfrak{s}.\mathcal{P}R^{rev})_0^{rev}$, so $Sd(\mathfrak{s}.\mathcal{P}R)^{rev}$ is star-connected at 0 with ray function ω_2

□

Lemma 4.0.13 *$Sd(s.*) : (Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, 0, \omega_1) \longleftrightarrow (Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}, 0, \omega_2)$ are maps of star-connected simplicial sets.*

Proof: Noticing that $\omega_2 = (s.*)_3 \circ \omega_1 \circ (s.*)_1$ for the simplicial maps $s.* : \mathfrak{s}.\mathcal{P}R \longleftrightarrow \mathfrak{s}.\mathcal{P}R^{rev}$, we find first of all that up to isomorphism

$$Sd(s.*)_1(\omega_1(P)) = (s.*)_3(\omega_1(P)) = (s.*)_3(\omega_1((P^*)^*)) = (s.*)_3(\omega_1((s.*)_1(P^*))) = \omega_2(P^*)$$

$$= \omega_2((s.*)_1(P)) = \omega_2(Sd(s.*)_0(P))$$

for every $P \in Sd(\mathfrak{s}.\mathcal{P}R)_0^{rev}$. Also,we have from Lemma 1.2.3.d of Chapter 2 that the functor $* \circ *$ is equivalent to the identity functor on $\mathcal{P}R$, so that $s. * \circ s.*$ is the identity and therefore

$$Sd(s.*)_1(\omega_2(P)) = (s.*)_3((s.*)_3 \circ \omega_1 \circ (s.*)_1(P)) = \omega_1((s.*)_1(P)) = \omega_1(Sd(s.*)_0(P))$$

for every $P \in Sd(\mathfrak{s}.\mathcal{P}R^{rev})_0^{rev}$. Thus $Sd(s.*) \circ \omega_1 = \omega_2 \circ Sd(s.*)$ and $Sd(s.*) \circ \omega_2 = \omega_1 \circ Sd(s.*)$, so $Sd(s.*)$ is a map of star-connected simplicial sets in each case.

□

Lemma 4.0.14 *$N(Q\mathcal{P}R)$ is star-connected at basepoint $0 \in \mathcal{P}R$; two different ray functions are given by $\widetilde{\omega}_1(P) = q_P \in N(Q\mathcal{P}R)_1$ and $\widetilde{\omega}_2(P) = \iota_P \in N(Q\mathcal{P}R)_1$.*

Proof: We can see just by using the notation for the nerve of a simplicial set that for face maps d_i on $N(Q\mathcal{P}R)_1$, $d_0(0 \cdots \overset{q_P}{\to} P) = P$ and $d_1(0 \cdots \overset{q_P}{\to} P) = 0$ for every $P \in Ob(\mathcal{P}R) = N(Q\mathcal{P}R)_0$. Similarly, $d_0(0 \cdots \overset{\iota_P}{\to} P) = P$ and $d_1(0 \cdots \overset{\iota_P}{\to} P) = 0$ for each such P. Therefore setting $\widetilde{\omega}_1(P) = q_P$ and $\widetilde{\omega}_2(P) = \iota_P$ defines two different ray functions so that $N(Q\mathcal{P}R)$ is star-connected.

□

Lemma 4.0.15 *$N.* : (N(Q\mathcal{P}R), 0, \widetilde{\omega}_1) \longleftrightarrow (N(Q\mathcal{P}R), 0, \widetilde{\omega}_2)$ are maps of star-connected simplicial sets.*

Proof: We calculate

$$(N.*)_1 \circ \widetilde{\omega}_1(P) = (N.*)_1(q_P) = q_P^* = \iota_{P^*} = \widetilde{\omega}_2(P^*) = \widetilde{\omega}_2((N.*)_0(P)),$$

and

$$(N.*)_1 \circ \widetilde{\omega}_2(P) = (N.*)_1(\iota_P) = (\iota_P)^* = q_{P^*} = \widetilde{\omega}_1(P^*) = \widetilde{\omega}_1((N.*)_0(P)).$$

Therefore, in either direction, $N.*$ is a map of star-connected simplicial sets by definition.

□

Now Lemma 4.0.22 of [11] implies that $N.* : T_{\widetilde{\omega}_1}(N(Q\mathcal{P}R)) \longleftrightarrow T_{\widetilde{\omega}_2}(N(Q\mathcal{P}R))$ for maximal trees $T_{\widetilde{\omega}_1}, T_{\widetilde{\omega}_2}$; consequently there exist homomorphisms of simplicial groups

$$G(N(Q\mathcal{P}R), T_{\widetilde{\omega}_1}) \overset{G(N.*)}{\longrightarrow} G(N(Q\mathcal{P}R), T_{\widetilde{\omega}_2}) \overset{G(N.*)}{\longrightarrow} G(N(Q\mathcal{P}R), T_{\widetilde{\omega}_1}).$$

Similarly there are homomorphisms of simplicial groups

$$G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, T_{\omega_1}) \overset{G(Sd(s.*))}{\longrightarrow} G(Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}, T_{\omega_2}) \overset{G(Sd(s.*))}{\longrightarrow} G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, T_{\omega_1}).$$

5 Connections between $N(Q\mathcal{P}R)$, $Sd(\mathfrak{s}.\mathcal{P}R)$ and $\mathfrak{s}.\mathcal{P}R$

Our goal is now to review the role of the maps H and I whose induced maps are part of the mapping

$$\begin{tikzcd}
\pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}) \arrow[r, "\theta^{-1}_{1*}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \arrow[r, "\zeta_{1*}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R)) \arrow[d, dashed, bend right, "\text{“preimage”}"] \\
\pi_1(\mathcal{G}.\mathcal{P}R) \arrow[u, "T_*"] & & \pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev})) \arrow[u, "GI_*"'] \arrow[d, "GH_*"'] \\
K_1(R) \arrow[u, "L"] \arrow[rr, dashed, "\xi"] & & \pi_1(G(N(Q\mathcal{P}R)))
\end{tikzcd}$$

as in [11] and to reestablish H, I and induced maps thereof as maps that can be used with the duality described in Chapter 2. Recall that $\mathfrak{s}.\mathcal{P}R$ is a reduced simplicial set with unique 0-simplex denoted 0, hence is clearly star-connected with ray function $\omega_0 \equiv 0_1 = s_0(0) = 0 \in \mathcal{P}R$.

We restate definitions from [10, 11] for H and I : Given A in any of the (equivalent) sets

$$\mathfrak{s}.\mathcal{P}R_{2n+1} = Sd(\mathfrak{s}.\mathcal{P}R)_n = Sd(\mathfrak{s}.\mathcal{P}R)^{rev}_n = Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev},$$

we have

$$H(A) = (a_1 \mid a_2 \mid \cdots \mid a_{n-k} \mid \cdots \mid a_n) \in N(Q\mathcal{P}R)_n$$

is a composition of morphisms

$$a_{n-k} = \begin{tikzcd} A_{k+1,2n-k} \cdots \arrow[rr] & & A_{k,2n-k+1} \\ & A_{k,2n-k} \arrow[ul, two heads, "p"] \arrow[ur, tail, "i"'] & \end{tikzcd},$$

for the appropriate i and p from the rows and columns defining A, for each $0 \le k \le n-1$. For this same A,

$$I(A) = \overbrace{d_0 d_0 \cdots d_0}^{n+1}(A) \in \mathfrak{s}.\mathcal{P}R_n = \mathfrak{s}.\mathcal{P}R_n^{rev},$$

where we are careful to apply the correct face maps depending on whether we are in the simplicial set or its reverse.

Theorem 5.0.16 *In the diagram*

$$\begin{tikzcd}
(\mathfrak{s}.\mathcal{P}R, 0, \omega_0) \arrow[r, "s.*"] & ((\mathfrak{s}.\mathcal{P}R)^{rev}, 0, \omega_0) \arrow[r, "s.*"] & (\mathfrak{s}.\mathcal{P}R, 0, \omega_0) \\
(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, 0, \omega_1) \arrow[u, "I"] \arrow[r, "Sd(s.*)"] \arrow[d, "H"] & (Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}, 0, \omega_2) \arrow[u, "I"] \arrow[r, "Sd(s.*)"] \arrow[d, "H"] & (Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, 0, \omega_1) \arrow[u, "I"] \arrow[d, "H"] \\
(N(Q\mathcal{P}R), 0, \widetilde{\omega}_1) \arrow[r, "N*"] & (N(Q\mathcal{P}R), 0, \widetilde{\omega}_2) \arrow[r, "N*"] & (N(Q\mathcal{P}R), 0, \widetilde{\omega}_1)
\end{tikzcd}$$

all arrows are maps of triples and all squares commute.

Proof: Let d_i denote face maps on $\mathfrak{s}.\mathcal{P}R$, d_i^{rev} face maps on $\mathfrak{s}.\mathcal{P}R^{rev}$. Since

$$I(\omega_1(P)) = d_0 d_0 \left(\begin{tikzcd} & & 0 \\ & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\ P \arrow[r, tail] & P \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads] \end{tikzcd} \right) = d_0 \left(\begin{tikzcd} & 0 \\ 0 \arrow[r, tail] & 0 \arrow[u, two heads] \end{tikzcd} \right) = 0 = \omega_0(I(P))$$

and (with $I(\omega_2(P)) = d_0^{rev} d_0^{rev}(\omega_2(P))$)

$$I(\omega_2(P)) = d_2 d_3 \left(\begin{tikzcd} & & P \\ & 0 \arrow[r, tail] & P \arrow[u, two heads] \\ 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads] \end{tikzcd} \right) = d_2 \left(\begin{tikzcd} & 0 \\ 0 \arrow[r, tail] & 0 \arrow[u, two heads] \end{tikzcd} \right) = 0 = \omega_0(I(P))$$

for any $P \in Sd(\mathfrak{s}.\mathcal{P}R)_0^{rev}$, we see that both versions of the map I are maps of triples. Given $A \in Sd(\mathfrak{s}.\mathcal{P}R)_n^{rev} = \mathfrak{s}.\mathcal{P}R_{2n+1}$ as a triangular commutative diagram with entries $A_{i,j} \in \mathcal{P}R,\ 0 \leq i < j \leq 2n+1$ we calculate

$$I(A) = \overbrace{d_0 d_0 \cdots d_0}^{n+1}(A) = \left(\begin{tikzcd} & & & & A_{2n,2n+1} \\ & & & A_{2n-1,2n} \arrow[r, tail] & A_{2n-1,2n+1} \arrow[u, two heads] \\ & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\ & A_{n+2,n+3} \arrow[r, tail] \arrow[uurr, dotted, no head] & \cdots \arrow[r, tail] & A_{n+2,2n} \arrow[r, tail] \arrow[u, two heads] & A_{n+2,2n+1} \arrow[u, two heads] \\ A_{n+1,n+2} \arrow[r, tail] & A_{n+1,n+3} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_{n+1,2n} \arrow[r, tail] \arrow[u, two heads] & A_{n+1,2n+1} \arrow[u, two heads] \end{tikzcd} \right)$$

in $\mathfrak{s}.\mathcal{P}R_n$, so that

$$s.*(I(A)) = \left(\begin{tikzcd} & & & & A^*_{n+1,n+2} \\ & & & A^*_{n+2,n+3} \arrow[r, tail] & A^*_{n+1,n+3} \arrow[u, two heads] \\ & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\ & A^*_{2n-1,2n} \arrow[r, tail] \arrow[uurr, dotted, no head] & \cdots \arrow[r, tail] & A^*_{n+2,2n} \arrow[r, tail] \arrow[u, two heads] & A^*_{n+1,2n} \arrow[u, two heads] \\ A^*_{2n,2n+1} \arrow[r, tail] & A^*_{2n-1,2n+1} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{n+2,2n+1} \arrow[r, tail] \arrow[u, two heads] & A^*_{n+1,2n+1} \arrow[u, two heads] \end{tikzcd} \right);$$

also $Sd(s.*)(A) = A^* \in \mathfrak{s}.\mathcal{P}R^{rev}_{2n+1} = Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}_n$, hence

$$I(Sd(s.*)(A)) = \overbrace{d_0^{rev} d_0^{rev} \cdots d_0^{rev}}^{n+1}(Sd(s.*)(A)) = d_{n+1} d_{n+2} \cdots d_{2n} d_{2n+1}(A^*)$$

$$= d_{n+1} \cdots d_{2n+1} \left(\begin{tikzcd}
 & & & & & A^*_{0,1} \\
 & & & & & \vdots \arrow[u, two heads] \\
 & & & A^*_{n+1,n+2} \arrow[r, tail] & \cdots \arrow[r, tail] & A^*_{0,n+2} \arrow[u, two heads] \\
 & & A^*_{n+2,n+3} \arrow[r, tail] & A^*_{n+1,n+3} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{0,n+3} \arrow[u, two heads] \\
 & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] & & \vdots \arrow[u, two heads] \\
 & \cdots \arrow[r, tail] & A^*_{n+2,2n} \arrow[r, tail] \arrow[u, two heads] & A^*_{n+1,2n} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{0,2n} \arrow[u, two heads] \\
A^*_{2n,2n+1} \arrow[r, tail] & \cdots \arrow[r, tail] & A^*_{n+2,2n+1} \arrow[r, tail] \arrow[u, two heads] & A^*_{n+1,2n+1} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{0,2n+1} \arrow[u, two heads]
\end{tikzcd} \right)$$

$$= \left(\begin{tikzcd}
 & & & & A^*_{n+1,n+2} \\
 & & & A^*_{n+2,n+3} \arrow[r, tail] & A^*_{n+1,n+3} \arrow[u, two heads] \\
 & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
 & A^*_{2n-1,2n} \arrow[r, tail] & \cdots \arrow[r, tail] & A^*_{n+2,2n} \arrow[r, tail] \arrow[u, two heads] & A^*_{n+1,2n} \arrow[u, two heads] \\
A^*_{2n,2n+1} \arrow[r, tail] & A^*_{2n-1,2n+1} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{n+2,2n+1} \arrow[r, tail] \arrow[u, two heads] & A^*_{n+1,2n+1} \arrow[u, two heads]
\end{tikzcd} \right) = s.*(I(A)).$$

When $A \in Sd(\mathfrak{s}.\mathcal{P}R^{rev})_n^{rev} = \mathfrak{s}.\mathcal{P}R^{rev}_{2n+1}$ we have

$$(s.*)_n(I(A)) = (s.*)_n(\overbrace{d_0^{rev} d_0^{rev} \cdots d_0^{rev}}^{n+1}(A)) = (s.*)_n(d_{n+1} d_{n+2} \cdots d_{2n+1}(A))^*$$

$$= (s.*)_n \left(\begin{tikzcd}
& & & & A_{n-1,n} \\
& & & A_{n-2,n-1} \arrow[r, tail] & A_{n-2,n} \arrow[u, two heads] \\
& & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
& A_{1,2} \arrow[r, tail] \arrow[uurr, dotted, dash] & \cdots \arrow[r, tail] & A_{1,n-1} \arrow[r, tail] \arrow[u, two heads] & A_{1,n} \arrow[u, two heads] \\
A_{0,1} \arrow[r, tail] & A_{0,2} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_{0,n-1} \arrow[r, tail] \arrow[u, two heads] & A_{0,n} \arrow[u, two heads]
\end{tikzcd} \right)$$

$$= \left(\begin{tikzcd}
& & & & A^*_{0,1} \\
& & & A^*_{1,2} \arrow[r, tail] & A^*_{0,2} \arrow[u, two heads] \\
& & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
& A^*_{n-2,n-1} \arrow[r, tail] \arrow[uurr, dotted, dash] & \cdots \arrow[r, tail] & A^*_{1,n-1} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,n-1} \arrow[u, two heads] \\
A^*_{n-1,n} \arrow[r, tail] & A^*_{n-2,n} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{1,n} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,n} \arrow[u, two heads]
\end{tikzcd} \right)$$

and

$$I(Sd(s.*)(A)) = \overbrace{d_0 d_0 \cdots d_0}^{n+1}(A^*)$$

$$
= d_0 \cdots d_0 \left(
\begin{tikzcd}
& & & & & & A^*_{0,1} \\
& & & & & A^*_{1,2} \arrow[r, tail] & A^*_{0,2} \arrow[u, two heads] \\
& & & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
& & & A^*_{n-2,n-1} \arrow[r, tail] & \cdots \arrow[r, tail] & A^*_{1,n-1} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,n-1} \arrow[u, two heads] \\
& & A^*_{n-1,n} \arrow[r, tail] & A^*_{n-2,n} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{1,n} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,n} \arrow[u, two heads] \\
& & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
A^*_{2n,2n+1} \arrow[r, tail] & \cdots \arrow[r, tail] & A^*_{n-1,2n+1} \arrow[r, tail] \arrow[u, two heads] & A^*_{n-2,2n+1} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{1,2n+1} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,2n+1} \arrow[u, two heads]
\end{tikzcd}
\right)
$$

$$
= \left(
\begin{tikzcd}
& & & & A^*_{0,1} \\
& & & A^*_{1,2} \arrow[r, tail] & A^*_{0,2} \arrow[u, two heads] \\
& & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
& A^*_{n-2,n-1} \arrow[r, tail] & \cdots \arrow[r, tail] & A^*_{1,n-1} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,n-1} \arrow[u, two heads] \\
A^*_{n-1,n} \arrow[r, tail] & A^*_{n-2,n} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A^*_{1,n} \arrow[r, tail] \arrow[u, two heads] & A^*_{0,n} \arrow[u, two heads]
\end{tikzcd}
\right) = s.*(I(A)).
$$

Thus the top two squares of the diagram are commutative squares of maps of triples.

Considering H, we find that

$$
H(\omega_1(P)) = H \left(
\begin{tikzcd}
& & 0 \\
& 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\
P \arrow[r, tail, "id_P"'] & P \arrow[r, tail, "id_P"'] \arrow[u, two heads] & P \arrow[u, two heads]
\end{tikzcd}
\right) =
\begin{tikzcd}
0 \cdots \arrow[rr] & & P \\
& P \arrow[ul] \arrow[ur, tail, "id_P"'] &
\end{tikzcd}
= q_P = \widetilde{\omega}_1(P)
$$

and (since $H_0 = id_{Ob(\mathcal{P}R)}$)

$$\widetilde{\omega}_1(H(P)) = \widetilde{\omega}_1(P) = H(\omega_1(P)).$$

Similarly, $\widetilde{\omega}_2(H(P)) = \widetilde{\omega}_2(P) = \iota_P$ and

$$H(\omega_2(P)) = H\left(\begin{tikzcd} & & P \\ & 0 \arrow[r, tail] & P \arrow[u, two heads] \\ 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads] \end{tikzcd}\right) = \begin{tikzcd} 0\cdots \arrow[rr] & & P \\ & 0 \arrow[ul] \arrow[ur, tail] & \end{tikzcd} = \iota_P = \widetilde{\omega}_2(P) = \widetilde{\omega}_2(H(P)).$$

Therefore both versions of H are maps of triples. Looking closer at H for a given $A \in \mathfrak{s}.\mathcal{P}R_{2n+1}$ will require us to consider for any $0 \le k \le n-1$ the commutative squares

$$\begin{tikzcd} A_{k+1,2n-k} \arrow[r, tail, "\tilde{i}"] & A_{k+1,2n-k+1} \\ A_{k,2n-k} \arrow[r, tail, "i"] \arrow[u, two heads, "p"] & A_{k,2n-k+1} \arrow[u, two heads, "\tilde{p}"'] \end{tikzcd} \quad and \quad \begin{tikzcd} A^*_{k+1,2n-k} \arrow[r, tail, "p^*"] & A^*_{k,2n-k} \\ A^*_{k+1,2n-k+1} \arrow[r, tail, "(\tilde{p})^*"] \arrow[u, two heads, "(\tilde{i})^*"] & A^*_{k,2n-k+1} \arrow[u, two heads, "i^*"'] \end{tikzcd}$$

in the triangular, commutative diagrams representing A, A^*, respectively, in $Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}$, as well as the (short) exact sequences

$$\begin{tikzcd} & A_{2n-k,2n-k+1} \\ & A_{k+1,2n-k+1} \arrow[u, two heads, "q"'] \\ A_{k,2n-k} \arrow[r, tail, "i"] & A_{k,2n-k+1} \arrow[u, two heads, "\tilde{p}"'] \end{tikzcd}, \quad \begin{tikzcd} & A_{2n-k,2n-k+1} \\ A_{k+1,2n-k+1} \arrow[r, tail, "\tilde{i}"] & A_{k+1,2n-k+1} \arrow[u, two heads, "q"'] \end{tikzcd}$$

where q is the appropriate composition by definition of A, and (dually) the short exact sequences

$$A^*_{2n-k,2n-k+1} \overset{(\tilde{p}\circ q)^*}{\rightarrowtail} A^*_{k,2n-k+1} \overset{i^*}{\twoheadrightarrow} A^*_{k,2n-k}$$

and

$$A^*_{2n-k,2n-k+1} \overset{q^*}{\rightarrowtail} A^*_{k+1,2n-k+1} \overset{(\tilde{i})^*}{\twoheadrightarrow} A^*_{k+1,2n-k} .$$

Since

$$H(A) = (a_1 \mid a_2 \mid \cdots \mid a_{n-k} \mid \cdots \mid a_n) \in N(Q\mathcal{P}R)_n$$

is a composition of morphisms

$$
\begin{tikzcd}
a_{n-k} = A_{k+1,2n-k} \cdots \arrow[rr] & & A_{k,2n-k+1} \\
& A_{k,2n-k} \arrow[ul, "p"] \arrow[ur, tail, "i"'] &
\end{tikzcd}
$$

with

$$
\begin{tikzcd}
N.*(a_{n-k}) = (a_{n-k})^* = A^*_{k+1,2n-k} \cdots \arrow[rr] & & A^*_{k,2n-k+1}\ , \\
& U_{(a_{n-k})^*} \arrow[ul, "p_{(a_{n-k})^*}"] \arrow[ur, tail, "i_{(a_{n-k})^*}"'] &
\end{tikzcd}
$$

but

$$H(Sd(s.*)(A)) = H(A^*) = (\tilde{a}_1 \mid \tilde{a}_2 \mid \cdots \mid \tilde{a}_{n-k} \mid \cdots \mid \tilde{a}_n)$$

has morphisms

$$
\begin{tikzcd}
\tilde{a}_{n-k} = A^*_{k+1,2n-k} \cdots \arrow[rr] & & A^{k,2n-k+1}\ , \\
& A^*_{k+1,2n-k+1} \arrow[ul, "(\tilde{i})^*"] \arrow[ur, tail, "(\tilde{p})^*"'] &
\end{tikzcd}
$$

we must construct an isomorphism $\Gamma : A^*_{k+1,2n-k+1} \to U_{(a_{n-k})^*}$ for which $p_{(a_{n-k})^*} \circ \Gamma = (\tilde{i})^*$ and $i_{(a_{n-k})^*} \circ \Gamma = (\tilde{p})^*$.

First, recall that

$$U_{(a_{n-k})^*} = \left\{ (\tau_1, \tau_2) \in A^*_{k+1,2n-k} \times A^*_{k,2n-k+1} \ \mid \ p^*(\tau_1) = i^*(\tau_2) \in A^*_{k,2n-k} \right\}$$

and define $\Gamma : A^*_{k+1,2n-k+1} \to A^*_{k+1,2n-k} \times A^*_{k,2n-k+1}$ by $\Gamma(w) = ((\tilde{i})^*(w), (\tilde{p})^*(w))$. From the commutative squares we see that $p^*((\tilde{i})^*(w)) = i^*((\tilde{p})^*(w))$ so that $\Gamma(w) \in U_{(a_{n-k})^*}$ $\forall\ w \in A^*_{k+1,2n-k+1}$. If $(\tau_1, \tau_2) = (0,0) = \Gamma(w)$ then $(\tilde{p})^*(w) = 0$, so $w = 0$ since $(\tilde{p})^*$ is injective. Thus Γ is injective.

Suppose $(\tau_1, \tau_2) \in U_{(a_{n-k})^*}$. Since $(\tilde{i})^*$ is surjective, $\exists\ w \in A^*_{k+1,2n-k+1}$ for which $(\tilde{i})^*(w) = \tau_1$. By definition of $U_{(a_{n-k})^*}$ and the commutative squares, we see

$$p^*(\tau_1) = p^*((\tilde{i})^*(w)) = i^*((\tilde{p})^*(w)) = i^*(\tau_2).$$

It follows that $(\tilde{p})^*(w) - \tau_2 \in ker(i^*)$. But with the exact sequences above we see that $ker(i^*) = im((q \circ \tilde{p})^*)$. Thus there is some $u \in A^*_{2n-k,2n-k+1}$ for which $(\tilde{p})^*(q^*(u)) = (\tilde{p})^*(\tau_1) - \tau_2$.

Set $v = w - q^*(u)$ for such u. Now $v \in A^*_{k+1,2n-k+1}$ with

$$(\tilde{i})^*(v) = (\tilde{i})^*(w) - (\tilde{i})^*(q^*(u)) = (\tilde{i})^*(w) - 0 = \tau_1$$

since the exact sequences show $im(q^*) = ker((\tilde{i})^*)$. By definition of u we have $(\tilde{p})^*(v) = (\tilde{p})^*(w) - (\tilde{p})^*(q^*(u)) = \tau_2$, so $\Gamma(v) = (\tau_1, \tau_2)$ in which case Γ is surjective.

By construction of Γ,

$$p_{(a_{n-k})^*} \circ \Gamma(w) = (\tilde{i})^*(w)$$

and

$$i_{(a_{n-k})^*} \circ \Gamma(w) = (\tilde{p})^*(w)$$

for every $w \in A^*_{k+1,2n-k+1}$. Therefore we have found an appropriate isomorphism from which to have $(a_{n-k})^* = \tilde{a}_{n-k}$ as morphisms in $Q\mathcal{P}R$ by Definition 5.1.2 of Chapter 2. By definition of $N.*$ we now see

$$N.*(H(A)) = N.*(a_1 \mid \cdots \mid a_n) = (a_1^* \mid \cdots \mid a_n^* = \tilde{a}_1 \mid \cdots \mid \tilde{a}_n) = H(Sd(s.*)(A))$$

for all $A \in \mathfrak{s}.\mathcal{P}R_{2n+1} = Sd(\mathfrak{s}.\mathcal{P}R)^{rev}_n$ (and equivalently all $A \in Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}_n$). We conclude that the bottom two squares of the diagram commute.

□

Corollary 5.0.17 *There is a diagram*

$$\begin{tikzcd}
G(\mathfrak{s}.\mathcal{P}R) \arrow[r, "G(s.*)"] & G((\mathfrak{s}.\mathcal{P}R)^{rev}) \arrow[r, "G(s.*)"] & G(\mathfrak{s}.\mathcal{P}R) \\
G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, T_{\omega_1}) \arrow[u, "GI"] \arrow[d, "GH"] \arrow[r, "G(Sd(s.*))"] & G(Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}, T_{\omega_2}) \arrow[u, "GI"] \arrow[d, "GH"] \arrow[r, "G(Sd(s.*))"] & G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, T_{\omega_1}) \arrow[u, "GI"] \arrow[d, "GH"] \\
G(N(Q\mathcal{P}R), T_{\tilde{\omega}_1}) \arrow[r, "G(N.*)"] & G(N(Q\mathcal{P}R), T_{\tilde{\omega}_2}) \arrow[r, "G(N.*)"] & G(N(Q\mathcal{P}R), T_{\tilde{\omega}_1})
\end{tikzcd}$$

in which all arrows are homomorphisms of simplicial groups and all squares commute.

Theorem 5.0.18 *([11] Theorems 6.0.5 and 6.0.8)All vertical arrows in the diagram in Theorem 5.0.16 are weak homotopy equivalences.*

Theorem 5.0.19 *In the diagram of induced homomorphisms*

$$\begin{tikzcd}
\pi_n(G(\mathfrak{s}.\mathcal{P}R)) \arrow[r, "G(s.*)_*"] & \pi_n(G(\mathfrak{s}.\mathcal{P}R^{rev})) \arrow[r, "G(s.*)_*"] & \pi_n(G(\mathfrak{s}.\mathcal{P}R)) \\
\pi_n(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, T_{\omega_1})) \arrow[u, "GI_*"] \arrow[d, "GH_*"] \arrow[r, "G(Sd(s.*))_*"] & \pi_n(G(Sd(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}, T_{\omega_2})) \arrow[u, "GI_*"] \arrow[d, "GH_*"] \arrow[r, "G(Sd(s.*))_*"] & \pi_n(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}, T_{\omega_1})) \arrow[u, "GI_*"] \arrow[d, "GH_*"] \\
\pi_n(G(N(Q\mathcal{P}R), T_{\widetilde{\omega}_1})) \arrow[r, "G(N.*)_*"] & \pi_n(G(N(Q\mathcal{P}R), T_{\widetilde{\omega}_2})) \arrow[r, "G(N.*)_*"] & \pi_n(G(N(Q\mathcal{P}R), T_{\widetilde{\omega}_1})),
\end{tikzcd}$$

the top row contains isomorphisms that are inverse to each other.

Proof: By Theorem 1.2.6, $G(\mathfrak{s}.^*)$ is a weak homotopy equivalence in both directions, so that $G(\mathfrak{s}.^*)_*$ is an isomorphism in each case. By Lemma 4.2.2 and Theorem 1.4.2 of Chapter 1 and Corollary 1.2.5 of this chapter, it follows that $G(\mathfrak{s}.^*)_*$ is its own inverse.

□

Corollary 5.0.20 *All horizontal rows in the diagram from Theorem 5.0.19 contain isomorphisms which are pairwise inverses of each other.*

Proof: From Corollary 5.0.17, Theorem 5.0.18 and Theorem 5.0.19, we calculate

$$GI_* \circ G(Sd(\mathfrak{s}.^*))_* \circ G(Sd(\mathfrak{s}.^*))_* = G(\mathfrak{s}.^*)_* \circ GI_* \circ G(Sd(\mathfrak{s}.^*))_* = G(\mathfrak{s}.^*)_* \circ G(\mathfrak{s}.^*)_* \circ GI_* = GI_*,$$

so that $G(Sd(\mathfrak{s}.^*))_* \circ G(Sd(\mathfrak{s}.^*))_*$ must be the identity (since GI_* is an isomorphism), hence $G(Sd(\mathfrak{s}.^*))_*$ is an isomorphism and is its own inverse. Similarly,

$$G(N.^*)_* \circ G(N.^*)_* \circ GH_* = G(N.^*)_* \circ GH_* \circ G(Sd(\mathfrak{s}.^*))_* = GH_* \circ G(Sd(\mathfrak{s}.^*))_* G(Sd(\mathfrak{s}.^*))_* = GH_*,$$

so that $G(N.^*)_*$ is an isomorphism and is its own inverse.

□

Chapter 4

Connections with Classical K-Groups

In this chapter, we first describe the maps in the composition

$$\begin{tikzcd}
\pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}) \arrow[r, "\theta_{1*}^{-1}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \arrow[r, "\zeta_{1*}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R)) \arrow[d, dashed, shift right=6ex, "\text{``preimage''}"] \\
\pi_1(\mathcal{G}.\mathcal{P}R) \arrow[u, "T_*"] & & \pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev})) \arrow[u, "GI_*"'] \arrow[d, "GH_*"'] \\
K_1(R) \arrow[u, "L"] \arrow[rr, dashed, "\xi"] & & \pi_1(G(N(Q\mathcal{P}R)))
\end{tikzcd}$$

by which Duflot shows that the map defined by $\xi(X, A) = [x(A)]$, with $x(A)$ as described in this chapter, is an isomorphism for the K_1 case. As we do this, we will correct a miscalculation in Duflot's work([11], pages 466 and 469). Then we will compare the above diagram with one that differs only by applying duality, replacing ζ_{1*} with the induced map $G(\mathfrak{s}.*)_{1*}$ of the weak homotopy equivalence $G(\mathfrak{s}.*)$ as in Theorem 1.2.6

of Chapter 3:

$$\begin{tikzcd}
\pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}) \arrow[r, "\theta_{1*}^{-1}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \arrow[r, "G(\mathfrak{s}.*)_{1*}"] \arrow[dr, phantom, "\text{``preimage''}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R)) \\
\pi_1(\mathcal{G}.\mathcal{P}R) \arrow[u, "T_*"] & & \pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev})) \arrow[u, "GI_*"'] \arrow[d, "GH_*"'] \\
K_1(R) \arrow[u, "L"] \arrow[rr, dashed, "\hat{\xi}"] & & \pi_1(G(N(Q\mathcal{P}R)))
\end{tikzcd}$$

The advantage of using $G(\mathfrak{s}.*)_*$ to compute in the upper right corner of this diagram is twofold:

1) $G(\mathfrak{s}.*)$ is functorially defined *at the simplicial level*, showing connections between simplices in simplicial sets, unlike the ad hoc definition of ζ_{1*} in [11].

2) As the induced map of a weak homotopy equivalence, $G(\mathfrak{s}.*)_*$ is defined as an isomorphism *in every simplicial dimension*, not just in dimension 1.

1 $L : K_1(R) \to \pi_1(\mathcal{G}.\mathcal{P}R)$

We describe the map $L : K_1(R) \to \pi_1(\mathcal{G}.\mathcal{P}R)$ by summarizing the results given by Nenashev in [12, 14]. View $K_1(R)$ as $K_1(R) \approx K_1^{det}(R)$. $K_1^{det}(R)$ is the "universal determinant functor" on the semisimple category $\mathcal{P}R$, which is a construction that allows us to view elements of $K_1(R)$ as pairs (P, α) where $P \in \mathcal{P}R$ and $\alpha \in Aut(P)$. Nenashev then defines a **double-short-exact sequence** in $\mathcal{P}R$, and these sequences become the generators of an abelian group, denoted $\mathcal{D}(R)$.

These double-short-exact sequences each contain two short exact sequences, which together constitute a 1-simplices of $\mathcal{G}.\mathcal{P}R$ as in Definition 2.1.1 of Chapter 3. These are put together with other 1-simplices to form "combinatorial loop objects" in $\mathcal{G}.\mathcal{P}R_1$. These loop objects bound 2-simplices in $\mathcal{G}.\mathcal{P}R$, and these 2-simplices form the backbone of a notion of homotopy inside $\mathcal{G}.\mathcal{P}R$. The homotopy class of this loop is denoted $m(l(\alpha))$, and a representative for such a class is $\mu(l(\alpha))$, so that we are concerned with elements

$m(l(\alpha)) = [\mu(l(\alpha))] \in \pi_1(\mathcal{G}.\mathcal{P}R)$. We represent this combinatorial loop object as the sequence

$$\mu(l(\alpha)) = \left\{ \left(\begin{array}{cc} \begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, "="'] \end{tikzcd}, & \begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, "\alpha"'] \end{tikzcd} \end{array} \right), \left(\begin{array}{cc} \begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, "="'] \end{tikzcd}, & \begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, "="'] \end{tikzcd} \end{array} \right) \right\} = \{z_1, z_2\}$$

of 1-simplices in $\mathcal{G}.\mathcal{P}R$.

Nenashev then shows ([12], Theorem 3.1) that there is an element $m(l) \in \pi_1(\mathcal{G}.\mathcal{P}R)$ corresponding to each element of $K_1(R)$, based on a result of Sherman's. Sherman had a result that involved loop objects of a certain form, and Nenashev showed that such loop objects are "freely homotopic" to certain of his $\mu(l)$, and thus are members of the classes $m(l)$. Furthermore, it is shown ([12], Theorem 6.2.(1)) that $l \mapsto m(l)$ is a surjective group homomorphism from $\mathcal{D}(R)$ to $\pi_1(\mathcal{G}.\mathcal{P}R)$, which pairs *generators* of $\mathcal{D}(R)$ in particular with homotopy classes in $\mathcal{G}.\mathcal{P}R_1$.

Nenashev completes the construction by showing ([12], Theorem 6.2.(2)) that there is a group isomorphism $(A, \alpha) \mapsto l(\alpha)$ from $K_1(R) := K_1^{det}(R)$ to $\mathcal{D}(R)$, so that composition yields (after he shows that m is an isomorphism in [14])

Theorem 1.0.1 *The map $L : K_1(R) \to \pi_1(\mathcal{G}.\mathcal{P}R)$ defined by $L(P, \alpha) = [\mu(l(\alpha))]$ is an isomorphism.*

2 $T : \mathcal{G}.\mathcal{P}R \to G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}$

Consider the reverse $\mathfrak{s}.\mathcal{P}R^{rev}$ again (i.e. Theorem 1.1.3 from Chapter 3 and Example 5.2.1 from Chapter 1). Let $G(\mathfrak{s}.\mathcal{P}R^{rev})$ be constructed via the twisting function t as in Definitions 4.1.2 and 4.1.4 in Chapter 1. From [17, 2] we know

$$d_i(t(D)) = \begin{cases} t(d_i^{(rev)} D) & 0 \le i \le n, \\ \\ \left[t(d_{n+1}^{(rev)} D) \right]^{-1} t(d_n^{(rev)} D) & i = n \end{cases}$$

(and $s_j(t(D)) = t(s_j^{rev}(D))$, for every j) for each $D \in \mathfrak{s}.\mathcal{P}R_{n+1} = \mathfrak{s}.\mathcal{P}R_{n+1}^{rev}$.

Lemma 2.0.2 *The map $T : \mathcal{G}.\mathcal{P}R \to G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}$ defined by*

$$T(x) = T(D_1, D_2) = [t(D_1)]^{-1} t(D_2),$$

for the pair of diagrams $x = (D_1, D_2)$ is simplicial map.

Proof: For $1 \leq i \leq n$,

$$
\begin{aligned}
d_i^{(rev)} T(x) &= d_i^{(rev)}([t(D_1)]^{-1} t(D_2)) \\
&= d_{n-i}([t(D_1)]^{-1} t(D_2)) \\
&= [d_{n-i}(t(D_1))]^{-1} d_{n-i}(t(D_2)) \\
&= [t(d_{n-i}^{(rev)} D_1)]^{-1} t(d_{n-i}^{(rev)} D_2) \\
&= [t(d_{n+1-(n-i)} D_1)]^{-1} t(d_{n+1-(n-i)} D_2) \\
&= [t(d_{i+1} D_1)]^{-1} t(d_{i+1} D_2) \\
&= T(d_{i+1} D_1, d_{i+1} D_2) \\
&= T(d_i x).
\end{aligned}
$$

Note that since they come from an element of $\mathcal{G}.\mathcal{P}R$, D_1 and D_2 have identical rows above the first row, so that $d_0 D_1 = d_0 D_2$, hence $t(d_0 D_1) = t(d_0 D_2) \in G(\mathfrak{s}.\mathcal{P}R)_{n-1}$. Thus

$$
\begin{aligned}
d_0^{(rev)} T(x) &= d_0^{(rev)}([t(D_1)]^{-1} t(D_2)) \\
&= [d_n(t(D_1))]^{-1} d_n(t(D_2)) \\
&= [[t(d_{n+1}^{(rev)} D_1)]^{-1} t(d_n^{(rev)} D_1)]^{-1} [t(d_{n+1}^{(rev)} D_2)]^{-1} t(d_n^{(rev)} D_2) \\
&= [[t(d_0 D_1)]^{-1} t(d_1 D_1)]^{-1} [t(d_0 D_2)]^{-1} t(d_1 D_2) \\
&= [t(d_1 D_1)]^{-1} t(d_1 D_2) \\
&= T(d_1 D_1, d_1 D_2) \\
&= T(d_0 x),
\end{aligned}
$$

so T commutes with the face maps. Similarly, we calculate for degeneracy maps

$$
\begin{aligned}
s_j^{(rev)}T(x) &= s_j^{(rev)}([t(D_1)]^{-1}t(D_2)) \\
&= s_{n-j}([t(D_1)]^{-1}t(D_2)) \\
&= [s_{n-j}(t(D_1))]^{-1}s_{n-j}(t(D_2)) \\
&= [t(s_{n-j}^{(rev)}D_1)]^{-1}t(s_{n-j}^{(rev)}D_2) \\
&= [t(s_{n+1-(n-j)}D_1)]^{-1}t(s_{n+1-(n-j)}D_2) \\
&= [t(s_{j+1}D_1)]^{-1}t(s_{j+1}D_2) \\
&= T(s_{j+1}D_1, s_{j+1}D_2) \\
&= T(s_jx)
\end{aligned}
$$

for each $0 \leq j \leq n$. Therefore this T is a simplicial map.

□

Furthermore, Duflot[11] shows that the induced map

$$T_* : \pi_n(\mathcal{G}.\mathcal{P}R) \to \pi_n(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev})$$

by $T_*([x]_{\mathcal{G}.\mathcal{P}R}) = [T(x)]_{G^{rev}}$ is an isomorphism, based on the results of Berger and Gillet-Grayson. That is,

Theorem 2.0.3 *T is a weak homotopy equivalence.*

Now we calculate the composition $T_* \circ L$ given the sequence

$$
\mu(l(\alpha)) = \left\{ \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "="'] \end{tikzcd}, \begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "\alpha"'] \end{tikzcd} \right), \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "="'] \end{tikzcd}, \begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "="'] \end{tikzcd} \right) \right\} = \{z_1, z_2\}
$$

as described for Nenashev's map L. Notice that in $\mathcal{G}.\mathcal{P}R$ we have $d_0z_1 = d_0z_2 = (P, P)$ and $d_1z_1 = d_1z_2 = (0, 0)$. Furthermore, the 2-simplex

$$
y = \left(\begin{tikzcd} & & 0 \\ & P \arrow[r, tail, "="] & P \arrow[u, two heads] \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "="'] \arrow[r, tail, "="] & P \arrow[u, two heads, "="'] \end{tikzcd}, \begin{tikzcd} & & 0 \\ & P \arrow[r, tail, "\alpha"] & P \arrow[u, two heads] \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "="'] \arrow[r, tail, "="] & P \arrow[u, two heads, "\alpha"'] \end{tikzcd} \right)
$$

has

$$d_0 y = \left(\begin{tikzcd} & 0 \\ P \arrow[r, tail, "="] & P \arrow[u, two heads] \end{tikzcd} , \begin{tikzcd} & 0 \\ P \arrow[r, tail, "="] & P \arrow[u, two heads] \end{tikzcd} \right) = s_0(P,P) = s_0 d_0 z_1 = s_0 d_0 z_2,$$

$d_1 y = z_2$ and $d_2 y = z_1$. Therefore $z_1 \sim z_2$ in $\mathcal{G}.\mathcal{P}R$ by Definition 1.3.1 of Chapter 1 (with the element y above as the homotopy element), so that we can choose the class $L(P,\alpha) := [z_1] \in \pi_1(\mathcal{G}.\mathcal{P}R)$ represented by

$$z_1 := (D_1, D_2) \in \mathfrak{s}.\mathcal{P}R_2^{rev} \times \mathfrak{s}.\mathcal{P}R_2^{rev}$$

as the representative element to send to T. In this case, note that

$$D_1 = \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "="'] \end{tikzcd} \right) = s_0(P) = s_1^{rev}(P),$$

so that $t(D_1) = 1 \in G(\mathfrak{s}.\mathcal{P}R^{rev})_1^{rev}$. It follows that

$$T_*(L(P,\alpha)) = [T(D_1,D_2)] = [(t(D_1))^{-1} t(D_2)] = [t(D_2)] = [t \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "\alpha"'] \end{tikzcd} \right)] \in \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}).$$

3 $\theta_1 : \overline{G(\mathfrak{s}.\mathcal{P}R^{rev})} \to \overline{G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}}$

Here we adopt the notation $\tilde{t}(u)$ for generators of $G(\mathfrak{s}.\mathcal{P}R^{rev})$ in order to distinguish them from the generators $t(u)$ of $G(\mathfrak{s}.\mathcal{P}R)$ and consider the map θ_q defined in Theorem 7.1.1 of [11]:

$$\theta_q : G(\mathfrak{s}.\mathcal{P}R^{rev})_q \to G(\mathfrak{s}.\mathcal{P}R^{rev})_q$$

by

$$\theta_q(\tilde{t}(u)) = (\tilde{t}(u))^{(-1)^q} s_0 d_q(\tilde{t}(u)) (s_1 d_q(\tilde{t}(u)))^{-1} \cdots (s_i d_q(\tilde{t}(u)))^{(-1)^i} \cdots (s_{q-1} d_q(\tilde{t}(u)))^{(-1)^{q-1}},$$

which is bijective for each q, maps $\overline{G(\mathfrak{s}.\mathcal{P}R^{rev})}$ to $G(\widetilde{\mathfrak{s}.\mathcal{P}R^{rev}})$ (recall Definition 2.2.1 of Chapter 1) and maps Z_q to Z_q.

For any simplicial group G with face maps d_i and given integer $q > 0$ we know

$$\overline{G^{rev}}_q = G_q^{rev} \cap ker(d_0^{rev}) \cap Ker(d_1^{rev}) \cap \cdots \cap ker(d_{q-1}^{rev})$$

$$= G_q \cap ker(d_q) \cap ker(d_{q-1}) \cap \cdots \cap ker(d_1) = \widetilde{G}_q.$$

so that $G(\widetilde{\mathfrak{s}.\mathcal{P}R^{rev}})_q = \overline{G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}}_q$, hence θ_q maps $\overline{G(\mathfrak{s}.\mathcal{P}R^{rev})}_q$ to $\overline{G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}}_q$. Similarly,

$$Z_q(G^{rev}) = \overline{G^{rev}}_q \cap ker(d_q^{rev}) = G_q^{rev} \cap ker(d_0^{rev}) \cap \cdots \cap ker(d_q^{rev})$$

$$= G_q \cap ker(d_q) \cap \cdots \cap ker(d_0) = \overline{G}_q \cap ker(d_q) = Z_q(G).$$

Thus θ_q maps $Z_q(G(\mathfrak{s}.\mathcal{P}R^{rev}))$ to $Z_q(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev})$. It follows that induced homomorphisms

$$\theta_{q*} : \pi_q(G(\mathfrak{s}.\mathcal{P}R^{rev})) \to \pi_q(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev})$$

are isomorphisms. We could use Proposition 5.2.4 of Chapter 1 to reformulate this, but instead of writing $G := (G^{rev})^{rev}$ too many times we will use the inverse isomorphism

$$\theta_{1*}^{-1} := (\theta_{1*})^{-1} : \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}) \to \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})).$$

4 $\theta_{1*}^{-1} \circ T_* \circ L$

Now by definition

$$\theta_1(\tilde{t}(u)) = (\tilde{t}(u))^{-1} s_0 d_1(\tilde{t}(u)),$$

and we use

$$T_*(L(P,\alpha)) = T_*([\mu(l(\alpha))]) = [\tilde{t}(D_2)]$$

where

$$D_2 = \left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "\alpha"'] \end{tikzcd}\right) \in \mathfrak{s}.\mathcal{P}R_2^{rev}.$$

We see

$$d_1\tilde{t}(D_2) = (\tilde{t}(d_2^{rev}D_2))^{-1}\tilde{t}(d_1^{rev}D_2) = (\tilde{t}(d_0D_2))^{-1}\tilde{t}(d_1D_2) = \tilde{t}(P)^{-1}\tilde{t}(P) = 1 \in G(\mathfrak{s}.\mathcal{P}R^{rev})_1,$$

so that

$$\theta_1(\tilde{t}(D_2)) = (\tilde{t}(D_2))^{-1}s_0(1) = (\tilde{t}(D_2))^{-1}(1) = (\tilde{t}(D_2))^{-1}.$$

It follows that

$$\theta_{1*}^{-1}(T(L(P,\alpha))) = \theta_{1*}^{-1}([\tilde{t}(D_2)]) = \theta_{1*}^{-1}([((\tilde{t}(D_2))^{-1})^{-1}]) = \theta_{1*}^{-1}([\theta_1((\tilde{t}(D_2))^{-1})])$$

$$= \theta_{1*}^{-1}(\theta_{1*}([(\tilde{t}(D_2))^{-1}]) = \theta_{1*}^{-1}(\theta_{1*}([\tilde{t}(D_2)]^{-1}) = [\tilde{t}(D_2)]^{-1}.$$

5 $\zeta_{1*} : \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \to \pi_1(G(\mathfrak{s}.\mathcal{P}R))$

The map

$$\zeta_{1*} : \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \to \pi_1(G(\mathfrak{s}.\mathcal{P}R)),$$

is defined by

$$[\tilde{t}\left(\begin{tikzcd} & A_{12} \\ A_{01} \arrow[r, tail, "k_1"] & A_{02} \arrow[u, two heads, "l_2"'] \end{tikzcd}\right)] \mapsto [t\left(\begin{tikzcd} & A_{01} \\ A_{12} \arrow[r, tail, "s_2"] & A_{02} \arrow[u, two heads, "p_1"'] \end{tikzcd}\right)]$$

where s_2 is a section for l_2 (i.e. $l_2 \circ s_2 = id_{A_{12}}$) and p_1 is defined by $p_1 = k_1^{-1} \circ (id_{A_{02}} - s_2 \circ l_2)$. Duflot shows that this ζ_1 is an isomorphism ([11], Theorem 10.0.21) and is independent of the choice of section s_2.

6 $\zeta_{1*} \circ \theta_{1*}^{-1} \circ T_* \circ L$

With

$$T_*(L(P,\alpha)) = [\tilde{t}(D_2)] = [\tilde{t}\left(\begin{tikzcd} & P \\ 0 \arrow[r, tail] & P \arrow[u, two heads, "\alpha"'] \end{tikzcd}\right)]$$

we see that $A_{01} = 0, k_1 : 0 \hookrightarrow P, A_{02} = A_{12} = P$, and $l_2 = \alpha$. Since α^{-1} is a section of $\alpha \in Aut(P)$ by definition, we see that

$$p_2 := k_1^{-1} \circ (id_P - \alpha^{-1} \circ \alpha) = k_1^{-1} \circ (id_P - id_P) \equiv 0.$$

Using Duflot's notation from [11], page 469, this is

$$\zeta_{1*}\theta_{1*}^{-1}T_*([\mu(l(\alpha))]) = \zeta_{1*}\theta_{1*}^{-1}([\tilde{t}(0 \to P \overset{\alpha}{\to} P)]) = [t(P \overset{\alpha^{-1}}{\to} P \to 0)]^{-1}.$$

Notice we have α^{-1} in place of α in the last expression, which represents a miscalculation on page 469 of[11] that is now corrected.

7 Computing GI_*^{-1}

Now our aim is to take the element

$$T_*(L(P,\alpha)) = [\tilde{t}(D_2)] = [\tilde{t}\left(\begin{tikzcd} & 0 \\ P \arrow[r, tail, "\alpha^{-1}"] & P \arrow[u, two heads] \end{tikzcd}\right)] = [\tilde{t}(D_2')]$$

of $\pi_1(G(\mathfrak{s}.\mathcal{P}R))$ and follow it back to $\pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev})$ via the isomorphism GI_{1*} induced by the weak homotopy equivalence GI. Duflot defines two elements $w_1(P,\alpha), w_2(P,\alpha) \in Sd(\mathfrak{s}.\mathcal{P}R)_2^{rev} = Sd(\mathfrak{s}.\mathcal{P}R)_2 = \mathfrak{s}.\mathcal{P}R_5$ which, due to the correction for page 469 of[11]. We adjust them by replacing α with α^{-1} (and using the notation in this paper, so from [11], page 466 we change: $T := \alpha \in Aut(P)$ and $X := P$):

$$w_1 := w_1(P,\alpha^{-1}) = \left(\begin{tikzcd}
 & & & & 0 \\
 & & & P \arrow[r, tail, "\alpha^{-1}"] & P \arrow[u, two heads] \\
 & & 0 \arrow[r, tail] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="'] & P \arrow[u, two heads, "="'] \\
 & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="'] & P \arrow[u, two heads, "="'] \\
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="'] & P \arrow[u, two heads, "="']
\end{tikzcd}\right)$$

and

$$w_2 := w_2(P,\alpha^{-1}) = \left(\begin{tikzcd}
 & & & & 0 \\
 & & & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\
 & & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & 0 \arrow[u, two heads, "="'] \\
 & P \arrow[r, tail, "="] & P \arrow[r, tail, "="] \arrow[u, two heads] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads] & P \arrow[u, two heads] \\
0 \arrow[r, tail] & P \arrow[r, tail, "="] \arrow[u, two heads, "="'] & P \arrow[r, tail, "="] \arrow[u, two heads, "="'] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="'] & P \arrow[u, two heads, "="']
\end{tikzcd}\right).$$

Now we calculate

$$I_2(w_1) = d_0d_0d_0(w_1) = d_0d_0 \left(\begin{tikzcd} & & & 0 \\ & & P \arrow[r, tail, "\alpha^{-1}"] & P \arrow[u, two heads, "="] \\ & 0 \arrow[r, tail] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="] & P \arrow[u, two heads, "="] \\ 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="] & P \arrow[u, two heads, "="] \end{tikzcd} \right) = d_0 \left(\begin{tikzcd} & & 0 \\ & P \arrow[r, tail, "\alpha^{-1}"] & P \arrow[u, two heads] \\ 0 \arrow[r, tail] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="] & P \arrow[u, two heads, "="] \end{tikzcd} \right)$$

$$= \left(\begin{tikzcd} & 0 \\ P \arrow[r, tail, "\alpha^{-1}"] & P \arrow[u, two heads] \end{tikzcd} \right) = D'_2 \in \mathfrak{s}.\mathcal{P}R_2$$

so that $GI(\tilde{t}(w_1)) = t(D'_2)$, in which case $GI_*([\tilde{t}(w_1)]^{-1}) = [t(D'_2)]^{-1}$. Similarly

$$I_2(w_2) = d_0d_0d_0(w_2) = d_0d_0 \left(\begin{tikzcd} & & & 0 \\ & & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\ & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & 0 \arrow[u, two heads] \\ P \arrow[r, tail, "="] & P \arrow[r, tail, "="] \arrow[u, two heads] & P \arrow[r, tail, "\alpha^{-1}"] \arrow[u, two heads, "="] & P \arrow[u, two heads] \end{tikzcd} \right) = d_0 \left(\begin{tikzcd} & & 0 \\ & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\ 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & 0 \arrow[u, two heads] \end{tikzcd} \right)$$

$$= \left(\begin{tikzcd} & 0 \\ 0 \arrow[r, tail] & 0 \arrow[u, two heads] \end{tikzcd} \right) = 0_2 \in \mathfrak{s}.\mathcal{P}R_2$$

so that $t(I_2(w_2)) = e_1$ and therefore $GI_*([\tilde{t}(w_2)]) = [e_1] = 1$. Now we have the result:

$$GI_*([\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}]) = GI_*([\tilde{t}(w_2)])(GI_*([\tilde{t}(w_1)]))^{-1} = [e_1]([t(D'_2)])^{-1} = [t(D'_2)]^{-1}.$$

Therefore we follow $[t(D'_2)]^{-1} \in \pi_1(G(\mathfrak{s}.\mathcal{P}R))$ back to the element $[\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}] \in \pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev}))$ via GI_*.

We pause here and note that

$$Q := \zeta_{1*} \circ \theta_{1*}^{-1} \circ T_* \circ L : K_1(R) \to \pi_1(G(\mathfrak{s}.\mathcal{P}R))$$

is an isomorphism, and this brings us to the maps GH, GI discussed in Chapter 3.

8 Computing GH_*

Consider the weak homotopy equivalence $H : Sd(\mathfrak{s}.\mathcal{P}R)^{rev} \to N(Q\mathcal{P}R)$ from [11], which induces weak homotopy equivalence GH and therefore results in isomorphism $GH_* : \pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev})) \to \pi_1(G(N(Q\mathcal{P}R)))$. Given an element

$$w = \left(\begin{tikzcd}
 & & & & & & A_{2n,2n+1} \\
 & & & \vdots & \vdots & & \vdots \arrow[u, two heads] \\
 & & \cdots \arrow[r, tail] & A_{k+1,2n-k} \arrow[u, two heads] \arrow[r, tail] & A_{k+1,2n-k+1} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_{k+1,2n+1} \arrow[u, two heads] \\
 & & \cdots \arrow[r, tail] & A_{k,2n-k} \arrow[u, two heads, "p"] \arrow[r, tail, "i"] & A_{k,2n-k+1} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_{k,2n+1} \arrow[u, two heads] \\
 & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] & & \vdots \arrow[u, two heads] \\
 & A_{1,2} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1,2n-k} \arrow[u, two heads] \arrow[r, tail] & A_{1,2n-k+1} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_{1,2n+1} \arrow[u, two heads] \\
A_{0,1} \arrow[r, tail] & A_{0,2} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_{0,2n-k} \arrow[u, two heads] \arrow[r, tail] & A_{0,2n-k+1} \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & A_{0,2n+1} \arrow[u, two heads]
\end{tikzcd}\right)$$

in $Sd(\mathfrak{s}.\mathcal{P}R)^{rev}_n = \mathfrak{s}.\mathcal{P}R_{2n+1}$ (and $0 \le k \le n-1$) this is defined by

$$H(w) = (a_{1=n-(n-1)} \mid a_2 \mid \cdots \mid a_{n-k} | \cdots | a_{n-0})$$

where

$$a_{n-k} = \begin{tikzcd}
A_{k+1,2n-k} \cdots \arrow[rr] & & A_{k,2n-k+1} \\
 & A_{k,2n-k} \arrow[ul, two heads, "p"] \arrow[ur, tail, "i"'] &
\end{tikzcd} .$$

For the elements $w_1, w_2 \in Sd(\mathfrak{s}.\mathcal{P}R)^{rev}_2$, we calculate $H_2(w_1) = (a_1|a_2)$ where

$$a_1 = a_{2-1} = \begin{tikzcd}
0 \cdots \arrow[rr] & & P \\
 & 0 \arrow[ul, two heads] \arrow[ur, tail] &
\end{tikzcd}$$

and

$$a_2 = a_{2-0} = \begin{tikzcd} P \cdots \arrow[rr] & & P\,. \\ & P \arrow[ul, "="] \arrow[ur, "\alpha^{-1}"'] & \end{tikzcd}$$

Using notation from [11], page 453, it follows that $H_2(w_1) = \iota_P|(\alpha^{-1})_!$. Similarly,

$$H_2(w_2) = (b_1|b_2) = \Big(\begin{tikzcd} 0 \cdots \arrow[rr] & & P \\ & P \arrow[ul] \arrow[ur, "="'] & \end{tikzcd} \Big| \begin{tikzcd} P \cdots \arrow[rr] & & P \\ & P \arrow[ul, "="] \arrow[ur, "\alpha^{-1}"'] & \end{tikzcd} \Big) = (q_P|(\alpha^{-1})_!).$$

Now we calculate the image of the induced map as

$$GH_*([\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}]) = [GH_1(\tilde{t}(w_2))(GH_1(\tilde{t}(w_1)))^{-1}]$$

$$= [t(H_2(w_2))(t(H_2(w_1)))^{-1}] = [t(q_P|\alpha^{-1})(t(\iota_P|\alpha^{-1}))^{-1}].$$

Using the notation of [11] again, we conclude

$$GH_*([\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}]) = [x(\alpha^{-1})].$$

9 Computing $\xi : (P, \alpha) \mapsto [x(\alpha)]$ Correctly

Duflot shows ([11], Lemma 9.0.8) that $[x(\alpha)][x(\beta)] = [x(\alpha \circ \beta)]$ in general for appropriate automorphisms α and β, so in particular $[x(\alpha^{-1})] = [x(\alpha)]^{-1}$. In [11], the isomorphism ξ is computed incorrectly but stated correctly, as we now confirm.

We have shown a composition of isomorphisms

$$\xi' := GH_* \circ (GI_*)^{-1} \circ \zeta_{1*} \circ \theta_{1*}^{-1} \circ T_* \circ L,$$

which is an isomorphism given by $(P, \alpha) \mapsto [x(\alpha)]^{-1}$ where $[x(\alpha)]$ is exactly the element described by Duflot in [11]. This does not change the conclusion that the map ξ is an isomorphism: composing with the "inversion" isomorphism N on the abelian group $\pi_1(G(N(Q\mathcal{P}R)))$ (recall Proposition 2.1.5 in Chapter 1) gives $\xi := N \circ \xi'$, defined by $(P, \alpha) \mapsto [x(\alpha)]$, which confirms the result that the explicit map $\xi : K_1(R) \to \pi_1(G(N(Q\mathcal{P}R)))$ defined in [11] is an isomorphism.

10 Computing $\hat{\xi}$

We now look to the alternative diagram

$$\begin{tikzcd}
\pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})^{rev}) \arrow[r, "\theta_{1*}^{-1}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \arrow[r, "G(\mathfrak{s}.*)_{1*}"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R)) \arrow[d, dashed, shift right=10ex, "\text{“preimage”}"] \\
\pi_1(\mathcal{G}.\mathcal{P}R) \arrow[u, "T_*"] & & \pi_1(G(Sd(\mathfrak{s}.\mathcal{P}R)^{rev})) \arrow[u, "GI_*"'] \arrow[d, "GH_*"'] \\
K_1(R) \arrow[u, "L"] \arrow[rr, dashed, "\hat{\xi}"] & & \pi_1(G(N(Q\mathcal{P}R)))
\end{tikzcd}$$

We see that the computation of $\hat{\xi}$ differs from that for ξ' only just after the calculation of the image of $\theta_{1*}^{-1} \circ T_* \circ L$ and before seeking a preimage of GI_{1*}, so we pick up the calculation at that point. The dual of the element

$$\theta_{1*}^{-1}(T(L(P,\alpha))) = \theta_{1*}^{-1}([\tilde{t}(D_2)]) = \theta_{1*}^{-1}([((\tilde{t}(D_2))^{-1})^{-1}]) = \theta_{1*}^{-1}([\theta_1((\tilde{t}(D_2))^{-1})])$$

$$= \theta_{1*}^{-1}(\theta_{1*}([(\tilde{t}(D_2))^{-1}]) = \theta_{1*}^{-1}(\theta_{1*}([\tilde{t}(D_2)]^{-1}) = [\tilde{t}(D_2)]^{-1} \in G(\mathfrak{s}.\mathcal{P}R^{rev})_1$$

is given by

$$G(\mathfrak{s}.*)_{1*}([\tilde{t}(D_2)]^{-1}) = [G(\mathfrak{s}.*)(\tilde{t}(D_2))]^{-1} = [t(\mathfrak{s}.*(D_2))]^{-1}$$

$$= [t \left(\begin{array}{ccc} & & 0 \\ & & \uparrow \\ P^* & \overset{\alpha^*}{\rightarrowtail} & P^* \end{array} \right)]^{-1} \in G(\mathfrak{s}.\mathcal{P}R)_1$$

Following this element back through GI_{1*}, we again adjust the elements $w_1(P,\alpha)$, $w_2(P,\alpha)$ proposed by Duflot, but this time we define

$$w_1 := w_1(P^*,\alpha^*) = \left(\begin{tikzcd}
 & & & & 0 \\
 & & & P^* \arrow[r, tail, "\alpha^*"] & P^* \arrow[u, two heads] \\
 & & 0 \arrow[r, tail] & P^* \arrow[r, tail, "\alpha^*"] \arrow[u, two heads, "="'] & P^* \arrow[u, two heads, "="'] \\
 & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & P^* \arrow[r, tail, "\alpha^*"] \arrow[u, two heads, "="'] & P^* \arrow[u, two heads, "="'] \\
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & P^* \arrow[r, tail, "\alpha^*"] \arrow[u, two heads, "="'] & P^* \arrow[u, two heads, "="']
\end{tikzcd}\right)$$

and

$$w_2 := w_2(P^*,\alpha^*) = \left(\begin{tikzcd}
 & & & & 0 \\
 & & & 0 \arrow[r, tail] & 0 \arrow[u, two heads] \\
 & & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads, "="'] & 0 \arrow[u, two heads, "="'] \\
 & P^* \arrow[r, tail, "="] & P^* \arrow[r, tail, "="] \arrow[u, two heads] & P^* \arrow[r, tail, "\alpha^*"] \arrow[u, two heads] & P^* \arrow[u, two heads] \\
0 \arrow[r, tail] & P^* \arrow[r, tail, "="] \arrow[u, two heads, "="'] & P^* \arrow[r, tail, "="] \arrow[u, two heads, "="'] & P^* \arrow[r, tail, "\alpha^*"] \arrow[u, two heads, "="'] & P^* \arrow[u, two heads, "="']
\end{tikzcd}\right).$$

Now similar to before we calculate

$$I_2(w_1) = d_0d_0d_0(w_1) = \left(\begin{tikzcd}
 & 0 \\
P^* \arrow[r, tail, "\alpha^*"] & P^* \arrow[u, two heads]
\end{tikzcd}\right) = \mathfrak{s}.*(D_2) \in \mathfrak{s}.\mathcal{P}R_2$$

so that $GI(\tilde{t}(w_1)) = t(\mathfrak{s}.*(D_2))$, in which case $GI_*([\tilde{t}(w_1)]^{-1}) = [t(\mathfrak{s}.*(D_2))]^{-1}$. Similarly

$$I_2(w_2) = d_0d_0d_0(w_2) = \left(\begin{tikzcd}
 & 0 \\
0 \arrow[r, tail] & 0 \arrow[u, two heads]
\end{tikzcd}\right) = 0_2 \in \mathfrak{s}.\mathcal{P}R_2$$

so that $t(I_2(w_2)) = e_1$ and therefore $GI_*([\tilde{t}(w_2)]) = [e_1] = 1$. Now we have the result:

$$GI_*([\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}]) = GI_*([\tilde{t}(w_2)])(GI_*([\tilde{t}(w_1)]))^{-1} = [e_1]([t(\mathfrak{s}. * (D_2))])^{-1} = [t(\mathfrak{s}. * (D_2))]^{-1}.$$

Notice that the map $\zeta_{1*} : \pi_1(G(\mathfrak{s}.\mathcal{P}R^{rev})) \to \pi_1(G(\mathfrak{s}.\mathcal{P}R))$ relies on a choice of section for the surjective map that is part of the 1-simplex used in the construction. Although Duflot showed that the construction of ζ_{1*} is independent of the choice of section, this is not sufficient to extend the idea to higher dimensions, and will not work for mapping $\pi_2(G(\mathfrak{s}.\mathcal{P}R^{rev}))$ to $\pi_2(G(\mathfrak{s}.\mathcal{P}R))$, should that be necessary. On the other hand, Theorem 1.2.6 of Chapter 3 gives us a weak homotopy equivalence with the same domain and range and which is by definition applicable in all dimensions. Therefore we map

$$G(\mathfrak{s}.*)_{1*}(\theta_{1*}^{-1}(T_*(L(P,\alpha)))) = G(\mathfrak{s}.*)_{1*}(\theta_{1*}^{-1}(T_*(D_1, D_2))) = G(\mathfrak{s}.*)_{1*}(\theta_{1*}^{-1}([\tilde{t}(D_2)])$$

$$= G(\mathfrak{s}.*)_{1*}([\tilde{t}(D_2)]^{-1}) = [t(\mathfrak{s}. * (D_2))]^{-1} = [t \left(\begin{tikzcd} & 0 \\ P^* \arrow[r, tail, "\alpha^*"] & P^* \arrow[u, two heads] \end{tikzcd} \right)]^{-1},$$

so that we have the alternative isomorphism

$$Q' := G(\mathfrak{s}.*)_{1*} \circ \theta_{1*}^{-1} \circ T_* \circ L : K_1(R) \to \pi_1(G(\mathfrak{s}.\mathcal{P}R)).$$

We will, in fact, use Q' along with the long exact sequence of a Kan fibration and the exact sequence of Milnor from Chapter 2 to construct an isomorphism for $K_2(R)$.

Now for these elements $w_1, w_2 \in Sd(\mathfrak{s}.\mathcal{P}R)_2^{rev}$, we calculate $H_2(w_1) = (a_1|a_2)$ where

$$a_1 = a_{2-1} = \begin{tikzcd} 0 \cdots \arrow[rr] & & P^* \\ & 0 \arrow[ul, two heads] \arrow[ur, tail] & \end{tikzcd}$$

and

$$a_2 = a_{2-0} = \begin{tikzcd} P^* \cdots \arrow[rr] & & P^* \\ & P^* \arrow[ul, two heads, "="] \arrow[ur, tail, "\alpha^*"'] & \end{tikzcd} .$$

We see quickly that $H_2(w_1) = \iota_{P^*}|(\alpha^*)_!$. Similarly,

$$H_2(w_2) = (b_1|b_2) = \left(\begin{tikzcd} 0 \cdots \arrow[rr] & & P^* \\ & P^* \arrow[ul, two heads] \arrow[ur, tail, "="'] & \end{tikzcd} \middle| \begin{tikzcd} P^* \cdots \arrow[rr] & & P^* \\ & P^* \arrow[ul, two heads, "="] \arrow[ur, tail, "\alpha^*"'] & \end{tikzcd} \right) = (q_{P^*}|(\alpha^*)_!).$$

Now we calculate the image of the induced map as

$$GH_*([\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}]) = [GH_1(\tilde{t}(w_2))(GH_1(\tilde{t}(w_1)))^{-1}]$$

$$= [t(H_2(w_2))(t(H_2(w_1)))^{-1}] = [t(q_{P^*}|\alpha^*)(t(\iota_{P^*}|\alpha^*))^{-1}].$$

Using the notation of [11] again, we conclude

$$GH_*([\tilde{t}(w_2)(\tilde{t}(w_1))^{-1}]) = [x(\alpha^*)].$$

It remains to compare $[x(\alpha^{-1})]$ with $[x(\alpha^*)]$ in $\pi_1(G(N(Q\mathcal{P}R)))$ to see if duality changes the image of the isomorphism in a straightforward way. It makes sense from Theorem 1.2.6 of Chapter 2 that these elements should in fact be equal up to a sign at worst, but showing this explicitly remains a topic of continuing research, and the issue does emerge again in the context of the commutative diagram introduced in Chapter 6.

Chapter 5

Working in $G(\mathfrak{s}.\mathcal{P}R)$

1 Introduction

This Chapter constitutes the main work of this dissertation. The main result is the construction of an explicit isomorphism $f : St(R) \to \pi_1(Y(R))$, where $Y(R)$ is a Kan Complex associated with the simplicial group $G(\mathfrak{s}.\mathcal{P}R)$. We begin by exploring analogs to Nenashev's work in [13], which gave a similar calculation for $\mathcal{G}.\mathcal{P}R$.

2 Homotopy in $G(\mathfrak{s}.\mathcal{P}R)$

2.1 $\mathfrak{i}_P$

We first turn our attention to the nerve construction on the group $Aut(P)$ for $P \in \mathcal{P}R$. Thus we use the common notation

$$\boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_n) \in N(Aut(P))_n$$

or simply $\alpha = (\alpha)$ for 1-simplices (and from now on we identify the single 0-simplex of $N(Aut(P))$ by P – recall Examples 5.1.1 and 5.1.2 of Chapter 1). Additionally we define

$$\mathbf{1}_P = (1_P|1_P|\cdots|1_P) \in N(Aut(P))_n$$

if $n > 0$, and $\mathbf{1}_P = P$ in dimension 0.

Using this, we identify $\boldsymbol{\alpha}$ with the element

$$\left(\begin{tikzcd}[column sep=small]
 & & & & P \\
 & & & 0 \arrow[r, tail] & P \arrow[u, two heads, "\alpha_n"] \\
 & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads] \\
 & 0 \arrow[r, tail] \arrow[urr, dotted, dash] & \cdots \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads, "\alpha_2"] \\
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads, "\alpha_1"]
\end{tikzcd}\right) \in \mathfrak{s}.\mathcal{P}R_{n+1}$$

and we will consider the particular generators $t(\boldsymbol{\alpha})$ and $t(\mathbf{1_P})$ of $G(\mathfrak{s}.\mathcal{P}R)_n$. The following can be verified by direct computation in $G(\mathfrak{s}.\mathcal{P}R)$.

Lemma 2.1.1 *Given any $P \in \mathcal{P}R$ there is a simplicial map $\mathfrak{i}_P : N(Aut(P)) \to G(\mathfrak{s}.\mathcal{P}R)$, given on n-simplices $\boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_n)$ by*

$$\mathfrak{i}_P(\boldsymbol{\alpha}) = t(\boldsymbol{\alpha})t(\mathbf{1}_P)^{-1}.$$

Definition 2.1.2 *(See [11]) A* ***short exact sequence of pairs***

$$(P', \alpha') \xrightarrow{f} (P, \alpha) \xrightarrow{g} (P'', \alpha'')$$

is a diagram

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, tail, "f"] \arrow[d, "\alpha'"'] & P \arrow[r, two heads, "g"] \arrow[d, "\alpha"'] & P'' \arrow[r] \arrow[d, "\alpha''"'] & 0 \\
0 \arrow[r] & P' \arrow[r, tail, "f"] & P \arrow[r, two heads, "g"] & P'' \arrow[r] & 0
\end{tikzcd}$$

in which all squares commute, α, α' and α'' are automorphisms and for which $P' \overset{f}{\hookrightarrow} P \overset{g}{\twoheadrightarrow} P''$ is a short exact sequence in $\mathcal{P}R$.

Given a short exact sequence

$$l : P' \overset{f}{\rightarrowtail} P \overset{g}{\twoheadrightarrow} P''$$

in $\mathcal{P}R$, $m > 0$, $\boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_m) \in N(Aut(P))_m$ and $\boldsymbol{\alpha'} = (\alpha'_1|\alpha'_2|\cdots|\alpha'_m) \in N(Aut(P'))_m$ for which

$$(P', \alpha'_i) \xrightarrow{f} (P, \alpha_i) \xrightarrow{g} (P'', 1_{P''})$$

is a short exact sequence of pairs for each $1 \leq i \leq m$, consider the element

$$(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) \in \mathfrak{s}.\mathcal{P}R_{m+2}$$

defined by

$$(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = \left(\begin{tikzcd}
 & & & & & P'' \\
 & & & & P' \arrow[r, tail, "f"] & P \arrow[u, two heads, "g"'] \\
 & & & 0 \arrow[r, tail] & P' \arrow[u, two heads, "\alpha'_m"'] \arrow[r, tail, "f"] & P \arrow[u, two heads, "\alpha_m"'] \\
 & & & \vdots \arrow[u, two heads] & \vdots \arrow[u, two heads, "\alpha'_{m-1}"'] & \vdots \arrow[u, two heads, "\alpha_{m-1}"'] \\
 & 0 \arrow[r, tail] \arrow[uurr, dotted, no head] & \cdots \arrow[r, tail] & 0 \arrow[r, tail] \arrow[u, two heads] & P' \arrow[u, two heads, "\alpha'_2"'] \arrow[r, tail, "f"] & P \arrow[u, two heads, "\alpha_2"'] \\
0 \arrow[r, tail] & 0 \arrow[u, two heads] \arrow[r, tail] & \cdots \arrow[r, tail] & 0 \arrow[u, two heads] \arrow[r, tail] & P' \arrow[u, two heads, "\alpha'_1"'] \arrow[r, tail, "f"] & P \arrow[u, two heads, "\alpha_1"']
\end{tikzcd} \right).$$

In case $m = 0$, this is the element $(P', P; l) \in \mathfrak{s}.\mathcal{P}R_2$ representing the short exact sequence itself:

$$(P', P; l) = \left(\begin{tikzcd}
 & P'' \\
P' \arrow[r, tail, "f"] & P \arrow[u, two heads, "g"']
\end{tikzcd} \right),$$

and we also define

$$(-, -; l) := P'' \in \mathfrak{s}.\mathcal{P}R_1.$$

Lemma 2.1.3 *Given a short exact sequence*

$$l : P' \overset{f}{\hookrightarrow} P \overset{g}{\twoheadrightarrow} P''$$

in $\mathcal{P}R$, $m > 0$, $\boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_m) \in N(Aut(P))_m$ *and* $\boldsymbol{\alpha'} = (\alpha'_1|\alpha'_2|\cdots|\alpha'_m) \in N(Aut(P'))_m$, *if*

$$(P', \alpha'_i) \xrightarrow{f} (P, \alpha_i) \xrightarrow{g} (P'', 1_{P''})$$

is a short exact sequence of pairs for each $1 \leq i \leq m$*, then* $t(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) \in G(\mathfrak{s}.\mathcal{P}R)_{m+1}$ *has*

$$d_{m+1}t(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = t(\boldsymbol{\alpha'})^{-1}t(\boldsymbol{\alpha}) \in G(\mathfrak{s}.\mathcal{P}R)_m$$

and

$$d_i t(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = t(d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l) \in G(\mathfrak{s}.\mathcal{P}R)_m$$

$\forall\ 0 \leq i \leq m.$

Proof: We see by definition that $d_0 t(P', P; l) = t(P'')$ and $d_1 t(P', P; l) = t(P')^{-1}t(P)$ for case $m = 1$, and by calculation in $\mathfrak{s}.\mathcal{P}R$ that $d_i(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = (d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l)\ \forall\ 0 \leq i \leq m$, in which case

$$d_i t(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = t(d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l)$$

for each $0 \leq i \leq m$ by definition. For $i = m + 1$ we see

$d_{m+1}(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = \boldsymbol{\alpha}$ and $d_{m+2}(\boldsymbol{\alpha'}; \boldsymbol{\alpha}; l) = \boldsymbol{\alpha'}$, so that by definition in $G(\mathfrak{s}.\mathcal{P}R)$ (i.e. Definition 4.1.2 of Chapter 1),

$$d_{m+1}t(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = t(d_{m+2}(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l))^{-1}t(d_{m+1}(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l)) = t(\boldsymbol{\alpha'})^{-1}t(\boldsymbol{\alpha})$$

Note also that $d_{m+1}t(\mathbf{1}_{P'}, \mathbf{1}_P; l) = t(\mathbf{1}_{P'})^{-1}t(\mathbf{1}_P) \neq 1 \in G(\mathfrak{s}.\mathcal{P}R)_m$.

□

Theorem 2.1.4 *Given* $m > 0, P, P' \in \mathcal{P}R, \boldsymbol{\alpha} \in N(Aut(P))_m, \boldsymbol{\alpha'} \in N(Aut(P'))_m$*, if*

$$l = P' \overset{f}{\hookrightarrow} P \overset{g}{\twoheadrightarrow} P''$$

is a short exact sequence for which $(P', \alpha'_i) \xrightarrow{f} (P, \alpha_i) \xrightarrow{g} (P'', 1_{P''})$ *is an exact sequence of pairs for each* $1 \leq i \leq m$*, then* $\exists\ w_m(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) \in G(\mathfrak{s}.\mathcal{P}R)_{m+1}$ *for which*

a) $d_i w_m(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = w_{m-1}(d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l)\ \forall\ 0 \leq i \leq m.$

b) $d_{m+1}w_m(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = \mathfrak{i}_{P'}(\boldsymbol{\alpha'})^{-1}\mathfrak{i}_P(\boldsymbol{\alpha}).$

Proof: First we define $w_0(P', P; l) := 1 \in G(\mathfrak{s}.\mathcal{P}R)_1$. In case $m = 1$ we have $\alpha \in Aut(P)$, $\alpha' \in Aut(P')$ and short exact sequence of pairs $(P', \alpha') \xrightarrow{f} (P, \alpha) \xrightarrow{g} (P'', 1_{P''})$ with corresponding short exact sequence l.

From Lemma 2.1.3 we have

$$u_1 = t(\alpha', \alpha; l) = t\left(\begin{tikzcd} & & P'' \\ & P' \arrow[r, tail, "f"] & P \arrow[u, two heads, "g"'] \\ 0 \arrow[r, tail] & P' \arrow[r, tail, "f"] \arrow[u, two heads, "\alpha'"'] & P \arrow[u, two heads, "\alpha"'] \end{tikzcd}\right), v_1 = t(1_{P'}, 1_P; l) = t\left(\begin{tikzcd} & & P'' \\ & P' \arrow[r, tail, "f"] & P \arrow[u, two heads, "g"'] \\ 0 \arrow[r, tail] & P' \arrow[r, tail, "f"] \arrow[u, two heads, "="'] & P \arrow[u, two heads, "="'] \end{tikzcd}\right) \in G(\mathfrak{s}.\mathcal{P}R)_2$$

with

$$d_2 u_1 = t(\alpha')^{-1} t(\alpha)$$

and

$$d_2 v_1 = t(1_{P'})^{-1} t(1_P).$$

On the other hand

$$d_0 u_1 = d_0 v_1 = t\left(\begin{tikzcd} & P'' \\ P' \arrow[r, tail, "f"] & P \arrow[u, two heads, "g"'] \end{tikzcd}\right)$$

and

$$d_1 u_1 = d_1 v_1 = t\left(\begin{tikzcd} & P'' \\ P' \arrow[r, tail, "f"] & P \arrow[u, two heads, "g"'] \end{tikzcd}\right).$$

With $\phi = t(P') \in G(\mathfrak{s}.\mathcal{P}R)_0$ set $z_1 = s_1 s_0 \phi$. Now let $w_1(\alpha', \alpha; l) = z_1 u_1 v_1^{-1} z_1^{-1} \in G(\mathfrak{s}.\mathcal{P}R)_2$. Then we calculate

$$d_0 w_1(\alpha', \alpha; l) = (d_0 z_1)(d_0 u_1)(d_0 v_1)^{-1}(d_0 z_1)^{-1} = 1 = w_0(P', P; l) = w_0(d_0\alpha', d_0\alpha; l)$$

and similarly

$$d_1 w_1(\alpha', \alpha; l) = 1 = w_0(P', P; l) = w_0(d_1\alpha', d_1\alpha; l).$$

Finally

$$d_2 w_1(\alpha', \alpha; l) = (d_2 z_1)(d_2 u_1)(d_2 v_1)^{-1}(d_2 z_1)^{-1} = (s_0\phi)(t(\alpha')^{-1}t(\alpha))(t(1_{P'})^{-1}t(1_P))^{-1}(s_0\phi)^{-1}.$$

But $s_0\phi = t(1_{P'})$ by definition, so

$$d_2 w_1(\alpha', \alpha; l) = t(1_{P'})t(\alpha')^{-1}t(\alpha)t(1_P)^{-1}t(1_{P'})t(1_{P'})^{-1}$$

$$= (t(\alpha')t(1_{P'})^{-1})^{-1}(t(\alpha)t(1_P)^{-1}) = \mathfrak{i}_{P'}(\alpha')^{-1}\mathfrak{i}_P(\alpha).$$

Note that in the special case $m = 1$ we have $w_1(\alpha', \alpha; l) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$, which gives us more information than we have for case $m \geq 2$.

For $m \geq 2, (\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) \in \mathfrak{s}.\mathcal{P}R_{m+2}$ define $m+1$-simplices

$$u_m = t(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l), v_m = t(\mathbf{1}_{P'}, \mathbf{1}_P; l), z_m = s_m s_{m-1} \cdots s_1 s_0 t(P')$$

in $G(\mathfrak{s}.\mathcal{P}R)$ and set $w_m = z_m u_m v_m^{-1} z_m^{-1}$. Then we find $d_i z_m = z_{m-1}\ \forall\ 0 \leq i \leq m+1$, and by Lemma 2.1.3 we know $d_i u_m = t(d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l)$ and $d_i v_m = t(\mathbf{1}_{P'}, \mathbf{1}_P)\ \forall\ 0 \leq i \leq m$. Thus

$$d_i w_m(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = (d_i z_m)(d_i u_m)(d_i v_m)^{-1}(d_i z_m)^{-1}$$

$$= z_{m-1} t(d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l) t(\mathbf{1}_{P'}, \mathbf{1}_P)^{-1} z_{m-1}^{-1} = w_{m-1}(d_i\boldsymbol{\alpha'}, d_i\boldsymbol{\alpha}; l)$$

whenever $0 \leq i \leq m$. For $i = m+1$ we notice again that $z_{m-1} = t(\mathbf{1}_{P'})$ so that with $d_{m+1}v_m = t(\mathbf{1}_{P'})^{-1}t(\mathbf{1}_P)$ and $d_{m+1}u_m = t(\boldsymbol{\alpha'})^{-1}t(\boldsymbol{\alpha})$ from Lemma 2.1.3, we have

$$d_{m+1} w_m(\boldsymbol{\alpha'}, \boldsymbol{\alpha}; l) = t(\mathbf{1}_{P'})t(\boldsymbol{\alpha'})^{-1}t(\boldsymbol{\alpha})(t(\mathbf{1}_{P'})^{-1}t(\mathbf{1}_P))^{-1}t(\mathbf{1}_{P'})^{-1}$$

$$= (t(\boldsymbol{\alpha'})t(\mathbf{1}_{P'})^{-1})^{-1}(t(\boldsymbol{\alpha})t(\mathbf{1}_P)^{-1}) = \mathfrak{i}_{P'}(\boldsymbol{\alpha'})^{-1}\mathfrak{i}_P(\boldsymbol{\alpha}).$$

□

Remark 2.1.5 *In case $P' = 0$ we notice in the proof above that $\phi = t(0) = t(s_0(0)) = 1$, so that $z_1 = 1$. Also $u_1 = v_1$. Therefore $P' = 0$ implies that $w_1(\alpha', \alpha; l) = 1 \in G(\mathfrak{s}.\mathcal{P}R)_2$.*

We have a few corollaries. The first is a partial analog (i.e. in case $m = 1$) to Lemma 2.3 of [13]. Recall Definition 2.2.1 and Lemma 2.2.2 from Chapter 1: $B_1 := im(\bar{d}_2) \lhd G_1$ for any simplicial group G, with $\bar{d}_2 = d_2|_{\bar{G}}$.

Corollary 2.1.6 *Given $P, P' \in \mathcal{P}R, \alpha \in Aut(P), \alpha' \in Aut(P')$, if $l = P' \overset{f}{\hookrightarrow} P \overset{g}{\twoheadrightarrow} P''$ is a short exact sequence for which $(P', \alpha') \overset{f}{\to} (P, \alpha) \overset{g}{\to} (P'', 1_{P''})$ is an exact sequence of pairs, then $\mathfrak{i}_{P'}(\alpha')$ and $\mathfrak{i}_P(\alpha)$ represent the same element of the group $G(\mathfrak{s}.\mathcal{P}R)_1/B_1$.*

Proof: As in the proof of Theorem 2.1.4, case $m = 1$ we construct

$$w_1(\alpha', \alpha; l) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$$

with $d_2 w_1(\alpha', \alpha; l) = (\mathfrak{i}_{P'}(\alpha'))^{-1}\mathfrak{i}_P(\alpha)$.

Therefore $(\mathfrak{i}_{P'}(\alpha'))^{-1}\mathfrak{i}_P(\alpha) \in B_1$ so that by Lemma 2.2.2 in Chapter 1 it follows that $[\mathfrak{i}_{P'}(\alpha')] = [\mathfrak{i}_P(\alpha)]$ inside $G(\mathfrak{s}.\mathcal{P}R)_1/B_1$.

□

Corollary 2.1.7 *Let $P' \in \mathcal{P}R, \alpha' \in Aut(P')$ and $P = P' \oplus Q$ with $Q \in \mathcal{P}R$ and consider the short exact sequence of pairs*

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r, tail, "i"] \arrow[d, "\alpha'"'] & P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha'\oplus 1"'] & Q \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & P' \arrow[r, tail, "i"] & P \arrow[r, two heads, "\pi"] & Q \arrow[r] & 0 \end{tikzcd}.$$

Then there is a $w(\alpha', \alpha' \oplus 1) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$ with $d_2 w(\alpha', \alpha' \oplus 1_Q) = \mathfrak{i}_{P'}(\alpha')^{-1}\mathfrak{i}_P(\alpha' \oplus 1_Q)$.

Proof: This follows immediately from Corollary 2.1.6 by applying it to the case $\alpha = \alpha' \oplus 1_Q$.

□

Consequently, $[\mathfrak{i}_{P'}(\alpha')] = [\mathfrak{i}_{P'\oplus Q}(\alpha' \oplus 1_Q)]$ inside $G(\mathfrak{s}.\mathcal{P}R)_1/B_1$. Therefore homotopy classes of images of the $\mathfrak{i}'s$ in the sense of Theorem 2.2.3 of Chapter 1 are stable under direct sums:

Corollary 2.1.8 *Given any $P' \in \mathcal{P}R, \alpha' \in Aut(P'), P = P' \oplus R^n, n \in \mathbb{N}$, there is a $\xi_n \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$ with $d_2\xi_n = \mathfrak{i}_{P'}(\alpha')^{-1}\mathfrak{i}_P(\alpha' \oplus 1_n)$ (using $1_n = id_{R^n}$).*

Proof: This follows from Theorem 2.1.4 and Corollary 2.1.7 by setting

$$\xi_n = w(\alpha', \alpha' \oplus 1_R)w(\alpha' \oplus 1_R, \alpha' \oplus 1_R \oplus 1_R) \cdots w(\alpha' \oplus 1_{n-1}, \alpha' \oplus 1_{n-1} \oplus 1_R) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$$

since $R \in \mathcal{P}R$ and

$$1_n = 1_R \oplus 1_R \oplus \cdots \oplus 1_R = 1_{n-1} \oplus 1_R.$$

That is, we calculate

$$d_2\xi_n = (\mathfrak{i}_{P'}(\alpha'))^{-1}\mathfrak{i}_{P'\oplus R}(\alpha' \oplus 1_R)(\mathfrak{i}_{P'\oplus R}(\alpha' \oplus 1_R))^{-1}\mathfrak{i}_{P'\oplus R^2}(\alpha' \oplus 1_2)$$

$$\cdots \mathfrak{i}_{P'\oplus R^{n-1}}(\alpha' \oplus 1_{n-1})(\mathfrak{i}_{P'\oplus R^{n-1}}(\alpha' \oplus 1_{n-1}))^{-1}\mathfrak{i}_{P'\oplus R^n}(\alpha' \oplus 1_n)$$

$$= (\mathfrak{i}_{P'}(\alpha'))^{-1}\mathfrak{i}_{P'\oplus R^n}(\alpha' \oplus 1_n).$$

□

2.2 Filtrations: The elements $X_m(F(P,\alpha))$

Definition 2.2.1 *Let $P \in \mathcal{P}R, \alpha \in Aut(P)$. An* ***admissible filtration*** *$F = F(P,\alpha)$ of the pair (P,α) with* ***length*** *n is a sequence*

$$F : P_0 = 0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n = P$$

of projective submodules P_i, with admissible inclusions (i.e. $P_i/P_{i-1} \in \mathcal{P}R \;\forall\; 2 \leq i \leq n$), such that:

1) $\alpha^{(i)} := \alpha|_{P_i} \in Aut(P_i) \;\forall\; 1 \leq i \leq n$, $\alpha^{(1)} = 1_{P_1}$ and $\alpha^{(n)} = \alpha$.

2) For each $2 \leq i \leq n$, the homomorphism induced by $\alpha^{(i)}$ on P_i/P_{i-1} is the identity.

We can show an analog to Lemma 2.2 of [13]:

Theorem 2.2.2 *Given $P \in \mathcal{P}R, \alpha \in Aut(P)$ and admissible filtration $F(P,\alpha)$, $\exists\; X_1(F(P,\alpha)) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$ with*

$$d_2X_1(F(P,\alpha)) = \mathfrak{i}_P(\alpha)$$

Proof: By Definition 2.2.1, for each $2 \leq i \leq n$ we have a short exact sequence of pairs $(P_{i-1}, \alpha^{(i-1)}) \to (P_i, \alpha^{(i)}) \to (P_i/P_{i-1}, 1_{P_i/P_{i-1}})$ corresponding to the short exact sequence $l_i : P_{i-1} \hookrightarrow P_i \twoheadrightarrow P_i/P_{i-1}$ with the canonical inclusion and projection. Therefore by Theorem 2.1.4 for each $2 \leq i \leq n$ there is a $w_i = w_1(\alpha^{(i-1)}, \alpha^{(i)}; l_i) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$ with $d_2w_i = (\mathfrak{i}_{P_{i-1}}(\alpha^{(i-1)}))^{-1}\mathfrak{i}_{P_i}(\alpha^{(i)})$.

Define $X_1(F(P,\alpha)) = w_2w_3\cdots w_n \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$. Then

$$d_2X_1(F(P,\alpha)) = (\mathfrak{i}_{P_1}(\alpha^{(1)}))^{-1}\mathfrak{i}_{P_2}(\alpha^{(2)})(\mathfrak{i}_{P_2}(\alpha^{(2)}))^{-1}\cdots$$

$$\cdots(\mathfrak{i}_{P_{n-2}}(\alpha^{(n-2)}))^{-1}\mathfrak{i}_{P_{n-1}}(\alpha^{(n-1)})(\mathfrak{i}_{P_{n-1}}(\alpha^{(n-1)}))^{-1}\mathfrak{i}_{P_n}(\alpha^{(n)})$$

$$= (\mathfrak{i}_{P_1}(\alpha^{(1)}))^{-1}\mathfrak{i}_{P_n}(\alpha^{(n)}).$$

Notice in particular that since $\alpha^{(1)} = 1_{P_1}$ by definition we must have

$$\mathfrak{i}_{P_1}(\alpha_1) = t(1_{P_1})t(1_{P_1})^{-1} = 1 \in G(\mathfrak{s}.\mathcal{P}R)_1,$$

as well as $P_n = P$ and $\alpha^{(n)} = \alpha$. Therefore

$$d_2X_1(F(P,\alpha)) = \mathfrak{i}_P(\alpha).$$

□

Definition 2.2.3 *Given $P \in \mathcal{P}R$, $\boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_m) \in N(Aut(P))_m$, let F be a sequence*

$$F : P_0 = 0 \subseteq P_1 \subseteq P_2 \cdots \subseteq P_{n-1} \subseteq P_n = P$$

of projective submodules and admissible inclusions such that F is an admissible filtration of the pair (P, α_i) for each $1 \leq i \leq m$. Then we say $F = F(P, \boldsymbol{\alpha})$ is an ***admissible filtration of the pair*** *$(P, \boldsymbol{\alpha}) = (P; \alpha_1|\alpha_2|\cdots|\alpha_m)$.*

And now we have an analog for Lemma 2.4 of [13] as well:

Lemma 2.2.4 *Suppose $\boldsymbol{\alpha} \in N(Aut(P))_m$ and $F = F(P, \boldsymbol{\alpha})$ is an admissible filtration of $(P, \boldsymbol{\alpha})$. Then there is an $X_m(F(P, \boldsymbol{\alpha})) \in G(\mathfrak{s}.\mathcal{P}R)_{m+1}$ for which:*

a) $d_k X_m(F(P, \boldsymbol{\alpha})) = X_{m-1}(F(P, d_k\boldsymbol{\alpha}))$ *for each* $0 \leq k \leq m$.

b) $d_{m+1} X_m(F(P, \boldsymbol{\alpha})) = \mathfrak{i}_P(\boldsymbol{\alpha})$

Proof: For $m = 0$, define $X_0(F(P, P)) = 1 \in G(\mathfrak{s}.\mathcal{P}R)_1$. In case $m = 1$ the result follows from Theorem 2.2.2; indeed, we have the stronger statement that $X_1(F(P, \alpha)) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$. The general assumption is that for each $\boldsymbol{\alpha} = (\alpha_1|\cdots|\alpha_m) \in N(Aut(P))_m$ the sequence F admits a diagram

$$\begin{tikzcd}
P_0 = 0 \arrow[r] & P_1 \arrow[r, tail] \arrow[d, "\alpha_1^{(1)}"] & P_2 \arrow[r, tail] \arrow[d, "\alpha_1^{(2)}"] & \cdots \arrow[r, tail] & P_{n-1} \arrow[r, tail] \arrow[d, "\alpha_1^{(n-1)}"] & P = P_n \arrow[d, "\alpha_1"] \\
0 \arrow[r] & P_1 \arrow[r, tail] \arrow[d, "\alpha_2^{(1)}"] & P_2 \arrow[r, tail] \arrow[d, "\alpha_2^{(2)}"] & \cdots \arrow[r, tail] & P_{n-1} \arrow[r, tail] \arrow[d, "\alpha_2^{(n-1)}"] & P \arrow[d, "\alpha_2"] \\
 & \vdots \arrow[d, "\alpha_{m-1}^{(1)}"] & \vdots \arrow[d, "\alpha_{m-1}^{(2)}"] & & \vdots \arrow[d, "\alpha_{m-1}^{(n-1)}"] & \vdots \arrow[d, "\alpha_{m-1}"] \\
0 \arrow[r] & P_1 \arrow[r, tail] \arrow[d, "\alpha_m^{(1)}"] & P_2 \arrow[r, tail] \arrow[d, "\alpha_m^{(2)}"] & \cdots \arrow[r, tail] & P_{n-1} \arrow[r, tail] \arrow[d, "\alpha_m^{(n-1)}"] & P \arrow[d, "\alpha_m"] \\
0 \arrow[r] & P_1 \arrow[r, tail] & P_2 \arrow[r, tail] & \cdots \arrow[r, tail] & P_{n-1} \arrow[r, tail] & P
\end{tikzcd}$$

in which all squares commute and the j^{th} horizontal sequence for $1 \leq j \leq m$ represents an admissible filtration $F(P, \alpha_j)$. Notice that such F will also be an admissible filtration of the pair $(P, \alpha_{j+1} \circ \alpha_j)$. So for each $2 \leq i \leq n$ and each $1 \leq j \leq m$ we have a short exact sequence of pairs

$$(P_{i-1}, \alpha_j^{(i-1)}) \to (P_i, \alpha_j^{(i)}) \to (P_i/P_{i-1}, 1_{P_i/P_{i-1}})$$

with corresponding short exact sequence l_i (note the same l_i corresponds to each j). Now for each $1 \leq i \leq n$ we have $\boldsymbol{\alpha^{(i)}} := (\alpha_1^{(i)}|\alpha_2^{(i)}|\cdots|\alpha_m^{(i)}) \in N(Aut(P_i))_m$ (i.e. consecutive vertical columns give pairs $\boldsymbol{\alpha^{(i-1)}}, \boldsymbol{\alpha^{(i)}}$ of m-simplices that satisfy the hypothesis for Theorem 2.1.4), with $\boldsymbol{\alpha^{(n)}} = \boldsymbol{\alpha}$. It follows that for each $2 \leq i \leq n$ $\exists$ $w_i = w_m(\boldsymbol{\alpha^{(i-1)}}, \boldsymbol{\alpha^{(i)}}; l_i) \in G(\mathfrak{s}.\mathcal{P}R)_{m+1}$ for which

$$d_k w_i = w_{m-1}(d_k\boldsymbol{\alpha^{(i-1)}}, d_k\boldsymbol{\alpha^{(i)}}; l_i)$$

for every $0 \leq k \leq m$ and

$$d_{m+1} w_i = \mathfrak{i}_{P_{i-1}}(\boldsymbol{\alpha^{(i-1)}})^{-1}\mathfrak{i}_{P_i}(\boldsymbol{\alpha^{(i)}}).$$

Now define $X_m(F(P,\boldsymbol{\alpha})) = w_2 w_3 \cdots w_n \in G(\mathfrak{s}.\mathcal{P}R)_{m+1}$ and calculate:

$$d_k X_m(F(P,\boldsymbol{\alpha})) = w_{m-1}(d_k\boldsymbol{\alpha^{(1)}}, d_k\boldsymbol{\alpha^{(2)}}; l_2) w_{m-1}(d_k\boldsymbol{\alpha^{(2)}}, d_k\boldsymbol{\alpha^{(3)}}; l_3)\cdots w_{m-1}(d_k\boldsymbol{\alpha^{(n-1)}}, d_k\boldsymbol{\alpha^{(n)}}; l_n)$$

$$= X_{m-1}(F(P, d_k\boldsymbol{\alpha}))$$

for $0 \leq k \leq m$, and

$$d_{m+1}X_m(F(P,\boldsymbol{\alpha})) = (\mathfrak{i}_{P_1}(\boldsymbol{\alpha^{(1)}}))^{-1}\mathfrak{i}_{P_2}(\boldsymbol{\alpha^{(2)}})(\mathfrak{i}_{P_2}(\boldsymbol{\alpha^{(2)}}))^{-1}\cdots\mathfrak{i}_{P_{n-1}}(\boldsymbol{\alpha^{(n-1)}})(\mathfrak{i}_{P_{n-1}}(\boldsymbol{\alpha^{(n-1)}}))^{-1}\mathfrak{i}_{P_n}(\boldsymbol{\alpha^{(n)}})$$

$$= (\mathfrak{i}_{P_1}(\boldsymbol{\alpha^{(1)}}))^{-1}\mathfrak{i}_{P_n}(\boldsymbol{\alpha^{(n)}}).$$

But by assumption F is an admissible filtration of (P,α_j) for each $1 \leq j \leq m$, so $\alpha_j^{(1)} = 1_{P_1}$ for each $1 \leq j \leq m$ by Definition 2.2.1, in which case $\boldsymbol{\alpha^{(1)}} = \mathbf{1}_{P_1}$. Also by definition $\boldsymbol{\alpha^{(n)}} = \boldsymbol{\alpha}$ with $P_n = P$. Therefore $d_{m+1}X_m(F(P,\boldsymbol{\alpha})) = (\mathfrak{i}_{P_1}(\mathbf{1}_{P_1}))^{-1}\mathfrak{i}_P(\boldsymbol{\alpha})$. But by definition (i.e. Lemma 2.1.1) we know

$$\mathfrak{i}_{P_1}(\mathbf{1}_{P_1}) = t(\mathbf{1}_{P_1})(t(\mathbf{1}_{P_1}))^{-1} = 1 \in G(\mathfrak{s}.\mathcal{P}R)_m.$$

It follows that

$$d_{m+1}X_m(F(P,\boldsymbol{\alpha})) = (1)\mathfrak{i}_P(\boldsymbol{\alpha}) = \mathfrak{i}_P(\boldsymbol{\alpha}).$$

□

2.3 Refinements of Admissible Filtrations

Definition 2.3.1 *(See [20]) Given a sequence*

$$F : P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$$

*of projective submodules for $P \in \mathcal{P}R$, a **refinement** of F is a sequence $\widetilde{F}$ which can be obtained from F by inserting a finite number of projective submodules into F. If F is an admissible filtration of (P, α), the restriction of $\alpha \in Aut(P)$ to each inserted submodule is an automorphism of that submodule and the inclusions around and including each inserted submodule are admissible, then we say $\widetilde{F}(P, \alpha)$ is an **admissible refinement** of the admissible filtration $F(P, \alpha)$.*

Lemma 2.3.2 *Any admissible refinement of an admissible filtration is an admissible filtration.*

Proof: Let

$$F : P_0 = 0 \subseteq P_1 \subseteq \cdots \subseteq P_{n-1} \subseteq P_n = P$$

be a sequence that is an admissible filtration of (P, α). Given $1 \leq i \leq n$, we show that the sequence

$$\widetilde{F} : P_0 = 0 \subseteq P_1 \subseteq \cdots P_{i-1} \subseteq \widetilde{P} \subseteq P_i \subseteq \cdots \subseteq P_{n-1} \subseteq P_n = P$$

obtained by inserting a single, α-invariant submodule $\widetilde{P} \in \mathcal{P}R$ as shown will be an admissible filtration of (P, α), provided the inclusions $P_{i-1} \subseteq \widetilde{P}, \widetilde{P} \subseteq P_i$ are admissible. The result will then follow by induction. Denote $\widetilde{\alpha} = \alpha|_{\widetilde{P}}$. We know by definition of admissible filtration F that $\alpha^{(i)}$ induces the identity map on P_i/P_{i-1}, so that with respect to cosets we have

$$\alpha^{(i)}(p) + P_{i-1} = p + P_{i-1} \in P_i/P_{i-1}$$

for every $p \in P_i$. Since $\widetilde{P} \subseteq P_i$ and $\alpha|_{\widetilde{P}} = \alpha^{(i)}|_{\widetilde{P}}$ it follows that $\widetilde{\alpha}$ induces the identity map on $\widetilde{P}/P_{i-1}$, so that

$$(P_{i-1}, \alpha^{(i-1)}) \to (\widetilde{P}, \widetilde{\alpha}) \to (\widetilde{P}/P_{i-1}, 1_{\widetilde{P}/P_{i-1}})$$

is a short exact sequence of pairs. But also we have that

$$\alpha^{(i)}(p) - p \in P_{i-1}$$

for each $p \in P_i$, and $P_{i-1} \subseteq \widetilde{P}$. Therefore

$$\alpha^{(i)}(p) - p \in \widetilde{P} \ \forall \ p \in P_i$$

hence $\alpha^{(i)}$ induces the identity map on $P_i/\widetilde{P}$ and

$$(\widetilde{P}, \widetilde{\alpha}) \to (P_i, \alpha^{(i)}) \to (P_i/\widetilde{P}, 1_{P_i/\widetilde{P}})$$

is a short exact sequence of pairs. Since these are precisely the short exact sequences that are inserted along with $\widetilde{P}$ and its inclusions and everything else about the filtration remains unchanged from $F(P, \alpha)$, it follows that the new sequence $\widetilde{F}$ is an admissible filtration of (P, α).

□

We first consider $\alpha \in Aut(P)$ with admissible filtration $F(P, \alpha) : P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n = P$, and a refinement of F by one additional, α-invariant submodule $\widetilde{P} \in \mathcal{P}R$ (and $\tilde{\alpha} := \alpha|_{\widetilde{P}}$):

$$\widetilde{F} : P_1 \subseteq P_2 \subseteq \cdots P_{i-1} \subseteq \widetilde{P} \subseteq P_i \subseteq \cdots \subseteq P_n = P,$$

such that $\tilde{P}/P_{i-1}, P_i/\tilde{P} \in \mathcal{P}R$.

We know $F(P, \alpha)$ gives exact sequences of pairs $l_j : (P_j, \alpha^{(j-1)}) \to (P_j, \alpha^{(j)}) \to (P_j/P_{j-1}, 1_{P_j/P_{j-1}}), 2 \leq j \leq n$ as in the proof of Theorem 2.2.2. From the proof of Lemma 2.3.2 we have exact sequences of pairs $\{\tilde{l}_j\}$ for $\widetilde{F}$ as well, and we see that

$$\tilde{l}_j = l_j \ \forall \ 2 \leq j \leq i-1,$$

$$\tilde{l}_i : (P_{i-1}, \alpha^{(i-1)}) \to (\widetilde{P}, \tilde{\alpha}) \to (\widetilde{P}/P_{i-1}, 1_{\widetilde{P}/P_{i-1}}),$$

$$\tilde{l}_{i+1} : (\widetilde{P}, \tilde{\alpha}) \to (P_i, \alpha^{(i)}) \to (P_i/\widetilde{P}, 1_{P_i/\widetilde{P}}),$$

and

$$\tilde{l}_j = l_{j-1} \ \forall \ i+2 \leq j \leq n+1.$$

From Lemma 2.2.4 we now have elements

$$X_1(F(P, \alpha)) = w_1(\alpha^{(1)}, \alpha^{(2)}; l_2) w_1(\alpha^{(2)}, \alpha^{(3)}; l_3) \cdots w_1(\alpha^{(i-2)}, \alpha^{(i-1)}; l_{i-1}) w_1(\alpha^{(i-1)}, \alpha^{(i)}, l_i)$$

$$\cdots w_1(\alpha^{(n-1)}, \alpha; l_n)$$

and

$$X_1(\widetilde{F}(P,\alpha)) = w_1(\alpha^{(1)},\alpha^{(2)};l_2)w_1(\alpha^{(2)},\alpha^{(3)};l_3)\cdots w_1(\alpha^{(i-2)},\tilde{\alpha};\tilde{l}_{i-1})w_1(\tilde{\alpha},\alpha^{(i)};\tilde{l}_i)w_1(\alpha^{(i)},\alpha^{(i+1)},l_{i+1})$$

$$\cdots w_1(\alpha^{(n-1)},\alpha;l_n)$$

in $\overline{G(\mathfrak{s}.\mathcal{P}R)}_2$.

The main results of this section require us to compare $X_1(F(P,\alpha))$ with $X_1(\widetilde{F}(P,\alpha))$. Let

$$A = w_1(\alpha^{(1)},\alpha^{(2)};l_2)w_1(\alpha^{(2)},\alpha^{(3)};l_3)\cdots w_1(\alpha^{(i-2)},\alpha^{(i-1)};l_{i-1}),$$

$$B = w_1(\alpha^{(i)},\alpha^{(i+1)},l_{i+1})\cdots w_1(\alpha^{(n-1)},\alpha;l_n),$$

$$C = w_1(\alpha^{(i-1)},\alpha^{(i)} : l_i),$$

$$C_1 = w_1(\alpha^{(i-1)},\tilde{\alpha};\tilde{l}_i)$$

and

$$C_2 = w_1(\tilde{\alpha},\alpha^{(i)};\tilde{l}_{i+1}).$$

Then $X_1(F(P,\alpha)) = ACB$ and $X_1(\widetilde{F}(P,\alpha)) = AC_1C_2B$. Thus we see that in order to compare $X_1(F(P,\alpha))$ to $X_1(\widetilde{F}(P,\alpha))$, we must first compare C to C_1C_2. Theorem 2.1.4 shows how to do this:

Lemma 2.3.3 *If*

$$\begin{tikzcd} P' \arrow[r, tail] \arrow[d, "\alpha'"] & \widetilde{P} \arrow[r, tail] \arrow[d, "\tilde{\alpha}"] & P \arrow[d, "\alpha"] \\ P' \arrow[r, tail] & \widetilde{P} \arrow[r, tail] & P \end{tikzcd}$$

is a commutative diagram of projective modules and automorphisms such that the horizontal rows are admissible inclusions, with exact sequences of pairs

$$l : (P',\alpha') \to (P,\alpha) \to (P/P', 1_{P/P'}),$$

$$\tilde{l}_1 : (P',\alpha') \to (\widetilde{P},\tilde{\alpha}) \to (\widetilde{P}/P', 1_{\widetilde{P}/P'})$$

and

$$\tilde{l}_2 : (\widetilde{P},\tilde{\alpha}) \to (P,\alpha) \to (P/\widetilde{P}, 1_{P/\widetilde{P}})$$

and corresponding elements $C = w_1(\alpha',\alpha;l)$, $C_1 = w_1(\alpha',\tilde{\alpha};\tilde{l}_1)$, *and* $C_2 = w_1(\tilde{\alpha},\alpha;\tilde{l}_2)$ *in* $\overline{G(\mathfrak{s}.\mathcal{P}R)}_2$, *then*

$[C] = [C_1C_2]$ *in* $G(\mathfrak{s}.\mathcal{P}R)_2/B_2$.

Proof: Define

$$(\alpha', \tilde{\alpha}, \alpha) = \left(\begin{tikzcd}
 & & & P/\widetilde{P} \\
 & & \widetilde{P}/P' \arrow[r, tail] & P/P' \arrow[u, two heads] \\
 & P' \arrow[r, tail] & \widetilde{P} \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads] \\
0 \arrow[r, tail] & P' \arrow[r, tail] \arrow[u, two heads, "\alpha'"] & \widetilde{P} \arrow[r, tail] \arrow[u, two heads, "\tilde{\alpha}"] & P \arrow[u, two heads, "\alpha"]
\end{tikzcd}\right) \in \mathfrak{s}.\mathcal{P}R_4.$$

Then

$$t(\alpha', \tilde{\alpha}, \alpha) \in G(\mathfrak{s}.\mathcal{P}R)_3$$

and similarly $t(1_{P'}, 1_{\widetilde{P}}, 1_P) \in G(\mathfrak{s}.\mathcal{P}R)_3$). Calculations show that

$$d_0 t(\alpha', \tilde{\alpha}, \alpha) = d_1 t(\alpha', \tilde{\alpha}, \alpha) = t\left(\begin{tikzcd}
 & & P/\widetilde{P} \\
 & \widetilde{P}/P' \arrow[r, tail] & P/P' \arrow[u, two heads] \\
P' \arrow[r, tail] & \widetilde{P} \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads]
\end{tikzcd}\right),$$

$$d_2 t(\alpha', \tilde{\alpha}, \alpha) = t(\tilde{\alpha}, \alpha; \tilde{l}_2),$$

and

$$d_3 t(\alpha', \tilde{\alpha}, \alpha) = (t(\alpha', \tilde{\alpha}; \tilde{l}_1))^{-1} t(\alpha', \alpha; l).$$

Likewise

$$d_0 t(1_{P'}, 1_{\widetilde{P}}, 1_P) = d_1 t(1_{P'}, 1_{\widetilde{P}}, 1_P) = t\left(\begin{tikzcd}
 & & P/\widetilde{P} \\
 & \widetilde{P}/P' \arrow[r, tail] & P/P' \arrow[u, two heads] \\
P' \arrow[r, tail] & \widetilde{P} \arrow[r, tail] \arrow[u, two heads] & P \arrow[u, two heads]
\end{tikzcd}\right),$$

$$d_2 t(1_{P'}, 1_{\widetilde{P}}, 1_P) = t(1_{\widetilde{P}}, 1_P; \tilde{l}_2),$$

and

$$d_3 t(1_{P'}, 1_{\widetilde{P}}, 1_P) = (t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1} t(1_{P'}, 1_P; l).$$

Let

$$x = (s_0 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1} s_2 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) t(\alpha', \tilde{\alpha}, \alpha)(t(1_{P'}, 1_{\widetilde{P}}, 1_P))^{-1}(s_2 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1} s_1 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1).$$

Then we calculate

$$d_0 x = (t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1}(d_0 s_2 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1)) d_0 t(\alpha', \tilde{\alpha}, \alpha)(d_0 t(1_{P'}, 1_{\widetilde{P}}, 1_P))^{-1}(d_0 s_2 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1} t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1)$$

$$= 1 \in G(\mathfrak{s}.\mathcal{P}R)_2$$

and

$$d_1 x = (t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1}(d_1 s_2 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1)) d_1 t(\alpha', \tilde{\alpha}, \alpha)(d_1 t(1_{P'}, 1_{\widetilde{P}}, 1_P))^{-1}(d_1 s_2 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1))^{-1} t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) = 1.$$

Consider $t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) \in G(\mathfrak{s}.\mathcal{P}R)_2$: from earlier calculations we have

$$d_0 s_0 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) = d_1 s_0 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) = d_2 s_0 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) = t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1),$$

and

$$d_3 s_0 t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1) = (z_1')^{-1} \tilde{z}_1$$

where

$$z_1', \tilde{z}_1 \in G(\mathfrak{s}.\mathcal{P}R)_2$$

are those elements used to express $w_1(\alpha', \tilde{\alpha}; \tilde{l}_1) = z_1' u_1 v_1 (z_1')^{-1}$ and $w_1(\tilde{\alpha}, \alpha; \tilde{l}_2) = \tilde{z}_1 \tilde{u_1} \tilde{v_1} (\tilde{z}_1)^{-1}$ as developed in the proof of Theorem 2.1.4. Notice that

$$w_1(\alpha', \alpha; l) = z_1' t(\alpha', \alpha; l)(t(1_{P'}, 1_P; l))^{-1}(z_1')^{-1}$$

for this same z_1'. We use these to calculate

$$d_2 x = t(\tilde{\alpha}, \alpha; \tilde{l}_2)(t(1_{\widetilde{P}}, 1_P; \tilde{l}_2))^{-1}$$

and

$$d_3x = ((z_1')^{-1}\tilde{z}_1)^{-1}t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1)t(\alpha', \tilde{\alpha}; \tilde{l}_1)^{-1}t(\alpha', \alpha; l)(t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1)^{-1}t(1_{P'}, 1_P; l))^{-1}(z_1')^{-1}\tilde{z}_1$$

$$= \tilde{z}_1^{-1}(z_1't(\alpha', \tilde{\alpha}; \tilde{l}_1)t(1_{P'}, 1_{\widetilde{P}}; \tilde{l}_1)^{-1}(z_1')^{-1})^{-1}(z_1't(\alpha', \alpha; l)t(1_{P'}, 1_P; l)^{-1}(z_1')^{-1})\tilde{z}_1$$

$$= \tilde{z}_1^{-1}w_1(\alpha', \tilde{\alpha}; \tilde{l}_1)^{-1}w_1(\alpha', \alpha; l)\tilde{z}_1.$$

Now set $\tilde{u} = x(s_2d_2x)^{-1}$:

$$d_0\tilde{u} = d_0x(s_1d_1d_0x)^{-1} = 1 \in G(\mathfrak{s}.\mathcal{P}R)_2;$$

$$d_1\tilde{u} = d_1x(s_1d_1d_1x)^{-1} = 1;$$

$$d_2\tilde{u} = d_2x(d_2x)^{-1} = 1;$$

$$d_3\tilde{u} = d_3x(d_2x)^{-1} = \tilde{z}_1^{-1}w_1(\alpha', \tilde{\alpha}; \tilde{l}_1)^{-1}w_1(\alpha', \alpha; l)\tilde{z}_1(t(\tilde{\alpha}, \alpha; \tilde{l}_2)(t(1_{\widetilde{P}}, 1_P; \tilde{l}_2))^{-1})^{-1}.$$

Therefore $\tilde{u} \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_3$ and we have

$$\tilde{z}_1^{-1}w_1(\alpha', \tilde{\alpha}; \tilde{l}_1)^{-1}w_1(\alpha', \alpha; l)\tilde{z}_1(t(\tilde{\alpha}, \alpha; \tilde{l}_2)(t(1_{\widetilde{P}}, 1_P; \tilde{l}_2))^{-1})^{-1} \in B_2 \triangleleft G(\mathfrak{s}.\mathcal{P}R)_2.$$

Moving to equivalence classes in $G(\mathfrak{s}.\mathcal{P}R)_2/B_2$ we see

$$[\tilde{z}_1^{-1}w_1(\alpha', \tilde{\alpha}; \tilde{l}_1)^{-1}w_1(\alpha', \alpha; l)\tilde{z}_1(t(\tilde{\alpha}, \alpha; \tilde{l}_2)(t(1_{\widetilde{P}}, 1_P; \tilde{l}_2))^{-1})^{-1}] = 1$$

so that

$$[w_1(\alpha', \tilde{\alpha}; \tilde{l}_1)^{-1}w_1(\alpha', \alpha; l)] = [\tilde{z}_1t(\tilde{\alpha}, \alpha; \tilde{l}_2)(t(1_{\widetilde{P}}, 1_P; \tilde{l}_2))^{-1}\tilde{z}_1^{-1}] = [w_1(\tilde{\alpha}, \alpha; \tilde{l}_2)].$$

Thus $[C_1^{-1}C] = [C_2]$ in $G(\mathfrak{s}.\mathcal{P}R)_2/B_2$. Calculating in the quotient group $G(\mathfrak{s}.\mathcal{P}R)_2/B_2$, it follows that $[C_1]^{-1}[C] = [C_2]$, so that $[C] = [C_1][C_2] = [C_1C_2]$.

□

As a corollary to the above lemma, we have one our main theorems of this section. We use already established notation and definitions.

Theorem 2.3.4 *Suppose that $\tilde{F}(P, \alpha)$ is an admissible refinement of the admissible filtration $F(P, \alpha)$. Then, given the elements $X_1(F(P, \alpha))$ and $X_1(\tilde{F}(P, \alpha)) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$,*

$$[X_1(F(P, \alpha))] = [X_1(\tilde{F}(P, \alpha))] \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2/B_2.$$

Proof: By Lemma 2.3.3, if the admissible filtration $\tilde{F}(P, \alpha)$ is obtained by inserting one projective module into the admissible filtration $F(P, \alpha)$, then in $\overline{G(s.\mathcal{P}F)}_2/B_2$,

$$[X_1(F(P, \alpha)] = [ACB],$$

with $A, C, B \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$ as defined immediately before the Lemma 2.3.3.

But, computing in the quotient group $\overline{G(s.\mathcal{P}F)}_2/B_2$, and using the lemma,

$$[ACB] = [A][C][B] = [A][C_1C_2][B] = [AC_1C_2B] = [X_1(\tilde{F}(P, \alpha))],$$

where C_1, C_2 are as defined immediately before the lemma.

By induction on the number of insertions to the original filtration F, we obtain the theorem.

□

2.4 Standard Filtrations

We now specialize to the case $P = R^N$ and $\alpha \in GL(N, R)$, $N \in \mathbb{N}$. We fix the standard basis $\beta_N = \{e_1, \ldots, e_N\}$ for R^N. If $I \subseteq \beta_N$, then $F(I)$ denotes the R-submodule of R^N spanned by the elements of I. We define $F(\emptyset) = \{0\}$. Note that $F(I)$ is always a free R-module on the set I. Moreover, if $I \subseteq J$, then the quotient module $F(J)/F(I)$ is a free R-module on the set $J - I$.

Definition 2.4.1 *Suppose*

$$I : \emptyset \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_S = \beta_N$$

is a chain of subsets of β_N. The ***standard filtration of*** *R^N defined by I is the filtration below, denoted by $F(I)$:*

$$0 \subseteq F(I_1) \subseteq F(I_2) \subseteq \cdots \subseteq F(I_S) = R^N.$$

By definition of these free R-modules, the inclusions in standard filtrations are always admissible. We do not require the inclusions to be strict.

Theorem 2.4.2 *Suppose $F_1 = F(I)$ and $F_2 = F(J)$ are admissible, standard filtrations of (R^N, α), corresponding to chains I and J of subsets of β_N. Then $[X_1(F_1(R^N, \alpha))] = [X_1(F_2(R^N, \alpha))]$ in $\overline{G(\mathfrak{s}.\mathcal{P}R)}_2/B_2$.*

Proof: Assume without loss of generality that I and J as in the hypothesis have the same number, S, of terms in their chains, where we append the empty set to the beginning of the shorter chain as necessary. Indeed, as in Remark 2.1.5, the element that results from Lemma 2.2.2 for this appended chain would be

$1 \in G(\mathfrak{s}.\mathcal{P}R)$ multiplied (finitely many times) by the element corresponding to the shorter chain, hence it would be exactly the same element. Let $\widetilde{F}$ be the filtration $\widetilde{F} = F(H)$ corresponding to the chain

$$H : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_S \cup J_S$$

of subsets of β_N (i.e. $H_0 = \emptyset$ and $H_i = I_i \cup J_i \ \forall\ 1 \leq i \leq S$).

$H_1 = I_1 \cup J_1$, and by assumption (Definition 2.2.1) we have $\alpha|_{F(I_1)} = 1_{F(I_1)}$ and $\alpha|_{F(J_1)} = 1_{F(J_1)}$. Thus

$$\alpha|_{F(I_1)+F(J_1)} = 1_{F(I_1)+F(J_1)},$$

and since $F(I_1) + F(J_1) = F(I_1 \cup J_1)$ for these free modules, it follows that $\alpha|_{F(H_1)} = 1_{F(H_1)}$. Again, $F(I_i) + F(J_i) = F(I_i \cup J_i) = F(H_i) \ \forall\ 1 \leq i \leq S$, and by definition we know $\alpha|_{F(I_i)} \in Aut(F(I_i))$ and $\alpha|_{F(J_i)} \in Aut(F(J_i))$, so $\alpha|_{F(H_i)} \in Aut(F(H_i))$.

By construction

$$H_i - H_{i-1} = (I_i \cup J_i) - (I_{i-1} \cup J_{i-1}).$$

We consider cosets $e + F(H_{i-1})$ in $F(H_i)/F(H_{i-1})$ for $e \in H_i - H_{i-1}$. If $e \in I_i$ then $e \notin I_{i-1}$, (since $e \notin I_{i-1} \cup J_{i-1}$) so that by definition of $F(I)$ as an admissible filtration of (R^N, α) we have $\alpha(e) = e + f$ where $f \in F(I_{i-1})$. But then $f \in F(I_{i-1}) + F(J_{i-1}) = F(H_{i-1})$, hence $\alpha(e) + F(H_{i-1}) = e + F(H_{i-1})$. Similarly, if $e \in J_i$ then $e \notin J_{i-1}$, so by definition of $F(J)$ we know $\alpha(e) + F(J_{i-1}) = e + F(J_{i-1})$, hence $\alpha(e) + F(H_{i-1}) = e + F(H_{i-1})$. We conclude that $\alpha|_{F(H_i)}$ induces the identity on the quotient module $F(H_i)/F(H_{i-1})$. Therefore $\widetilde{F} = F(H)$ is a standard, admissible filtration of (R^N, α), constructed from $F_1(R^N, \alpha)$ and $F_2(R^N, \alpha)$.

Now define a sequence of standard, admissible filtrations $F_{a,b} = F(H_{a,b})$ of (R^N, α), and similarly $F'_{a,b} = F(H'_{a,b})$, for each $1 \leq a \leq S-1$ and each $1 \leq b \leq a$, corresponding to chains

$$H_{a,b} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a} \cup J_{S-a-1} \subseteq \cdots \subseteq I_{S-a+b-1} \cup J_{S-a-1}$$

$$\subseteq I_{S-a+b-1} \cup J_{S-a} \subseteq I_{S-a+b} \cup J_{S-a} \subseteq I_{S-a+b+1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-1} \cup J_{S-a} \subseteq I_S \cup J_S (= \beta_N = I_S)$$

for $F_{a,b}$, and

$$H'_{a,b} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a} \cup J_{S-a-1} \subseteq \cdots \subseteq I_{S-a+b-1} \cup J_{S-a-1}$$

$$\subseteq I_{S-a+b} \cup J_{S-a} \subseteq I_{S-a+b+1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-1} \cup J_{S-a} \subseteq I_S$$

for $F'_{a,b}$.

It is a straightforward exercise to verify that the resulting filtrations are admissible, and since the chain $H'_{a,b}$ can be obtained from $H_{a,b}$ by deleting the term $I_{S-a+b-1} \cup J_{S-a}$, each filtration $F_{a,b}$ is a refinement of the corresponding $F'_{a,b}$. But we also have

$$H_{a,b+1} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a} \cup J_{S-a-1} \subseteq \cdots \subseteq I_{S-a+b-1} \cup J_{S-a-1}$$

$$\subseteq I_{S-a+b} \cup J_{S-a-1} \subseteq I_{S-a+b} \cup J_{S-a} \subseteq I_{S-a+b+1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-1} \cup J_{S-a} \subseteq I_S,$$

so that $H'_{a,b}$ can be obtained from $H_{a,b+1}$ by deleting the term $I_{S-a+b} \cup J_{S-a-1}$. Therefore $F_{a,b+1}$ is a refinement for $F'_{a,b}$.

Notice in particular that for each $1 \leq a \leq S-1$ we have

$$H_{a,a} : \emptyset \subseteq I_1 \cup J_1 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a} \cup J_{S-a-1} \subseteq \cdots \subseteq I_{S-1} \cup J_{S-a-1} \subseteq I_{S-1} \cup J_{S-a} \subseteq I_S$$

with

$$H'_{a,a} : \emptyset \subseteq I_1 \cup J_1 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a} \cup J_{S-a-1} \subseteq \cdots \subseteq I_{S-1} \cup J_{S-a-1} \subseteq I_S,$$

and

$$H_{a+1,1} : \emptyset \subseteq I_1 \cup J_1 \subseteq \cdots \subseteq I_{S-(a+1)-1} \cup J_{S-(a+1)-1} \subseteq I_{S-a-1} \cup J_{S-a-2} \subseteq I_{S-a-1} \cup J_{S-a-1}$$

$$\subseteq I_{S-a} \cup J_{S-a-1} \subseteq I_{S-a+1} \cup J_{S-a-1} \subseteq \cdots \subseteq I_{S-1} \cup J_{S-a-1} \subseteq I_S.$$

Thus $H'_{a,a}$ can be obtained from $H_{a+1,1}$ by deleting the term $I_{S-a-1} \cup J_{S-a-2}$, in which case $F_{a+1,1}$ is a refinement of $F'_{a,a}$. Furthermore,

$$H_{1,1} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots I_{S-2} \cup J_{S-2} \subseteq I_{S-1} \cup J_{S-2} \subseteq I_{S-1} \cup J_{S-1} \subseteq I_S$$

so that deleting the term $I_{S-1} \cup J_{S-2}$ from $H_{1,1}$ gives H. Therefore $F_{1,1}$ is a refinement of $\widetilde{F}$.

Now using $J_0 := \emptyset$ we calculate

$$H_{S-1,b} : \emptyset \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_b \subseteq I_b \cup J_1 \subseteq I_{b+1} \cup J_1 \subseteq \cdots \subseteq I_{S-1} \cup J_1 \subseteq I_S \;\forall\; 1 \leq b \leq S-1,$$

so that

$$H_{S-1,S-1} : \emptyset \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{S-1} \subseteq I_{S-1} \cup J_1 \subseteq I_S.$$

Therefore

$$H'_{S-1,S-1} : \emptyset \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{S-1} \subseteq I_S,$$

in which case $F_1 = F'_{S-1,S-1}$ as admissible filtrations of (R^N, α). Now by Theorem 2.3.4 we have

$$[X_1(\widetilde{F}(R^N, \alpha))] = [X_1(F_{1,1}(R^N, \alpha))],$$

$$[X_1(F'_{a,b}(R^N, \alpha))] = [X_1(F_{a,b}(R^N, \alpha))] \; \forall \; 1 \leq a \leq S-1, 1 \leq b \leq a,$$

$$[X_1(F'_{a,b}(R^N, \alpha))] = [X_1(F_{a,b+1}(R^N, \alpha))] \; \forall \; 1 \leq a \leq S-1, 1 \leq b \leq a,$$

$$[X_1(F'_{a,a}(R^N, \alpha))] = [X_1(F_{a+1,1}(R^N, \alpha))] \; \forall \; 1 \leq a \leq S-1,$$

and

$$[X_1(F_1(R^N, \alpha))] = [X_1(F'_{S-1,S-1}(R^N, \alpha))].$$

These equalities imply (by transitivity of the equivalence relation) that $[X_1(\widetilde{F}(R^N, \alpha))] = [X_1(F_1(R^N, \alpha))]$ in $\overline{G(\mathfrak{s}.\mathcal{P}R)_2}/B_2$.

Going back to the chain H, we make a symmetric argument by defining chains $K_{a,b}$ with corresponding filtrations $\bar{F}_{a,b}$ in a different way, switching the roles of I and J :

$$K_{a,b} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a-1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a+b-1}$$

$$\subseteq I_{S-a} \cup J_{S-a+b-1} \subseteq I_{S-a} \cup J_{S-a+b} \subseteq I_{S-a} \cup J_{S-a+b+1} \subseteq \cdots \subseteq I_{S-a} \cup J_{S-1} \subseteq I_S \cup J_S (= \beta_N = J_S),$$

with corresponding

$$K'_{a,b} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a-1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a+b-1}$$

$$\subseteq I_{S-a} \cup J_{S-a+b} \subseteq I_{S-a} \cup J_{S-a+b+1} \subseteq \cdots \subseteq I_{S-a} \cup J_{S-1} \subseteq J_S$$

(with corresponding filtrations $\bar{F'}_{a,b}$), which can be obtained from $K_{a,b}$ by deleting the term $I_{S-a} \cup J_{S-a+b-1}$. Also,

$$K_{a,b+1} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a-1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a+b-1}$$

$$\subseteq I_{S-a-1} \cup J_{S-a+b} \subseteq I_{S-a} \cup J_{S-a+b} \subseteq I_{S-a} \cup J_{S-a+b+1} \subseteq \cdots \subseteq I_{S-a} \cup J_{S-1} \subseteq J_S,$$

so that $K'_{a,b}$ can be obtained from $K_{a,b+1}$ by deleting the term $I_{S-a-1} \cup J_{S-a+b}$. Then

$$K_{a,a} : \emptyset \subseteq I_1 \cup J_1 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a-1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-1} \subseteq I_{S-a} \cup J_{S-1} \subseteq J_S$$

gives

$$K'_{a,a} : \emptyset \subseteq I_1 \cup J_1 \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-a-1} \subseteq I_{S-a-1} \cup J_{S-a} \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-1} \subseteq J_S,$$

and

$$K_{a+1,1} : \emptyset \subseteq I_1 \cup J_1 \subseteq \cdots \subseteq I_{S-(a+1)-1} \cup J_{S-(a+1)-1} \subseteq I_{S-a-2} \cup J_{S-a-1} \subseteq I_{S-a-1} \cup J_{S-a-1}$$

$$\subseteq I_{S-a-1} \cup J_{S-a} \subseteq I_{S-a-1} \cup J_{S-a+1} \subseteq \cdots \subseteq I_{S-a-1} \cup J_{S-1} \subseteq J_S.$$

Thus $K'_{a,a}$ can be obtained from $K_{a+1,1}$ by deleting the term $I_{S-a-2} \cup J_{S-a-1}$, in which case $\bar{F}_{a+1,1}$ is a refinement of $\bar{F}'_{a,a}$. Furthermore,

$$K_{1,1} : \emptyset \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2 \subseteq \cdots I_{S-2} \cup J_{S-2} \subseteq I_{S-2} \cup J_{S-1} \subseteq I_{S-1} \cup J_{S-1} \subseteq J_S$$

so that deleting the term $I_{S-2} \cup J_{S-1}$ from $K_{1,1}$ gives H. Therefore $\bar{F}_{1,1}$ is a refinement of $\widetilde{F}$.

Finally, we calculate

$$K'_{S-1,S-1} : \emptyset \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_{S-1} \subseteq J_S,$$

so that $\bar{F}'_{S-1,S-1} = F_2$. Therefore we conclude that $[X_1(\widetilde{F}(R^N, \alpha))] = [X_1(F_2(R^N, \alpha))]$ in $\overline{G(\mathfrak{s}.\mathcal{P}R)}_2/B_2$ just as we did with the $H_{a,b}$ for F_1. It follows by transitivity of the equivalence relation that $[X_1(F_1(R^N, \alpha))] = [X_1(F_2(R^N, \alpha))]$ in $\overline{G(\mathfrak{s}.\mathcal{P}R)}_2/B_2$.

□

3 Homotopy Fibers

3.1 Definitions

We now cite a standard construction used in simplicial homotopy theory that will act as a central figure in our isomorphism, as it will allow us to represent the Steinberg Relations of K-Theory completely in terms of simplicial homotopy theory. Work such as that of [15] provides a good review.

Definition 3.1.1 *Given a simplicial group G, define a simplicial group G^I by m-simplices*

$$G^I_m = \{(g_0, g_1, \dots, g_m) \in G_{m+1} \times G_{m+1} \times \dots \times G_{m+1} \mid d_i g_i = d_i g_{i-1} \ \forall\ 1 \leq i \leq m\}$$

(and componentwise multiplication as the operation) with face maps defined on

$$\boldsymbol{g} = (g_0, g_1, \dots, g_m)$$

by

$$d_j \boldsymbol{g} = (d_{j+1} g_0, d_{j+1} g_1, \dots, d_{j+1} g_{j-1}, d_j g_{j+1}, \dots, d_j g_m)$$

and degeneracy maps defined by

$$s_j \boldsymbol{g} = (s_{j+1} g_0, s_{j+1} g_1, \dots, s_{j+1} g_j, s_j g_j, s_j g_{j+1}, \dots, s_j g_m),$$

$\forall\ 0 \leq j \leq m$.

Note in particular

$$d_0 \boldsymbol{g} = (d_0 g_1, d_0 g_2, \dots, d_0 g_m)$$

and

$$d_m \boldsymbol{g} = (d_{m+1} g_0, d_{m+1} g_1, \dots, d_{m+1} g_{m-1}).$$

The following construction is well known; we cite [15] as a reference.

Lemma 3.1.2 *Let G be a simplicial group.*

a) $\exists$ homomorphisms of simplicial groups, that are also Kan Fibrations, $\partial_0, \partial_1 : G^I \to G$, given on m-simplices $\boldsymbol{g} = (g_0, g_1, \dots, g_m) \in G^I_m$ by

$$\partial_0(\boldsymbol{g}) = d_0 g_0$$

and

$$\partial_1(\boldsymbol{g}) = d_{m+1} g_m.$$

b) The diagonal simplicial map $D : G \to G \times G$ factors as

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, dashed, "D"'] & G^I \arrow[d, "(\partial_0,\partial_1)"] \\ & G \times G \end{tikzcd}$$

where ρ is a homotopy inverse for both ∂_0 and ∂_1, defined on m-simplices $g \in G$, by

$$\rho(g) = (s_0 g, s_1 g, \ldots, s_m g).$$

To be more precise,

$$\partial_i \circ \rho = id_G; i = 0, 1$$

and

$$\rho \circ \partial_i \text{ is homotopic as a simplicial map to } id_{G^I}; i = 0, 1.$$

We in fact have the stronger result that the map

$$(\partial_0, \partial_1) : G^I \to G \times G,$$

through which the diagonal map factors, is a Kan Fibration.

Definition 3.1.3 *Given $P \in \mathcal{P}R$ and simplicial group $G = G(\mathfrak{s}.\mathcal{P}R)$, denote the pullback of the diagram*

$$\begin{tikzcd} & N(Aut(P)) \arrow[d, "\mathfrak{i}_P"] \\ G^I \arrow[r, "\partial_1"] & G \end{tikzcd} .$$

by $\mathfrak{I}_P$. That is, $\mathfrak{I}_P$ has m-simplices

$$\mathfrak{I}_{P,m} = \left\{ (\boldsymbol{\alpha}, \boldsymbol{g}) \in N(Aut(P))_m \times G^I_m \mid \boldsymbol{g} = (g_0, \ldots, g_m) \text{ has } d_{m+1} g_m = \mathfrak{i}_P(\boldsymbol{\alpha}) \right\}$$

with face maps defined by $d_j(\boldsymbol{\alpha}, \boldsymbol{g}) = (d_j \boldsymbol{\alpha}, d_j \boldsymbol{g})$ and degeneracy maps s_j defined similarly.

We have the following well-known lemma as well, again citing [15]:

Lemma 3.1.4 *The simplicial sets $\mathfrak{I}_P$ and $N(Aut(P))$ are homotopy equivalent.*

Proof: Define

$$\lambda_P : N(Aut(P)) \to \mathfrak{I}_P$$

by

$$\lambda_P(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}, \rho(\mathfrak{i}_P(\boldsymbol{\alpha}))).$$

By Lemma 2.1.1 we know that $\mathfrak{i}_P$ is a simplicial map, and by Theorem 3.1.2 we know ρ is a simplicial map, so the composite $\rho \circ \mathfrak{i}_P$ is a simplicial map. Since the identity is always a simplicial map, the definitions of the face and degeneracy maps on $\mathfrak{I}_P$ imply that λ_P must be a simplicial map.

Define

$$\partial_P^* : \mathfrak{I}_P \to N(Aut(P))$$

by

$$\partial_P^*(\boldsymbol{\alpha}, \boldsymbol{g}) = \boldsymbol{\alpha}.$$

This is the projection from $N(Aut(P)) \times G^I$ to $N(Aut(P))$, so that it preserves images of face and degeneracy maps. Therefore ∂_P^* is a simplicial map.

We see that ∂_P^* is a fibration, since it is the pullback of the fibration ∂_1,

$$\partial_P^* \circ \lambda_P = id_{N(Aut(P))},$$

and we cite [15] (but omit proof) that

$$\lambda \circ \partial_P^* \text{ is homotopic as a simplicial map to } id_{\mathfrak{I}_P}.$$

By Definition 1.4.3 of Chapter 1, it follows that $N(Aut(P))$ and $\mathfrak{I}_P$ are homotopy equivalent.

□

Now define a simplicial map $p_P : \mathfrak{I}_P \to G(\mathfrak{s}.\mathcal{P}R)$ on m-simplices

$$(\boldsymbol{\alpha}, \boldsymbol{g}) = (\alpha_1 | \cdots | \alpha_m; g_0, \ldots, g_m)$$

by $p_P(\boldsymbol{\alpha}, \boldsymbol{g}) = d_0 g_0 = \partial_0 \circ \pi_2(\boldsymbol{\alpha}, \boldsymbol{g})$ (π_2 the usual projection from $N(Aut(P)) \times G(\mathfrak{s}.\mathcal{P}R)^I$ to $G(\mathfrak{s}.\mathcal{P}R)^I$).

We have

Lemma 3.1.5 *The map p_P is a fibration of simplicial sets.*

Definition 3.1.6 *The* ***homotopy fiber of*** $\mathfrak{i}_P$ *is the simplicial subcomplex*

$$Y_P \subseteq \mathfrak{I}_P \subseteq N(Aut(P)) \times G(\mathfrak{s}.\mathcal{P}R)^I$$

whose m-simplices are

$$Y_{P,m} = p_P^{-1}(1)_m = \{(\boldsymbol{\alpha}, \boldsymbol{g}) \in \mathfrak{I}_{P,m} \mid p_P(\boldsymbol{\alpha}, \boldsymbol{g}) = 1 \in G(\mathfrak{s}.\mathcal{P}R)_m\}$$

$$= \{(\alpha_1|\cdots|\alpha_m; g_0, \ldots, g_m) \mid d_0 g_0 = 1 \in G(\mathfrak{s}.\mathcal{P}R)_m\}.$$

Note that face and degeneracy maps are given by those corresponding to the Cartesian Product (recall Definition 3.3.1 from Chapter 1).

Lemma 3.1.7 *Y_P is a Kan Complex.*

Proof: The map p_P is a fibration, thus using Proposition 7.3 of May (i.e. Proposition 3.2.1 of Chapter 1), Y_P is a Kan complex.

□

This fact allows us to use the canonical construction of $\pi_1(Y_P, \Phi)$ (with $\phi_0 = (*, 1) \in Y_{P,0}$) in conjunction with the chain complex construction of $\pi_2(G(\mathfrak{s}.\mathcal{P}R))$. The result will allow us to see the Steinberg relations from the point of view of homotopy classes in this homotopy fiber.

4 Steinberg Relations in $\pi_1(Y_N(R))$

Here we focus on $P = R^N \in \mathcal{P}R$ and denote $\mathfrak{I}_N := \mathfrak{I}_{R^N}$, $\mathfrak{i}_N := \mathfrak{i}_{R^N}$ and $Y_N := Y_{R^N}$; with analogous definitions for the fibration $p_N : \mathfrak{I}_N \to N(GL(N,R))$ and the homotopy equivalence $\lambda_N : N(GL(N,R)) \to \mathfrak{I}_N$. We want to use Y_N as an analog for $St(N,R)$, even so far as to have a direct limit $Y(R)$ analogous to $St(R)$ and $GL(R)$. To that end, we first consider the concept of stabilization of the simplicial sets $Y_N, N \in \mathbb{N}$.

4.1 Stability of $Y_N \overset{\subseteq}{\to} \mathfrak{I}_N \overset{p_N}{\to} G(\mathfrak{s}.\mathcal{P}R)$

We want to see what happens as a result of the embedding $R^N \hookrightarrow R^{N+1}$ with $R^{N+1} = R^N \oplus R$. Given $m > 0, \boldsymbol{\alpha} = (\alpha_1|\alpha_2|\cdots|\alpha_m) \in N(GL(N,R))_m$, we somewhat ambiguously denote $\mathbf{1} = (1_R|1_R|\cdots|1_R)$ (and realize $1_{N+1} := 1_N \oplus 1_R$) and define $\boldsymbol{\alpha} \oplus \mathbf{1} \in N(GL(N+1,R))_m$ by

$$\boldsymbol{\alpha} \oplus \mathbf{1} = (\alpha_1 \oplus 1_R|\alpha_2 \oplus 1_R|\cdots|\alpha_m \oplus 1_R).$$

We will consider "stability maps," all denoted σ, between the various simplicial sets associated to R^N and R^{N+1}. For instance,

Lemma 4.1.1 *$\sigma : N(GL(N,R)) \to N(GL(N+1,R))$ defined by $\sigma(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \oplus \mathbf{1}$ is a simplicial map.*

Proof: We calculate

$$d_0\sigma_m(\boldsymbol{\alpha}) = (\alpha_2 \oplus 1_R|\alpha_3 \oplus 1_R|\cdots|\alpha_m \oplus 1_R) = (\alpha_2|\alpha_3|\cdots|\alpha_m) \oplus \mathbf{1} = (d_0\boldsymbol{\alpha}) \oplus \mathbf{1} = \sigma_{m-1}(d_0\boldsymbol{\alpha}),$$

$$d_m\sigma_m(\boldsymbol{\alpha}) = (\alpha_1 \oplus 1_R|\alpha_2 \oplus 1_R|\cdots|\alpha_{m-1} \oplus 1_R) = (\alpha_1|\alpha_2|\cdots|\alpha_{m-1}) \oplus \mathbf{1} = (d_m\boldsymbol{\alpha}) \oplus \mathbf{1} = \sigma_{m-1}(d_m\boldsymbol{\alpha}),$$

and for each $1 \le i \le m-1$,

$$d_i\sigma_m(\boldsymbol{\alpha}) = (\alpha_1 \oplus 1_R|\alpha_2 \oplus 1_R|\cdots|(\alpha_{i+1} \oplus 1_R) \circ (\alpha_i \oplus 1_R)|\alpha_{i+2} \oplus 1_R|\cdots|\alpha_m \oplus 1_R)$$

$$= (\alpha_1 \oplus 1_R|\alpha_2 \oplus 1_R|\cdots|(\alpha_{i+1} \circ \alpha_i) \oplus 1_R|\alpha_{i+2} \oplus 1_R|\cdots|\alpha_m \oplus 1_R)$$

$$= (\alpha_1|\alpha_2|\cdots|\alpha_{i+1} \circ \alpha_i|\alpha_{i+2}|\cdots|\alpha_m) \oplus \mathbf{1} = (d_i\boldsymbol{\alpha}) \oplus \mathbf{1} = \sigma_{m-1}(d_i\boldsymbol{\alpha}).$$

Also, for $0 \le j \le m$,

$$s_j\sigma_m(\boldsymbol{\alpha}) = (\alpha_1 \oplus 1_R|\alpha_2 \oplus 1_R|\cdots|\alpha_j \oplus 1_R|1_{N+1}|\alpha_{j+1} \oplus 1_R|\cdots|\alpha_m \oplus 1_R)$$

$$= (\alpha_1|\alpha_2|\cdots|\alpha_j \oplus 1_R|1_N \oplus 1_R|\alpha_{j+1} \oplus 1_R|\cdots|\alpha_m \oplus 1_R)$$

$$= (\alpha_1|\alpha_2|\cdots|\alpha_j|1_N|\alpha_{j+1}|\cdots|\alpha_m) \oplus \mathbf{1} = (s_j\boldsymbol{\alpha}) \oplus \mathbf{1} = \sigma_{m+1}(s_j\boldsymbol{\alpha}).$$

Therefore σ is a simplicial map by definition.

□

For the short exact sequence $l : R^N \hookrightarrow R^{N+1} \twoheadrightarrow R$, where $R^N \subseteq R^{N+1}$ is the embedding $x \mapsto (x, 0)$, we see that there are short exact sequences of pairs $(R^N, \alpha_i) \to (R^{N+1}, \alpha_i \oplus 1_R) \to (R, 1_R)$ for each $1 \le i \le m$, in which case we have $w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) \in G(\mathfrak{s}.\mathcal{P}R)_{m+1}$ by Theorem 2.1.4.

Let $\boldsymbol{\xi} = (\boldsymbol{\alpha}; \boldsymbol{g}) = (\alpha_1|\cdots|\alpha_m; g_0, \ldots, g_m) \in \mathfrak{I}_{N,m}$ and define $\boldsymbol{\xi} \oplus \mathbf{1} = (\boldsymbol{\alpha} \oplus \mathbf{1}; \tilde{\boldsymbol{g}})$ where $\tilde{\boldsymbol{g}} = (\tilde{g}_0, \ldots, \tilde{g}_m)$ is defined by

$$\tilde{g}_i = g_i(s_m s_{m-1} \cdots s_{i+1} d_{i+1} d_{i+2} \cdots d_{m-1} d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

for $0 \le i \le m-1$, and

$$\tilde{g}_m = g_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l).$$

Proposition 4.1.2 *Given $\boldsymbol{\xi} = (\boldsymbol{\alpha}; \boldsymbol{g}) \in \mathfrak{I}_N$ as above,*

1) $\boldsymbol{\xi} \oplus \mathbf{1} \in \mathfrak{I}_{N+1,m}$.

2) The map $\sigma : \mathfrak{I}_N \to \mathfrak{I}_{N+1}$ defined by $\sigma(\boldsymbol{\xi}) = \boldsymbol{\xi} \oplus \mathbf{1}$ is a simplicial map.

3) $\sigma|_{Y_N} : Y_N \to Y_{N+1}$.

4) The following diagrams commute:

$$\begin{tikzcd}
Y_N \arrow[r, tail, "\subseteq"] \arrow[d, "\sigma"] & \mathfrak{I}_N \arrow[r, "p_N"] \arrow[d, "\sigma"] & G(\mathfrak{s}.\mathcal{P}R) \arrow[d, "="] \\
Y_{N+1} \arrow[r, tail, "\subseteq"] & \mathfrak{I}_{N+1} \arrow[r, "p_{N+1}"] & G(\mathfrak{s}.\mathcal{P}R)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{I}_N \arrow[r, "\partial_N^*"] \arrow[d, "\sigma"] & N(GL(N,R)) \arrow[d, "\sigma"] \\
\mathfrak{I}_{N+1} \arrow[r, "\partial_{N+1}^*"] & N(GL(N+1,R))
\end{tikzcd} .$$

5) The diagram below commutes "up to homotopy"; i.e., $\lambda_{N+1} \circ \sigma$ is homotopic as a simplicial map to $\sigma \circ \lambda_N$:

$$\begin{tikzcd}
N(GL(N,R)) \arrow[r, "\lambda_N"] \arrow[d, "\sigma"] & \mathfrak{I}_N \arrow[d, "\sigma"] \\
N(GL(N+1,R)) \arrow[r, "\lambda_{N+1}"] & \mathfrak{I}_{N+1}
\end{tikzcd} .$$

Proof: We first calculate

$$d_i\tilde{g}_i = (d_i g_i)(d_i s_m s_{m-1} \cdots s_{i+1} d_{i+1} d_{i+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= (d_i g_i)(s_{m-1} s_{m-2} \cdots s_i d_i d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= (d_i g_i)(s_{m-1} s_{m-2} \cdots s_i d_i s_i d_i d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= (d_i g_i)(d_i s_m s_{m-1} \cdots s_{i+1} s_i d_i d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= (d_i g_{i-1})(d_i s_m s_{m-1} \cdots s_i d_i d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= d_i(g_{i-1} s_m s_{m-1} \cdots s_i d_i d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1} : l))$$

$$= d_i\tilde{g}_{i-1}$$

for each $1 \leq i \leq q-1$, and since $\tilde{g}_{m-1} = g_{m-1}s_m d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$ we have

$$d_m \tilde{g}_m = (d_m g_m)(d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= (d_m g_m)(d_m s_m d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = (d_m g_{m-1})(d_m s_m d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l))$$

$$= d_m(g_{m-1} s_m d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = d_m \tilde{g}_{m-1}.$$

Therefore $\tilde{\boldsymbol{g}} \in G(\mathfrak{s}.\mathcal{P}R)^I_m$. Also, $\boldsymbol{g} \in G(\mathfrak{s}.\mathcal{P}R)^I_m$ so that by Theorem 2.1.4 we see

$$d_{m+1}\tilde{g}_m = (d_{m+1}g_m)(d_{m+1}w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = \mathfrak{i}_N(\boldsymbol{\alpha})(\mathfrak{i}_N(\boldsymbol{\alpha}))^{-1}\mathfrak{i}_{N+1}(\boldsymbol{\alpha} \oplus \mathbf{1}) = \mathfrak{i}_{N+1}(\boldsymbol{\alpha} \oplus \mathbf{1}).$$

(1) now follows by definition of $\mathfrak{I}_{N+1}$.

Now given $1 \leq i \leq m-1$, we use superscripts again for $\boldsymbol{g} = (g^{(0)}, \ldots, g^{(m)}) \in G(\mathfrak{s}.\mathcal{P}R)^I_m$ and denote the images of face maps by

$$d_i\boldsymbol{g} = (d_{i+1}g^{(0)}, \ldots, d_{i+1}g^{(i-1)}, d_i g^{(i+1)}, \ldots, d_i g^{(m)}) = (g_i^{(0)}, \ldots, g_i^{(i-1)}, g_i^{(i)}, \ldots, g_i^{(m-1)}) := \boldsymbol{g}_i.$$

Now we have $\tilde{\boldsymbol{g}}_i$ defined by

$$\tilde{g}_i^{(j)} = g_i^{(j)} s_{m-1} \cdots s_{j+1} d_{j+1} \cdots d_{m-1} w_{m-1}(d_i\boldsymbol{\alpha}, d_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

for $0 \leq i \leq m$ and $0 \leq j \leq m-2$, and

$$\tilde{g}_i^{(m-1)} = g_i^{(m-1)} w_{m-1}(d_i\boldsymbol{\alpha}, d_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

for each $0 \leq i \leq m$. We must show

$$\sigma_{m-1}(d_i\boldsymbol{\xi}) = (d_i\boldsymbol{\alpha} \oplus \mathbf{1}; \tilde{\boldsymbol{g}}_i) = (d_i(\boldsymbol{\alpha} \oplus \mathbf{1}); d_i\tilde{\boldsymbol{g}}) = d_i\sigma_m(\boldsymbol{\xi}).$$

We can see that $d_i(\boldsymbol{\alpha} \oplus \mathbf{1}) = (d_i\boldsymbol{\alpha}) \oplus \mathbf{1} \in N(GL(N+1,R))_{m-1}$ for each $0 \leq i \leq m$ by direct calculation. Note that

$$g_i^{(j)} = \begin{cases} d_{i+1}g^{(j)}, & 0 \leq j \leq i-1 \\ \\ d_i g^{(j+1)}, & i \leq j \leq m-1 \end{cases}$$

when $1 \le i \le m-1$,

$$g_0^{(j)} = d_0 g^{(j+1)}, 0 \le j \le m-1,$$

and

$$g_m^{(j)} = d_{m+1} g^{(j)}, 0 \le j \le m-1.$$

Now if $1 \le i \le m-1$ and $0 \le j \le i-1$ we see

$$\tilde{g}_i^{(j)} = d_{i+1} g^{(j)} s_{m-1} \cdots s_{j+1} d_{j+1} \cdots d_{m-1} w_{m-1}(d_i \boldsymbol{\alpha}, d_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

$$= d_{i+1} g^{(j)} s_{m-1} \cdots s_{i+1} s_i \cdots s_{j+1} d_{j+1} \cdots d_{i-1} d_i \cdots d_{m-1} d_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_{i+1} g^{(j)} s_{m-1} \cdots s_{i+1} s_i \cdots s_{j+1} d_{j+1} \cdots d_{i-1} d_i d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_{i+1} g^{(j)} s_{m-1} \cdots s_{i+1} d_{i+1} s_{i+1} s_i \cdots s_{j+1} d_{j+1} \cdots d_{m-1} w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_{i+1} g^{(j)} d_{i+1} s_m \cdots s_{i+2} s_{i+1} s_i \cdots s_{j+1} d_{j+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_{i+1}(g^{(j)} s_m \cdots s_{j+1} d_{j+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = d_{i+1} \tilde{g}^{(j)}.$$

If $1 \le i \le j \le m-2$ then

$$\tilde{g}_i^{(j)} = d_i g^{(j+1)} s_{m-1} \cdots s_{j+1} d_{j+1} \cdots d_{m-1} d_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_i g^{(j+1)} s_{m-1} \cdots s_{j+1} d_i d_{j+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_i g^{(j+1)} d_i s_m \cdots s_{j+2} d_{j+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_i(g^{(j+1)} s_m \cdots s_{j+2} d_{j+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = d_i \tilde{g}^{(j+1)}.$$

Furthermore,

$$\tilde{g}_0^{(j)} = d_0 g^{(j+1)} s_{m-1} \cdots s_{j+1} d_{j+1} \cdots d_{m-1} d_0 w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_0 g^{(j+1)} s_{m-1} \cdots s_{j+1} d_0 d_{j+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_0 g^{(j+1)} d_0 s_m \cdots s_{j+2} d_{j+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_0(g^{(j+1)} s_m \cdots s_{j+2} d_{j+2} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = d_0 \tilde{g}^{(j+1)}$$

for each $0 \le j \le m-2$ and

$$\tilde{g}_m^{(j)} = d_{m+1} g^{(j)} s_{m-1} \cdots s_{j+1} d_{j+1} \cdots d_{m-1} d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_{m+1} g^{(j)} d_{m+1} s_m s_{m-1} \cdots s_{j+1} d_{j+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= d_{m+1}(g^{(j)} s_m \cdots s_{j+1} d_{j+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = d_{m+1} \tilde{g}^{(j)}$$

for each $0 \le j \le m-2$. Also,

$$\tilde{g}_0^{(m-1)} = d_0 g^{(m)} d_0 w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = d_0 \tilde{g}^{(m-1)},$$

$$\tilde{g}_m^{(m-1)} = d_{m+1} g^{(m-1)} d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = d_{m+1} g^{(m-1)} d_{m+1} s_m d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = d_{m+1} \tilde{g}^{(m-1)}$$

and lastly, for $1 \le i \le m-1$ we have

$$\tilde{g}_i^{(m-1)} = d_i g^{(m)} d_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = d_i \tilde{g}^{(m)}.$$

It follows that

$$d_i \tilde{\boldsymbol{g}} = (d_{i+1} \tilde{g}^{(0)}, \ldots, d_{i+1} \tilde{g}^{(i-1)}, d_i \tilde{g}^{(i+1)}, \ldots, d_i \tilde{g}^{(m)}) = (\tilde{g}_i^{(0)}, \ldots, \tilde{g}_i^{(i-1)}, \tilde{g}_i^{(i)}, \ldots, \tilde{g}_i^{(m-1)}) = \tilde{\boldsymbol{g}}_i$$

so that $d_i \sigma_m(\boldsymbol{\xi}) = \sigma_{m-1}(d_i \boldsymbol{\xi})$.

We proceed in a similar manner for the degeneracy maps. Notice that for any $0 \le i \le m$,

$$s_i \boldsymbol{\alpha} = (\alpha_1 | \alpha_2 | \cdots | \alpha_i | 1_N | \alpha_{i+1} | \cdots | \alpha_m)$$

and clearly $s_i(\boldsymbol{\alpha} \oplus \mathbf{1}) = s_i \boldsymbol{\alpha} \oplus \mathbf{1}$. We can also calculate

$$s_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = w_{m+1}(s_i \boldsymbol{\alpha}, s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

from Lemma 2.1.1 and Theorem 2.1.4.

Denote the images of $\boldsymbol{g}$ under the degeneracy maps by

$$s_i \boldsymbol{g} = (s_{i+1} g^{(0)}, \ldots, s_{i+1} g^{(i)}, s_i g^{(i)}, s_i g^{(i+1)}, \ldots, s_i g^{(m)}) = (g_i^{(0)}, \ldots, g_i^{(m+1)}) = \boldsymbol{g}_i$$

so that $s_i\boldsymbol{\xi} = (s_i\boldsymbol{\alpha}; \boldsymbol{g}_i)$ and

$$\sigma_{m+1}(s_i\boldsymbol{\xi}) = (s_i\boldsymbol{\xi}) \oplus \mathbf{1} = (s_i\boldsymbol{\alpha} \oplus \mathbf{1}; \tilde{\boldsymbol{g}}_i)$$

where $\tilde{\boldsymbol{g}}_i$ is defined by

$$\tilde{g}_i^{(j)} = g_i^{(j)} s_{m+1} \cdots s_{j+1} d_{j+1} \cdots d_{m+1} w_{m+1}(s_i\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

for $0 \leq i \leq m$ and $0 \leq j \leq m$, and

$$\tilde{g}_i^{(m+1)} = g_i^{(m+1)} w_{m+1}(s_i\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

for each $0 \leq i \leq m$. Similar to the case for the face maps, we must show $s_i\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{g}}_i$ for each $0 \leq i \leq m$.

If $j \leq i$ then

$$s_{i+1}\tilde{g}^{(j)} = s_{i+1} g^{(j)} s_{i+1} s_m \cdots s_{i+1} \cdots s_{j+1} d_{j+1} \cdots d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= s_{i+1} g^{(j)} s_{m+1} \cdots s_{i+2} s_{i+1} \cdots s_{j+1} d_{j+1} \cdots d_{i+1} \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= s_{i+1} g^{(j)} s_{m+1} \cdots s_{i+2} s_{i+1} \cdots s_{j+1} d_{j+1} \cdots d_{i+1} \cdots d_m d_{i+1} s_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= s_{i+1} g^{(j)} s_{m+1} \cdots s_{i+2} s_{i+1} \cdots s_{j+1} d_{j+1} \cdots d_{i+1} d_{i+2} \cdots d_{m+1} w_{m+1}(s_i\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

$$= g_i^{(j)} s_{m+1} \cdots s_{j+1} d_{j+1} \cdots d_{m+1} w_{m+1}(s_i\boldsymbol{\alpha}, (s_i\boldsymbol{\alpha}) \oplus \mathbf{1}; l) = \tilde{g}_i^{(j)}.$$

If $i+1 \leq j \leq m$ then

$$s_i\tilde{g}^{(j-1)} = s_i g^{(j-1)} s_i s_m \cdots s_j d_j \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= s_i g^{(j-1)} s_{m+1} \cdots s_{j+1} s_i d_j \cdots d_m w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= s_i g^{(j-1)} s_{m+1} \cdots s_{j+1} d_{j+1} \cdots d_{m+1} s_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)$$

$$= s_i g^{(j-1)} s_{m+1} \cdots s_{j+1} d_{j+1} \cdots d_{m+1} w_{m+1}(s_i\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l)$$

$$= g_i^{(j)} s_{m+1} \cdots s_{j+1} d_{j+1} \cdots d_{m+1} w_{m+1}(s_i\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha}) \oplus \mathbf{1}; l) = \tilde{g}_i^{(j)},$$

and for $0 \leq i \leq m$ we have

$$s_i\tilde{g}^{(m)} = s_i g^{(m)} s_i w_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = s_i g^{(j-1)} w_{m+1}(s_i\boldsymbol{\alpha}, s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); l) = g_i^{(m+1)} w_{m+1}(s_i\boldsymbol{\alpha}, (s_i\boldsymbol{\alpha}) \oplus \mathbf{1}; l) = \tilde{g}_i^{(m+1)}.$$

Since

$$s_i\tilde{\boldsymbol{g}} = (s_{i+1}\tilde{g}^{(0)}, \ldots, s_{i+1}\tilde{g}^{(i)}, s_i\tilde{g}^{(i)}, \ldots, s_i\tilde{g}^{(m)})$$

it follows that

$$s_i\sigma_m(\boldsymbol{\xi}) = s_i(\boldsymbol{\xi} \oplus \mathbf{1}) = (s_i(\boldsymbol{\alpha} \oplus \mathbf{1}); s_i\tilde{\boldsymbol{g}}) = ((s_i\boldsymbol{\alpha}) \oplus \mathbf{1}; \tilde{\boldsymbol{g}}_i) = \sigma_{m+1}(s_i\boldsymbol{\alpha}; s_i\boldsymbol{g}) = \sigma_{m+1}(s_i\boldsymbol{\xi}).$$

Therefore $\sigma : Y_N \to Y_{N+1}$ is a simplicial map by Definition 1.1.2 of Chapter 1, and we have (2). In addition to the properties already verified for $\boldsymbol{\xi} \oplus \mathbf{1} \in G(\mathfrak{s}.\mathcal{P}R)^I_m$ above, we also see that

$$d_0\tilde{g}_0 = (d_0g_0)(d_0s_ms_{m-1}\cdots s_1d_1d_2\cdots d_mw_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)).$$

However, by Theorem 2.1.4 we see

$$d_1d_2\cdots d_mw_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = w_1(d_1d_2\cdots d_m\boldsymbol{\alpha}, d_1d_2\cdots d_m(\boldsymbol{\alpha} \oplus \mathbf{1}); l) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2.$$

Therefore

$$d_0d_1d_2\cdots d_mw_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l) = 1 \in G(\mathfrak{s}.\mathcal{P}R)_1$$

and

$$d_0\tilde{g}_0 = (d_0g_0)(s_{m-1}s_{m-2}\cdots s_0d_0d_1d_2\cdots d_mw_m(\boldsymbol{\alpha}, \boldsymbol{\alpha} \oplus \mathbf{1}; l)) = d_0g_0.$$

If $\boldsymbol{\xi} \in Y_{N,m}$ then $d_0g_0 = 1$ by definition, hence

$$d_0\tilde{g}_0 = d_0g_0 = 1$$

so that $\boldsymbol{\xi} \oplus \mathbf{1} \in Y_{N+1,m}$ and we have (3).

For (4), we interpret the conclusion of (3): the left side of the diagram

$$\begin{tikzcd}
Y_N \arrow[r, tail, "\subseteq"] \arrow[d, "\sigma"] & \mathfrak{I}_N \arrow[r, "p_N"] \arrow[d, "\sigma"] & G(\mathfrak{s}.\mathcal{P}R) \arrow[d, "="] \\
Y_{N+1} \arrow[r, tail, "\subseteq"] & \mathfrak{I}_{N+1} \arrow[r, "p_{N+1}"] & G(\mathfrak{s}.\mathcal{P}R)
\end{tikzcd}$$

is exactly the fact that the restriction of σ to the homotopy fiber is the map σ between homotopy fibers. In order to verify this, we showed that $d_0\tilde{g}_0 = d_0 g_0$, so that

$$p_N(\boldsymbol{\xi}) = p_N(\boldsymbol{\alpha}; \boldsymbol{g}) = d_0 g_0 = d_0\tilde{g}_0 = p_{N+1}(\boldsymbol{\alpha}\oplus\mathbf{1}; \boldsymbol{\xi}\oplus\mathbf{1}) = p_{N+1}(\boldsymbol{\xi}\oplus\mathbf{1}).$$

Therefore the right square in the diagram commutes. Also by definition (i.e. the proof of Lemma 3.1.4 and Lemma 4.1.1) we see

$$\sigma(\partial^*(\boldsymbol{\xi})) = \sigma(\partial^*(\boldsymbol{\alpha}; \boldsymbol{g})) = \sigma(\boldsymbol{\alpha}) = \boldsymbol{\alpha}\oplus\mathbf{1} = \partial^*(\boldsymbol{\alpha}\oplus\mathbf{1}; \tilde{\boldsymbol{g}}) = \partial^*(\boldsymbol{\xi}\oplus\mathbf{1}) = \partial^*(\sigma(\boldsymbol{\xi})),$$

so that the diagram

$$\begin{array}{ccc} \mathfrak{I}_N & \xrightarrow{\partial^*} & N(GL(N,R)) \\ \big\downarrow{\scriptstyle\sigma} & & \big\downarrow{\scriptstyle\sigma} \\ \mathfrak{I}_{N+1} & \xrightarrow{\partial^*} & N(GL(N+1,R)) \end{array}$$

commutes.

Finally, compose the relation implied by (4) with λ_N :

$$\sigma\circ\partial_N^*\circ\lambda_N = \partial_{N+1}^*\circ\sigma\circ\lambda_N$$

and apply Lemma 3.1.4 so that $\partial_N^*\circ\lambda_N = id_{N(GL(N,R))}$, hence

$$\sigma = \partial_{N+1}^*\circ\sigma\circ\lambda_N.$$

Now put both sides of the above equation into λ_{N+1} :

$$\lambda_{N+1}\circ\sigma = \lambda_{N+1}\circ\partial_{N+1}^*\circ\sigma\circ\lambda_N;$$

since $\lambda_{N+1}\circ\partial_{N+1}^*$ is homotopic to $id_{\mathfrak{I}_{N+1}}$ by Lemma 3.1.4 and homotopy is preserved by composition, it follows that $\lambda_{N+1}\circ\sigma$ is homotopic to $\sigma\circ\lambda_N$. Thus we have the diagram of (5) commuting "up to homotopy". $\square$

We can now extend to simplicial maps $\sigma_j^i = \overbrace{\sigma\circ\sigma\circ\cdots\circ\sigma}^{j-i} : Y_i \to Y_j, i \le j$, and define a direct limit as in [20]:

$$Y(R) = \varinjlim_{N,\sigma} Y_N.$$

4.2 Filtration-Independent Elements of $\pi_1(Y_N)$

Lemma 4.2.1 *Let $N \in \mathbb{N}, E_1, E_2 \in Aut(R^N)$ and suppose there is a filtration F of R^N that is an admissible filtration of both (R^N, E_1) and (R^N, E_2). Then:*

a) F is an admissible filtration of $(R^N, E_1 \circ E_2)$ as well as (R^N, E_1^{-1}), (R^N, E_2^{-1}) and $(R^N, 1_N)$.

b) $(E_1; 1, X_1(F(R^N, E_1))), (E_2; 1, X_1(F(R^N, E_2))) \in \widetilde{Y}_{N,1}$ with respect to $\phi_0 = (*; 1)$.

c) $[E_1; 1, X_1(F(R^N, E_1))] \bullet [E_2; 1, X_1(F(R^N, E_2))] = [E_1 \circ E_2; 1, X_1(F(R^N, E_1 \circ E_2))] \in \pi_1(Y_N, \Phi)$ via extender

$$z_{E_1 E_2} := (E_2|E_1; 1, s_2 d_2 X_2(F(R^N; E_2|E_1)), X_2(F(R^N; E_2|E_1))) \in Y_{N,2}.$$

d) $[E_1; 1, X_1(F(R^N, E_1))]^{-1} = [E_1^{-1}; 1, X_1(F(R^N, E_1^{-1}))]$.

Proof: By Definition 2.2.1 there are diagrams

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, tail] \arrow[d, "(E_1)^{(1)}"] & P_2 \arrow[r, tail] \arrow[d, "(E_1)^{(2)}"] & \cdots \arrow[r, tail] & P_n = R^N \arrow[d, "E_1"] \\ 0 \arrow[r] & P_1 \arrow[r, tail] & P_2 \arrow[r, tail] & \cdots \arrow[r, tail] & P_n = R^N \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, tail] \arrow[d, "(E_2)^{(1)}"] & P_2 \arrow[r, tail] \arrow[d, "(E_2)^{(2)}"] & \cdots \arrow[r, tail] & P_n = R^N \arrow[d, "E_2"] \\ 0 \arrow[r] & P_1 \arrow[r, tail] & P_2 \arrow[r, tail] & \cdots \arrow[r, tail] & P_n = R^N \end{tikzcd}.$$

We also have $(E_1)^{(i)}, (E_2)^{(i)} \in Aut(P_i)$, hence

$$(E_1 \circ E_2)^{(i)} = (E_1 \circ E_2)|_{P_i} = E_1|_{P_i} \circ E_2|_{P_i} = (E_1)^{(i)} \circ (E_2)^{(i)} \in Aut(P_i)$$

for each $1 \leq i \leq n$. In particular we know by definition that $(E_1)^{(1)} = (E_2)^{(1)} = 1_{P_1}$, so that $(E_1 \circ E_2)^{(1)} = 1_{P_1}$. Also $(E_1)^{(n)} = E_1$ and $(E_2)^{(n)} = E_2$, so that $(E_1 \circ E_2)^{(n)} = E_1 \circ E_2$. Therefore the diagram

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, tail] \arrow[d, "(E_1 \circ E_2)^{(1)}"] & P_2 \arrow[r, tail] \arrow[d, "(E_1 \circ E_2)^{(2)}"] & \cdots \arrow[r, tail] & P_n = R^N \arrow[d, "(E_1 \circ E_2)^{(n)}"] \\ 0 \arrow[r] & P_1 \arrow[r, tail] & P_2 \arrow[r, tail] & \cdots \arrow[r, tail] & P_n = R^N \end{tikzcd}$$

satisfies part (1) of Definition 2.2.1. Furthermore, from the short exact sequences of pairs

$$(P_i, (E_1)^{(i)}) \to (P_{i+1}, (E_1)^{(i+1)}) \to (P_{i+1}/P_i, 1_{P_{i+1}/P_i})$$

and

$$(P_i, (E_2)^{(i)}) \to (P_{i+1}, (E_2)^{(i+1)}) \to (P_{i+1}/P_i, 1_{P_{i+1}/P_i})$$

we see that the same inclusions and projections allow a short exact sequence of pairs

$$(P_i, (E_1 \circ E_2)^{(i)}) \to (P_{i+1}, (E_1 \circ E_2)^{(i+1)}) \to (P_{i+1}/P_i, 1_{P_{i+1}/P_i}),$$

which gives part (2) of Definition 2.2.1. Reversing directions on the automorphisms in the above diagrams shows that any subsequence that is an admissible filtration for (R^N, E_1) will also be an admissible filtration of (R^N, E_1^{-1}). This proves (a).

Now for the admissible filtration $F(R^N, E_1)$ (and similarly for $F(R^N, E_2)$) we have

$$X_1(F(R^N, E_1)) \in \overline{G(\mathfrak{s}.\mathcal{P}R)_2}$$

from Theorem 2.2.2. Consider the element

$$\boldsymbol{g} = (g_0, g_1) = (1, X_1(F(R^N, E_1))) \in G(\mathfrak{s}.\mathcal{P}R)_2 \times G(\mathfrak{s}.\mathcal{P}R)_2.$$

From Theorem 2.2.2 we see

$$d_1 g_1 = 1 = d_1(1) = d_1 g_0,$$

in which case $\boldsymbol{g} \in G(\mathfrak{s}.\mathcal{P}R)_1^I$. Notice also that $d_2 g^1 = \mathfrak{i}_{R^N}(E_1)$ and $d_0 g^0 = 1 \in G(\mathfrak{s}.\mathcal{P}R)_1$. Therefore we have

$$X(E_1) := (E_1; \boldsymbol{g}) \in Y_{N,1} \subseteq \mathfrak{I}_{N,1}$$

by Definitions 3.1.3 and 3.1.6. Furthermore, we calculate

$$d_0 \boldsymbol{g} = d_1 \boldsymbol{g} = 1 \in G(\mathfrak{s}.\mathcal{P}R)_1 = G(\mathfrak{s}.\mathcal{P}R)_0^I,$$

so that

$$d_0 X(E_1) = (d_0 E_1; d_0 \boldsymbol{g}) = (R^N; 1) = \phi_0$$

and

$$d_1 X(E_1) = (d_1 E_1 : d_1 \boldsymbol{g}) = (R^N; 1) = \phi_0,$$

so $X(E_1) \in \widetilde{Y}_{N,1}$ by Definition 1.3.4 of Chapter 1. The same process with respect to $F(R^N, E_2)$ shows

$$X(E_2) := (E_2; \boldsymbol{g'}) \in \widetilde{Y}_{N,1}$$

when $\boldsymbol{g'} = (g'_0, g'_1) = (1, X_1(F(R^N, E_2)))$. This proves (b).

We construct the product

$$[X(E_1)] \bullet [X(E_2)] \in \pi_1(Y_N, \Phi)$$

by the canonical definition in [2] (i.e. Definition 1.3.6 of Chapter 1). Consider the list

$$C_{E_1E_2} = (X(E_1), -, X(E_2))$$

in $Y_{N,1}$. Since $X(E_1), X(E_2) \in \widetilde{Y}_{N,1}$, this list is a compatible list of 2-simplices in $Y_{N,1}$. We construct an extender for this list.

Since F is a filtration of both (R^N, E_1) and (R^N, E_2), it is by Definition 2.2.3 a filtration of $(R^N, \boldsymbol{E'})$ where $\boldsymbol{E'} = (E_2|E_1) \in N(Aut(R^N))_2$. Thus by Lemma 2.2.4 we have

$$X_2(F(R^N; \boldsymbol{E'})) \in G(\mathfrak{s}.\mathcal{P}R)_3.$$

Consider the element

$$\boldsymbol{g''} = (g''_0, g''_1, g''_2) = (1, s_2d_2X_2(F(R^N, \boldsymbol{E'})), X_2(F(R^N, \boldsymbol{E'}))) \in G_3 \times G_3 \times G_3.$$

Then $d_1g''_0 = 1 \in G(\mathfrak{s}.\mathcal{P}R)_2$, and since

$$d_2X_2(F(R^N, \boldsymbol{E'})) = X_1(F(R^N, E_2)) \in \overline{G(\mathfrak{s}.\mathcal{P}R)}_2$$

we see that

$$d_1s_2d_2X_2(F(R^N, \boldsymbol{E'})) = s_1d_1d_2X_2(F(R^N, \boldsymbol{E'})) = s_1(1) = 1 \in G(\mathfrak{s}.\mathcal{P}R)_2,$$

in which case we have $d_1g''_0 = d_1g''_1$. Also

$$d_2g''_1 = d_2X_2(F(R^N, \boldsymbol{E'})) = d_2s_2d_2X_2(F(R^N, \boldsymbol{E'})).$$

Thus $d_2g''_1 = d_2g''_2$, hence $\boldsymbol{g''} \in G(\mathfrak{s}.\mathcal{P}R)^I_2$. Note also that $d_0g_0 = 1 \in G(\mathfrak{s}.\mathcal{P}R)_2$ and that Lemma 2.2.4 gives $d_3g_2 = \mathfrak{i}_{R^N}(\boldsymbol{E'})$, so that we can have the element

$$z_{E_1E_2} := (\boldsymbol{E'}, \boldsymbol{g''}) = (E_2|E_1; 1, s_2d_2X_2(F(R^N; E_2|E_1)), X_2(F(R^N; E_2|E_1))) \in Y_{N,2} \subseteq \mathfrak{I}_{N,2}.$$

Now we check

$$d_0 z_{E_1E_2} = (d_0\boldsymbol{E'}; d_0\boldsymbol{g''}) = (E_1; d_0g_1'', d_0g_2'')$$

$$= (E_1; 1, X_1(F(R^N, E_1))) = X(E_1)$$

and

$$d_2 z_{E_1E_2} = (d_2\boldsymbol{E'}; d_2\boldsymbol{g''}) = (E_2; d_3g_0'', d_3g_1'')$$

$$= (E_2; 1, X_1(F(R^N, E_2))) = X(E_2),$$

so that $z_{E_1E_2}$ extends the list C_{E_1,E_2}. But now by Definition 1.3.6 of Chapter 1 we have

$$[X(E_1)][X(E_2)] = [d_1 z_{E_1E_2}] = [d_1\boldsymbol{E'}; d_1\boldsymbol{g''}]$$

$$= [E_1 \circ E_2; d_2g_0'', d_1g_2''] = [E_1 \circ E_2; 1, X_1(F(R^N, E_1 \circ E_2))]$$

for (c).

Now (d) follows by checking

$$X_1(F(R^N, E_1 \circ E_1^{-1}))) = X_1(F(R^N, 1_N))) = 1 \in G(\mathfrak{s}.\mathcal{P}R))_2$$

by definition, hence (c) implies

$$[E_1 \circ E_1^{-1}; 1, X_1(F(R^N, E_1 \circ E_1^{-1}))] = [1_N; 1, 1] = 1 \in \pi(Y_N, \Phi).$$

□

Theorem 4.2.2 *If F is an admissible filtration of (R^N, E), $E \in GL(N, R)$, and $\tilde{F}$ is an admissible refinement of F, then for $\xi = (E; 1, X_1(F(R^N, E))), \tilde{\xi} = (E; 1, X_1(\widetilde{F}(R^N, E))) \in \widetilde{Y}_{N,1}$ we have $\xi \sim \tilde{\xi}$ in the sense of [2], Chapter 3.*

Proof: We proceed as before for refinements, assuming a refinement by one, E-invariant submodule so that the result follows by induction. From Lemma 4.2.1 we know that $\xi, \tilde{\xi} \in \widetilde{Y}_{N,1}$. We will construct an element $y \in Y_{N,2}$ that meets the definition (i.e. Definition 1.3.1 of Chapter 1).

Recalling the constructions for Theorem 2.3.4, set

$$X_1 = X_1(F(R^N, E)) = ACB, \tilde{X}_1 = X_1(\tilde{F}(R^N, E)) = AC_1C_2B$$

With $A, B, C, C_1, C_2 \in \overline{G(\mathfrak{s}.\mathcal{P}R)_2}$ as for Theorem 2.3.4 (i.e. with $P = R^N$ and $\alpha = E$). We know from Lemma 2.3.3 that $\exists\ u \in \overline{G(\mathfrak{s}.\mathcal{P}R)_3}$ with $d_3 u = C_1 C_2 C^{-1}$. Let $v = X_2(F(R^N; E|1_N)) \in G(\mathfrak{s}.\mathcal{P}R)_3$ (using the filtration F for the admissible $F(R^N; E|1_N)$; recall $d_1 v \in \overline{G(\mathfrak{s}.\mathcal{P}R)_2}$) and set

$$\boldsymbol{g} = (g_0, g_1, g_2) = (1, (s_2 A)u(s_2 A)^{-1} s_2 d_2 v, v) \in G_3 \times G_3 \times G_3.$$

Then

$$d_1 g_1 = (s_1 d_1 A)(d_0 u)(s_1 d_1 A)^{-1}(s_1 d_1 d_1 v) = (s_1 d_1 A)(1)(s_1 d_1 A)^{-1}(1)(s_1(1)) = 1 = d_1 g_0$$

and

$$d_2 g_1 = A(d_2 u)A^{-1}(d_2 v) = A(1)A^{-1}(d_2 v) = d_2 v = d_2 g_2$$

so that $\boldsymbol{g} \in G_2^I$. Also $d_0 g_0 = 1$ and

$$d_3 g_2 = d_3 v = \mathfrak{i}_N(E|1),$$

so that the element

$$y = (E|1_N; \boldsymbol{g})$$

is a 2-simplex in the homotopy fiber Y_N by Definition 3.1.6. Furthermore, we see that $d_2 g_0 = 1$ and

$$d_1 g_2 = d_1 v = X_1(F(R^N; E)),$$

so that

$$d_1 y = (E; d_2 g_0, d_1 g_2) = (E; 1, X_1) = \xi.$$

Similarly, $d_3 g_0 = 1$ and

$$d_3 g_1 = A(d_3 u)A^{-1}(d_2 v) = AC_1C_2C^{-1}A^{-1}ACB = AC_1C_2B = X_1(\tilde{F}(R^N, E)),$$

in which case

$$d_2 y = (E; d_3 g_0, d_3 g_1) = (E; 1, \tilde{X}_1) = \tilde{\xi}.$$

Finally, $d_0 g_0 = 1$ and

$$d_0 g_2 = d_0 v = X_1(F(R^N; 1_N)) = 1 \in G(\mathfrak{s}.\mathcal{P}R)_2,$$

hence

$$d_0 y = (1_N; 1, 1) = 1 = s_0 d_0 \xi = s_0 d_0 \tilde{\xi}.$$

Thus y is the required "homotopy" from ξ to $\tilde{\xi}$ according to Definition 1.3.1 of Chapter 1.

□

An identical process to the proof of Theorem 2.4.2 gives

Theorem 4.2.3 *Suppose that F_1 and F_2 are standard, admissible filtrations of $E \in GL(N, R)$. Then for $\xi_1 = (E; 1, X_1(F_1(R^N; E)))$ and $\xi_2 = (E; 1, X_1(F_2(R^N; E)))$ we have $[\xi_1] = [\xi_2] \in \pi_1(Y_N)$. That is, the class $[\xi]$ for $\xi = (E; 1, X_1(F(R^N; E)))$ is independent of the choice of filtration F for the corresponding matrix E so long as that filtration is standard.*

Proof: In order to prove Theorem 2.4.2, a chain $\widetilde{F}$ and a sequence of chains $H_{a,b}$ were constructed through which we found

$$[X_1(\widetilde{F}(R^N, E))] = [X_1(F_{1,1}(R^N, E))],$$

$$[X_1(F'_{a,b}(R^N, E))] = [X_1(F_{a,b}(R^N, E))] \ \forall \ 1 \le a \le S-1, 1 \le b \le a,$$

$$[X_1(F'_{a,a}(R^N, E))] = [X_1(F_{a+1,1}(R^N, E))] \ \forall \ 1 \le a \le S-1,$$

and

$$[X_1(F_1(R^N, E))] = [X_1(F'_{S-1,S-1}(R^N, E))]$$

via Theorem 2.3.4 because of the resulting refinements. Since these refinements are still intact, Theorem 4.2.2 implies

$$[(E; 1, X_1(\widetilde{F}(R^N, E)))] = [(E; 1, X_1(F_{1,1}(R^N, E)))],$$

$$[(E; 1, X_1(F'_{a,b}(R^N, E)))] = [(E; 1, X_1(F_{a,b}(R^N, E)))] \ \forall \ 1 \le a \le S-1, 1 \le b \le a,$$

$$[(E; 1, X_1(F'_{a,a}(R^N, E)))] = [(E; 1, X_1(F_{a+1,1}(R^N, E)))] \ \forall \ 1 \le a \le S-1,$$

and

$$[\xi_1] = [(E; 1, X_1(F'_{S-1,S-1}(R^N, E)))].$$

These equalities imply (by transitivity of the equivalence relation) that $[(E; 1, X_1(\widetilde{F}(R^N, E))] = [\xi_1]$ in $\pi_1(Y_N)$. Another, similar sequence of chains allows $[(E; 1, X_1(\widetilde{F}(R^N, E)))] = [\xi_2]$ so that by transitivity we have $[\xi_1] = [\xi_2] \in \pi_1(Y_N)$.

□

4.3 The Steinberg Relations

4.3.1 Standard filtrations for elementary matrices

Recall Definition 3.1.2 from Chapter 2 for the elementary matrices $e_{ij}^N(a) \in GL(N,R)$ given $i \neq j$ and $N \geq 3$.

Lemma 4.3.1 *Given elementary matrices $e_{ij}^N(a), e_{kl}^N(b)$, there is a standard filtration*

$$F : 0 = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{n-1} \subseteq P_n = R^N$$

of free submodules of R^N that is a standard, admissible filtration of both $(R^N, e_{ij}^N(a))$ and $(R^N, e_{kl}^N(b))$.

Proof: We choose the standard basis for R^N :

$$R^N = \langle e_1, e_2, \ldots, e_N \rangle .$$

Thus we write

$$e_{ij}^N(a)[e_h] = \begin{cases} e_h, & h \neq j \\ \\ ae_i + e_j, & h = j \end{cases}$$

for the images of basis elements.

We begin by ordering the indices $\{i,j,k,l\}$ as $\{M_1 > M_2 > M_3 > M_4\}$ and thinking of them in pairs: for each $1 \leq h \leq 4$, we use M_h and consider $\widetilde{M_h}$ so that one of the matrices is $e_{M_h \widetilde{M_h}}$ (or $e_{\widetilde{M_h} M_h}$).

When $M_1, M_4 \in \{i,k\}$ we have either $\widetilde{M_1} = M_3 \in \{j,l\}$ or $\widetilde{M_1} = M_2 \in \{j,l\}$; we show the case for $\widetilde{M_1} = M_3$, since the other case is similar by replacing M_3 with M_2 and vice versa. Thus we denote the images by

$$e_{M_1 \widetilde{M_1}}(e_h) = \begin{cases} e_h, & h \neq M_3 \\ \\ r_1 e_{M_1} + e_{M_3}, & h = M_3 \end{cases}$$

and

$$e_{M_4 \widetilde{M_4}}(e_h) = \begin{cases} e_h, & h \neq M_2 \\ \\ r_4 e_{M_4} + e_{M_2}, & h = M_2 \end{cases}$$

.

Let

$$P_1 = \langle e_1, \ldots, e_{M_4}, \ldots, e_{M_3 - 1}, e_{M_2 + 1}, \ldots, e_{M_1}, \ldots, e_N \rangle ,$$

$$P_2 = \langle e_1, \ldots, e_{M_4}, \ldots, e_{M_3}, e_{M_2}, \ldots, e_N \rangle \,,$$

and $P_3 = R^N$, with inclusions preserving the specified order of basis elements. On the generators e_h of P_1 we see that $e_{M_1\widetilde{M}_1}(e_h) = e_h$ since $h \neq \widetilde{M}_1$ and $e_{M_4\widetilde{M}_4}(e_h) = e_h$ since $h \neq \widetilde{M}_4$. On P_2, we have $e_{M_1M_3}(e_h) = e_h \ \forall\ h \neq M_3$ and $e_{M_1M_3}(e_{M_3}) = r_1e_{M_1} + e_{M_3} \in P_2$; similarly $e_{M_4M_2}(e_h) = e_h \ \forall\ h \neq M_2$ and $e_{M_4M_2}(e_{M_2}) = r_4e_{M_4} + e_{M_2} \in P_2$. Thus $e^{(1)}_{M_1\widetilde{M}_1} = e^{(1)}_{M_4\widetilde{M}_4} = 1_{P_1}$ and $e^{(2)}_{M_1\widetilde{M}_1}, e^{(2)}_{M_4\widetilde{M}_4} \in Aut(P_2)$. On the quotient module $P_2/P_1 = \langle \overline{e}_{M_3}, \overline{e}_{M_2} \rangle \in \mathcal{P}R$ we calculate on equivalence classes and see

$$e_{M_1M_3}(\overline{e}_{M_3}) = \overline{r_1e_{M_1} + e_{M_3}} = \overline{0 + e_{M_3}} = \overline{e}_{M_3}$$

and $e_{M_1M_3}(\overline{e}_{M_2}) = \overline{e}_{M_2}$. Similarly $e_{M_4M_2}(\overline{e}_{M_3}) = \overline{e}_{M_3}$ and

$$e_{M_4M_2}(\overline{e}_{M_2}) = \overline{r_4e_{M_4} + e_{M_2}} = \overline{0 + e_{M_2}} = \overline{e}_{M_2}.$$

On $P_3/P_2 = \langle \overline{e}_{M_3+1}, \ldots, \overline{e}_{M_2-1} \rangle \in \mathcal{P}R$ we have $e_{M_1M_3}(\overline{e}_h) = \overline{e}_h \ \forall\ h \neq M_3$ and $e_{M_4M_2}(\overline{e}_h) = \overline{e}_h \ \forall\ h \neq M_2$. We conclude that induced maps on these quotients are the identity, so that

$$(P_1, e^{(1)}_{M_1\widetilde{M}_1}) \to (P_2, e^{(2)}_{M_1\widetilde{M}_1}) \to (P_2/P_1, 1),$$

$$(P_1, e^{(1)}_{M_4\widetilde{M}_4}) \to (P_2, e^{(2)}_{M_4\widetilde{M}_4}) \to (P_2/P_1, 1),$$

$$(P_2, e^{(2)}_{M_1\widetilde{M}_1}) \to (P_3 = R^N, e_{M_1\widetilde{M}_1}) \to (P_3/P_2, 1),$$

and

$$(P_2, e^{(2)}_{M_4\widetilde{M}_4}) \to (P_3 = R^N, e_{M_4\widetilde{M}_4}) \to (P_3/P_2, 1)$$

are short exact sequences of pairs, in which case this choice of $P_1 \subseteq P_2 \subseteq P_3$ is an admissible filtration of both $(R^N, e_{M_1\widetilde{M}_1})$ and $(R^N, e_{M_4\widetilde{M}_4})$.

When $M_1 \in \{i, k\}$ and $M_4 \in \{j, l\}$ we have $\widetilde{M}_1 \in \{M_2, M_3, M_4\}$. In case $\widetilde{M}_1 = M_2$ we note that $\widetilde{M}_3 = M_4$ and set $P_1 = \langle e_{M_2+1}, \ldots, e_{M_1}, \ldots, e_N \rangle$, $P_2 = \langle e_{M_2}, \ldots, \ldots, e_N \rangle$, $P_3 = \langle e_{M_4+1}, \ldots, e_{M_3}, \ldots, e_N \rangle$ and $P_4 = \langle e_1, \ldots, e_N \rangle = R^N$ with inclusions that preserve the order of the basis elements. Since $e_{M_1\widetilde{M}_1}(e_h) = e_h \ \forall\ h > M_2$ and $e_{M_3\widetilde{M}_3}(e_h) = e_h \ \forall\ h > M_2 > M_4$, we know that the restrictions of both matrices to P_1 are the identity. On the other hand, $e_{M_1M_2}(e_{M_2}) = r_1e_{M_1} + e_{M_2} \in P_2$ and $e_{M_3M_4}(e_{M_2}) = e_{M_2}$, so that $e_{M_1M_2}, e_{M_3M_4} \in Aut(P_2)$. Also, $e_{M_1M_2}, e_{M_3M_4} \in Aut(P_3)$ since both matrices map $e_h \mapsto e_h$ for any $M_4 < h < M_2$.

We have quotient modules $P_2/P_1 = \langle \overline{e}_{M_2} \rangle$, $P_3/P_2 = \langle \overline{e}_{M_4+1}, \ldots, \overline{e}_{M_2-1} \rangle$, and $P_4/P_3 = \langle \overline{e}_1, \ldots, \overline{e}_4 \rangle$, which are all clearly finitely generated and projective (since they are free modules). Both matrices induce the identity on P_3/P_2 by definition. On P_2/P_1 we see that

$$e_{M_1M_2}(\overline{e}_{M_2}) = \overline{r_1 e_{M_1} + e_{M_2}} = \overline{0 + e_{M_2}} = \overline{e}_{M_2},$$

while $e_{M_3M_4}$ induces the identity by definition. Similarly, on P_2/P_1 we see that

$$e_{M_3M_4}(\overline{e}_{M_4}) = \overline{r_3 e_{M_3} + e_{M_4}} = \overline{0 + e_{M_4}} = \overline{e}_{M_4},$$

while $e_{M_1M_2}$ induces the identity by definition. Therefore with short exact sequences of pairs

$$(P_1, e^{(1)}_{M_1\widetilde{M}_1}) \to (P_2, e^{(2)}_{M_1\widetilde{M}_1}) \to (P_2/P_1, 1),$$

$$(P_1, e^{(1)}_{M_2\widetilde{M}_2}) \to (P_2, e^{(2)}_{M_2\widetilde{M}_2}) \to (P_2/P_1, 1),$$

$$(P_2, e^{(2)}_{M_1\widetilde{M}_1}) \to (P_3, e^{(3)}_{M_1\widetilde{M}_1}) \to (P_3/P_2, 1),$$

$$(P_2, e^{(2)}_{M_2\widetilde{M}_2}) \to (P_3, e^{(3)}_{M_2\widetilde{M}_2}) \to (P_3/P_2, 1),$$

$$(P_2, e^{(3)}_{M_1\widetilde{M}_1}) \to (P_4 = R^N, e_{M_1\widetilde{M}_1}) \to (P_4/P_3, 1),$$

and

$$(P_2, e^{(3)}_{M_2\widetilde{M}_2}) \to (P_3 = R^N, e_{M_2\widetilde{M}_2}) \to (P_4/P_3, 1)$$

we have that $P_1 \subseteq P_2 \subseteq P_3 \subseteq P_4$ is an admissible filtration of both $(R^N, e_{M_1\widetilde{M}_1})$ and $(R^N, e_{M_2\widetilde{M}_2})$.

Similar to the case for $\widetilde{M}_1 = M_2$, in case $\widetilde{M}_1 = M_3$ the sequence $P_1 \subseteq P_2 \subseteq P_3$ with $P_1 = \langle e_{M_3+1}, \ldots, e_N \rangle$, $P_2 = \langle e_{M_4+1}, \ldots, e_N \rangle$ and $P_3 = R^N$ is an admissible filtration for both $(R^N, e_{M_1\widetilde{M}_1})$ and $(R^N, e_{M_2\widetilde{M}_2})$. In fact, this same filtration $P_1 \subseteq P_2 \subseteq P_3$ is also an admissible filtration of both $(R^N, e_{M_1\widetilde{M}_1})$ and $(R^N, e_{M_2\widetilde{M}_2})$ in case $\widetilde{M}_1 = M_4, \widetilde{M}_2 = M_3$ since $M_3 > M_4$. When $\widetilde{M}_1 = M_4$ but $\widetilde{M}_3 = M_2$, we use $P_1 = \langle e_{M_4+1}, \ldots, e_{M_3}, \ldots, e_{M_2-1} \rangle$, $P_2 = \langle e_{M_4+1}, \ldots, e_N \rangle$ and $P_3 = R^N$.

The remaining cases can be verified as an exercise as follows.

When $M_1 \in \{j, l\}$ and $M_1 = \widetilde{M}_4$, the sequence with $P_1 = \langle e_1, \ldots, e_{M_2-1} \rangle$, $P_2 = \langle e_1, \ldots, e_{M_1} \rangle$, $P_3 = R^N$ is an admissible filtration of $(R^N, e_{M_4M_1})$ and $(R^N, e_{M_3M_2})$, while the sequence given by $P_1 = \langle e_{M_3+1}, \ldots, e_{M_2} \rangle$, $P_2 = \langle e_1, \ldots, e_{M_1-1} \rangle$, $P_3 = R^N$ is an admissible filtration of $(R^N, e_{M_4M_1})$ and $(R^N, e_{M_2M_3})$. If $M_1 = \widetilde{M}_2$ then the sequence given by $P_1 = \langle e_1, \ldots, e_{M_3-1} \rangle$, $P_2 = \langle e_1, \ldots, e_{M_1-1} \rangle$, $P_3 = R^N$ is an admissible filtration of $(R^N, e_{M_2M_1})$ and $(R^N, e_{M_4M_3})$.

The final two cases are $M_1 = \widetilde{M}_2$ and $M_1 = \widetilde{M}_3$. In these cases, the sequence with $P_1 = \langle e_{M_4+1}, \ldots, e_{M_3} \rangle$, $P_2 = \langle e_1, \ldots, e_{M_1-1} \rangle$, $P_3 = R^N$ is an admissible filtration of $(R^N, e_{M_2M_1})$ and $(R^N, e_{M_3M_4})$, while the sequence given by $P_1 = \langle e_{M_4+1}, \ldots, e_{M_2} \rangle$, $P_2 = \langle e_1, \ldots, e_{M_1-1} \rangle$, $P_3 = R^N$ is an admissible filtration of $(R^N, e_{M_3M_1})$ and $(R^N, e_{M_2M_4})$. Notice that these calculations account for the cases where at least one of the matrices is upper-triangular as well as the case both being lower-triangular and all other cases.

□

4.3.2 Steinberg generators and relations in $\pi_1(Y_N)$

As usual, let $e^N_{ij}(a) \in GL(N, R); i \neq j; i, j \leq N$ be an $N \times N$ elementary matrix associated to the element $a \in R$. Of course, $e^N_{ij}(a) \oplus 1 = e^{N+1}_{ij}(a)$.

Definition 4.3.2 *Suppose that $i, j \leq N, i \neq j$ and $a \in R$. Define elements $X^N_{ij}(a) \in \pi_1(Y_N(R))$ by*

$$X^N_{ij}(a) = [(e^N_{ij}; 1, X_1(F(R^N, e^N_{ij}(a))))],$$

where $F(R^N, e^N_{ij}(a))$ is any standard admissible filtration of $(R^N, e^N_{ij}(a))$.

Combining Lemma 4.3.1 with Theorem 4.2.3 allows us to see that $X^N_{ij}(a)$ has a defining admissible filtration and that this element of $\pi_1(Y_N)$ is independent of the choice of the standard admissible filtration chosen to define it.

Theorem 4.3.3 *Given any $i, j, k, l \leq N$ such that $i \neq j, k \neq l$, and $a, b \in R$, we have the following computations in $\pi_1(Y_N)$:*

1. $X^N_{ij}(a)X^N_{ij}(b) = X^N_{ij}(a + b)$
2. $X^N_{ij}(a)X^N_{jl}(b)X^N_{ij}(a)^{-1}X^N_{jl}(b)^{-1} = X^N_{il}(ab)$, *if* $i \neq l$
3. $X^N_{ij}(a)X^N_{kl}(b) = X^N_{kl}(b)X^N_{ij}(a)$, *if* $i \neq l, j \neq k$
4. *If $\sigma : Y_N \to Y_{N+1}$ is as in Proposition 4.1.2, then $\sigma_*(X^N_{ij}(a)) = X^{N+1}_{ij}(a)$.*

Proof: Given elementary matrices $e^N_{st}(a), e^N_{uv}(b) \in Aut(R^N)$, $\{s \neq t\} \subset \{i, j, k, l\}$, $\{u \neq v\} \subseteq \{i, j, k, l\}$, $a, b \in R$, Lemma 4.3.1 guarantees a subsequence F that is a filtration for both and by Lemma 4.2.1.(a) and (b) we know that there are corresponding equivalence classes

$$X^N_{st}(a) := X(e^N_{st}(a)) = [e^N_{st}(a); 1, X_1(F(R^N, e^N_{st}(a)))] \in \pi_1(Y_N).$$

Then Lemma 4.2.1.(c) allows us to calculate directly in each case (we omit some of the "bullets" that indicate group operation in homotopy groups as in Chapter 1, and understand concatenation to stand for this):

1. Since $e_{ij}^N(a) \circ e_{ij}^N(b) := e_{ij}^N(a)e_{ij}^N(b) = e_{ij}^N(a+b)$ for elementary matrices $e_{ij}^N(a)$ and $e_{ij}^N(b)$ we let $E_1 = e_{ij}^N(a)$ and $E_2 = e_{ij}^N(b)$ so that

$$X_{ij}^N(a)X_{ij}^N(b) = [e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))][e_{ij}^N(b); 1, X_1(F(R^N, e_{ij}^N(b)))]$$

$$= [e_{ij}^N(a) \circ e_{ij}^N(b); 1, X_1(F(R^N, e_{ij}^N(a) \circ e_{ij}^N(b)))] = [e_{ij}^N(a+b); 1, X_1(F(R^N, e_{ij}^N(a+b)))]$$

$$= X_{ij}^N(a+b)$$

2. We use associativity,

$$e_{ij}^N(a)e_{jl}^N(b)(e_{ij}^N(a))^{-1}(e_{jl}^N(b))^{-1} = (e_{ij}^N(a)e_{jl}^N(b))((e_{ij}^N(a))^{-1}(e_{jl}^N(b))^{-1}) = e_{il}^N(ab),$$

and Lemma 4.2.1.(d).

$$X_{ij}^N(a)X_{jl}^N(b)(X_{ij}^N(a))^{-1}(X_{jl}^N(b))^{-1} = ([e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))]$$

$$\bullet [e_{jl}^N(b); 1, X_1(F(R^N, e_{jl}^N(b)))])([e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))]^{-1}[e_{jl}^N(b); 1, X_1(F(R^N, e_{jl}^N(b)))]^{-1})$$

$$= ([e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))][e_{jl}^N(b); 1, X_1(F(R^N, e_{jl}^N(b)))])$$

$$\bullet ([e_{ij}^N(a)^{-1}; 1, X_1(F(R^N, e_{ij}^N(a)^{-1}))][e_{jl}^N(b)^{-1}; 1, X_1(F(R^N, e_{jl}^N(b)^{-1}))])$$

$$= [e_{ij}^N(a)e_{jl}^N(b); 1, X_1(F(R^N, e_{ij}^N(a)e_{jl}^N(b)))]$$

$$\bullet [(e_{ij}^N(a))^{-1}(e_{jl}^N(b))^{-1}; 1, X_1(F(R^N, (e_{ij}^N(a))^{-1}(e_{jl}^N(b))^{-1}))]$$

$$= [e_{ij}^N(a)e_{jl}^N(b)(e_{ij}^N(a))^{-1}(e_{jl}^N(b))^{-1}; 1, X_1(F(R^N, e_{ij}^N(a)e_{jl}^N(b)(e_{ij}^N(a))^{-1}(e_{jl}^N(b))^{-1}))]$$

$$= [e_{il}^N(ab); 1, X_1(F(R^N, e_{il}^N(ab)))]$$

$$= X_{il}^N(ab).$$

3. Once again we calculate directly using $e_{ij}(a)e_{kl}(b) = e_{kl}(b)e_{ij}(a)$:

$$X_{ij}^N(a)X_{kl}^N(b) = [e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))][e_{kl}^N(b); 1, X_1(F(R^N, e_{kl}^N(b)))]$$

$$= [e^N_{ij}(a)e^N_{kl}(b); 1, X_1(F(R^N, e^N_{ij}(a)e^N_{kl}(b)))]$$

$$= [e^N_{kl}(b)e^N_{ij}(a); 1, X_1(F(R^N, e^N_{kl}(b)e^N_{ij}(a)))]$$

$$= [e^N_{kl}(b); 1, X_1(F(R^N, e^N_{kl}(b)))][e^N_{ij}(a); 1, X_1(F(R^N, e^N_{ij}(a)))]$$

$$= X^N_{kl}(b)X^N_{ij}(a).$$

4. Suppose that F is any standard admissible filtration of $(R^N, e^N_{ij}(a))$. Then

$$\sigma(e^N_{ij}; 1, X_1(F(R^N, e^N_{ij}(a)))) = (e^{N+1}_{ij}(a); \tilde{g}_0, \tilde{g}_1),$$

where

$$\tilde{g}_0 = s_1 d_1 w_1(e^N_{ij}(a), e^{N+1}_{ij}(a); l),$$

l being the short exact sequence $(R^N, e^N_{ij}(a)) \to (R^{N+1}, e^{N+1}_{ij}(a)) \to (R, 1)$, and

$$\tilde{g}_1 = X_1(F(R^N, e^N_{ij}(a)))w_1(e^N_{ij}(a), e^{N+1}_{ij}(a); l) := X_1((F \oplus 1)(R^{N+1}, e^{N+1}_{ij}(a)))$$

where $F \oplus 1$ is the standard admissible filtration of $(R^{N+1}, e^{N+1}_{ij}(a))$ obtained by inserting the usual inclusion $R^N \subset R^{N+1}$ at the end of F. Therefore, we may conclude that $\sigma_*(X^N_{ij}(a)) = X^{N+1}_{ij}(a)$.

□

As an immediate corollary to the last theorem above we have

Theorem 4.3.4 *For every N there is a homomorphism of groups*

$$f_N : St(N, R) \to \pi_1(Y_N),$$

defined on the usual Steinberg generators $x^N_{ij}(a)$by

$$f_N(x^N_{ij}(a)) = X^N_{ij}(a),$$

such that the following diagram commutes:

$$\begin{tikzcd} St(N,R) \arrow[r, "\iota_{N,N+1}"] \arrow[d, "f_N"] & St(N+1,R) \arrow[d, "f_{N+1}"] \\ \pi_1(Y_N) \arrow[r, "\sigma_*"] & \pi_1(Y_{N+1}) \end{tikzcd} .$$

The top arrow is the usual "stabilization" homomorphism for the Steinberg groups as seen in Chapter 2.

Thus, passing to the direct limit there exists a homomorphism of groups

$$f : St(R) \to \varinjlim_{N,\sigma_*} \pi_1(Y_N) = \pi_1(\varinjlim_{N,\sigma} Y_N) = \pi_1(Y(R)),$$

such that

$$f(x_{ij}(a)) = (j_N)_*(X_{ij}^N(a)),$$

and N *is such that* $i, j \leq N$, $j_N : Y_N \to Y(R)$ *is one of the maps defining the direct limit.*

Chapter 6

Connecting $K_2(R)$ to $\pi_2(G(N(Q\mathcal{P}R)))$

1 Connecting the Exact Sequences

In this chapter, we drop the subscript N when referring to the direct limit, understanding that statements made with respect to this limit are with regard to the proper maps "for sufficient N". Therefore we have maps such as $\lambda_{1*} : \pi_1(N(GL(R))) \to \pi_1(\mathfrak{I}(R))$ in the direct limit, for example.

We use the isomorphisms developed in Chapters 4 and 5 to define an isomorphism $\tilde{f} : K_2 \to \pi_2(G(\mathfrak{s}.\mathcal{P}R))$, which will then combine with the theory of Chapter 3 to allow an isomorphism from K_2 to $\pi_2(G(N(Q\mathcal{P}R)))$. Consider what we can describe explicitly so far, using the defined maps and the long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & 1 \arrow[r] & K_2(R) \arrow[r, "i_K"] \arrow[d, dashed, "\tilde{f}"] & St(R) \arrow[r, "\phi"] \arrow[d, "f"] & GL(R) \arrow[r, "\pi"] \arrow[d, "\lambda_{1*}"] & K_1(R) \arrow[d, "Q'"] \\
\cdots \arrow[r, "i_{Y*}"] & \pi_2(\mathfrak{I}(R)) \arrow[r, "p_*"] & \pi_2(G(\mathfrak{s}.\mathcal{P}R)) \arrow[r, "d_\sharp"] & \pi_1(Y(R)) \arrow[r, "i_{Y*}"] & \pi_1(\mathfrak{I}(R)) \arrow[r, "p_*"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R))
\end{tikzcd}.$$

The top sequence is the exact sequence for K-groups involving $K_2(R)$ as in Chapter 5 of [8], and the bottom sequence is the long exact sequence of the Kan Fibration $p : \mathfrak{I} \to G(\mathfrak{s}.\mathcal{P}R)$ as in Lemma 3.1.5 of Chapter 5 and Definition 3.2.3 of Chapter 1.

By Lemma 3.1.4 of Chapter 5 we know $\pi_n(\mathfrak{I}(R)) \approx \pi_n(N(GL(R)))$ $\forall n$. But since $GL(R)$ is a group we know from Example 5.1.2 of Chapter 1 that $N(GL(R))$ is reduced, hence $\pi_0(N(GL(R))) := 1$. Also $N(GL(R))$ has the property that $\pi_n(N(GL(R))) := 1$ $\forall$ $n > 1$ as in Example 5.1.3 of Chapter 1. Therefore

$$\pi_0(\mathfrak{I}(R)) \approx \pi_0(N(GL(R))) = 1$$

and

$$\pi_2(\mathfrak{I}(R)) \approx \pi_2(N(GL(R))) = 1.$$

Now we can work with the exact sequences

$$\begin{tikzcd}
1 \arrow[r] & K_2(R) \arrow[r, "i_K"] \arrow[d, dashed, "\tilde{f}"] & St(R) \arrow[r, "\phi"] \arrow[d, "f"] & GL(R) \arrow[r, "\pi"] \arrow[d, "\lambda_{1*}"] & K_1(R) \arrow[d, "Q'"] \\
1 \arrow[r, "p_*"] & \pi_2(G(\mathfrak{s}.\mathcal{P}R)) \arrow[r, "d_\sharp"] & \pi_1(Y(R)) \arrow[r, "i_{Y*}"] & \pi_1(\mathfrak{I}(R)) \arrow[r, "p_*"] & \pi_1(G(\mathfrak{s}.\mathcal{P}R))
\end{tikzcd}$$

and the following results.

Theorem 1.0.1 *The following diagram commutes:*

$$\begin{tikzcd}
St(R) \arrow[r, "\phi"] \arrow[d, "f"] & GL(R) \arrow[d, "\lambda_{1*}"] \\
\pi_1(Y(R)) \arrow[r, "i_{Y1*}"] & \pi_1(\mathfrak{I}(R))
\end{tikzcd}.$$

Proof: With sufficient $N \in \mathbb{N}$ for the direct limits involved, we show

$$i_{Y1*}(f_N(x_{ij}^N(a))) = [e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))] = [\lambda(e_{ij}^N(a))] = [\lambda(\phi(x_{ij}^N(a)))],$$

where F is a standard, admissible filtration of $(R^N, e_{ij}^N(a))$ (hence a standard, admissible filtration of $(R^N, 1_N)$ and $(R^N, e_{ij}^N(a)|1_N)$ as well). We do this directly in $\mathfrak{I}_N$: that is, for

$$x_1 = (e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a))))$$

and

$$x_2 = (e_{ij}^N(a); s_0\mathfrak{i}_N(e_{ij}^N(a)), s_1\mathfrak{i}_N(e_{ij}^N(a)))$$

we have $x_1 \sim x_2 \in \mathfrak{I}_{N1}$ via

$$y = (e_{ij}^N(a)|1_N : s_0X_1(F(R^N, e_{ij}^N(a))), s_1X_1(F(R^N, e_{ij}^N(a))), X_2(F(R^N, e_{ij}^N(a)|1_N)))$$

$$= (e_{ij}^N(a)|1_N; h_0, h_1, h_2).$$

Indeed, we have $d_0x_1 = d_0x_2 = (R^N; 1)$ and $d_1x_1 = d_1x_2 = (R^N; 1)$; also,

$$d_1h_1 = X_1(F(R^N, e_{ij}^N(a))) = d_1h_0 = d_2h_2 = d_2h_1$$

and $d_3h_2 = \mathfrak{i}_{N2}(e_{ij}^N(a))$. Therefore $y \in \mathfrak{I}_{N2}$ and we calculate

$$d_0y = (1_N; d_0h_1, d_0h_2) = (1_N; 1, X_1(F(R^N, 1_N))) = (1_N; 1, 1) = s_0d_0x_1 = s_0d_0x_2,$$

$$d_1y = (1_N \circ e_{ij}^N(a); d_2h_0, d_1h_2) = (e_{ij}^N(a); 1, X_1(F(R^N, e_{ij}^N(a)))) = x_1,$$

and

$$d_2y = (e_{ij}^N(a); d_3h_0, d_3h_1) = (e_{ij}^N(a); s_0\mathfrak{i}_{N1}(e_{ij}^N(a)), s_1\mathfrak{i}_{N1}(e_{ij}^N(a))) = x_2.$$

Therefore y is a homotopy from x_1 to x_2 by Definition 1.3.1 of Chapter 1. Extending this from generators to groups, it follows that $i_{Y1*} \circ f = \lambda_* \circ \phi$ and the diagram commutes.

□

We note from this that by definition of the homotopy equivalences we have $\lambda_{1*} := (\partial^*)_*^{-1}$. Also, by Theorem 1.2.6 from Chapter 3, it makes sense that the rightmost square of the diagram with the long exact sequences should commute up to a sign at worst. However, we have not confirmed this explicitly yet and this data is not needed for our final result.

Theorem 1.0.2 *The map $f : St(R) \to \pi_1(Y(R))$ from Lemma 4.3.4 of Chapter 5 is an isomorphism.*

Proof: We argue in a parallel fashion to Nenashev ([13], page 230), although we are careful to notice that our homotopy fiber $Y(R)$, derived from $G(\mathfrak{s}.\mathcal{P}R)$, is not the same as the one Nenashev derived from $\mathcal{G}.\mathcal{P}R$.

Nevertheless, similar properties hold: we note that as induced maps from weak homotopy equivalences we have $\lambda_{1*}^{-1} = (\partial_1^*)_*$, so that from Theorem 1.0.1 the diagram

$$\begin{tikzcd}
St(R) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & E(R) \\
& \pi_1(Y(R)) \arrow[ur, "(\partial_1^*)_* \circ i_{Y1*}"'] &
\end{tikzcd}$$

makes sense and commutes.

From [10] and [1] we recognize that Quillen's +-construction has a *Universal Property*: if $\widetilde{Y}(R)$ is the topological homotopy fiber identified with our simplicial homotopy fiber $Y(R)$, then there is a topological homotopy fibration (see [6])

$$\widetilde{Y}(R) \xrightarrow{\mathfrak{j}} |N(GL(R))| \xrightarrow{+} |N(GL(R))|^+,$$

and there is a continuous function $\hat{f}$ for which, up to homotopy, the diagram

$$\begin{tikzcd}
\widetilde{Y}(R) \arrow[r, "\mathfrak{j}"] \arrow[d, "\hat{f}"] & {|N(GL(R))|} \arrow[r, "+"] \arrow[d, "="] & {|N(GL(R))|^+} \arrow[d, "|p_1\circ\lambda_1|^+"] \\
{|Y(R)|} \arrow[r, "|\partial_1^*\circ i_{Y1}|"] & {|N(GL(R))|} \arrow[r, "|p_1\circ\lambda_1|\circ +"] & {|G(\mathfrak{s}.\mathcal{P}R)|^+}
\end{tikzcd}$$

commutes. [10] also gives functoriality of geometric realization, as well as the +-construction, so that $|p_1 \circ \lambda_1|$ is a weak homotopy equivalence. It follows that $\hat{f}$ must also be a weak homotopy equivalence. Thus

$$\hat{f}_* : \pi_1(\widetilde{Y}(R)) \to \pi_1(|Y(R)|) := \pi_1(Y(R))$$

is an isomorphism.

The work of Loday and Suslin (as reflected in [10]) tells us that there is an isomorphism $\theta : \pi_1(\widetilde{Y}(R)) \to St(R)$ for which the diagram

$$\begin{tikzcd}
St(R) \arrow[rr, "\phi"] & & GL(R) \\
& \pi_1(\widetilde{Y}(R)) \arrow[ul, "\theta"] \arrow[ur, "\mathfrak{j}_*"'] &
\end{tikzcd}$$

commutes. We claim that because of this, the diagram

$$\begin{tikzcd}
St(R) \arrow[rr, "\theta\circ\hat{f}_*^{-1}\circ f"] \arrow[dr, "\phi"'] & & St(R) \arrow[dl, "\phi"] \\
& E(R) &
\end{tikzcd}$$

also commutes. Indeed, by the diagrams we have so far,

$$\phi \circ \theta = \mathfrak{j}_* = (\partial_1^*)_* \circ i_{Y1*} \circ \hat{f}_*,$$

so now

$$\phi \circ \theta \circ \hat{f}_*^{-1} = (\partial_1^*)_* \circ i_{Y1*},$$

hence by Theorem 1.0.1

$$\phi \circ \theta \circ \hat{f}_*^{-1} \circ f = (\partial_1^*)_* \circ i_{Y1*} \circ f = (\partial_1^*)_* \circ \lambda_{1*} \circ \phi = \phi.$$

Now [8] tells us that the only endomorphism of $St(R)$ for which the last diagram above commutes is the identity endomorphism on $St(R)$. Thus

$$\theta \circ \hat{f}_*^{-1} \circ f = id_{St(R)},$$

in which case the map f must be an isomorphism. In fact, from a topological standpoint we can now write $f := \hat{f}_* \circ \theta^{-1}$.

□

From Lemma 3.1.4 of Chapter 5 we know that $\lambda_* : \pi_1(N(GL(R))) \to \pi_1(\mathfrak{I}(R))$ is an isomorphism, and from Example 5.1.3 of Chapter 1 we know that $\pi_1(N(GL(R))) := GL(R)$, so that $\lambda_{1*} : GL(R) \approx \pi_1(\mathfrak{I}(R))$. Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & K_2(R) \arrow[r, "i_K"] \arrow[d, dashed, "\tilde{f}"] & St(R) \arrow[r, "\phi"] \arrow[d, "f"] & GL(R) \arrow[d, "\lambda_{1*}"] \\
1 \arrow[r] & \pi_2(G(\mathfrak{s}.\mathcal{P}R)) \arrow[r, "d_\sharp"] & \pi_1(Y(R)) \arrow[r, "i_{Y*}"] & \pi_1(\mathfrak{I}(R))
\end{tikzcd}$$

Using the isomorphisms f and λ_*, we construct the map $\tilde{f}$ via diagram-chasing, then show that it is an isomorphism. Note that since $\pi_2(\mathfrak{I}(R)) = 1$, from Definition 3.2.3 of Chapter 1 we know $d_\sharp : \pi_2(G(\mathfrak{s}.\mathcal{P}R)) \to \pi_1(Y(R))$ must be injective.

Given $z \in K_2(R)$, the inclusion map i_K makes $z \in St(R)$. By Theorem 1.0.1,

$$i_{Y*}(f(z)) = \lambda_{1*}(\phi(z)).$$

But by exactness,

$$\lambda_{1*}(\phi(z)) = \lambda_{1*}(\phi(i_K(z))) = \lambda_{1*}(1) = 1$$

since $i_K(z) \in ker(\phi)$. Therefore $f(z) \in ker(i_{Y*})$. But $ker(i_{Y*}) = im(d_\sharp)$ by exactness, so there must be an element $v \in \pi_2(G(\mathfrak{s}.\mathcal{P}R))$ for which $f(z) = d_\sharp(v)$. Therefore we set

$$\tilde{f}(z) := v.$$

Since $d_\sharp$ is injective as noted, this element v must be uniquely assigned for z, in which case $\tilde{f}$ is well-defined. Suppose that $\tilde{f}(z_1 z_2) = y \in \pi_2(G(\mathfrak{s}.\mathcal{P}R))$. Then

$$d_\sharp(y) = f(z_1 z_2) = f(z_1)f(z_2) = d_\sharp(y_1)d_\sharp(y_2)$$

for some $y_1, y_2 \in \pi_2(G(\mathfrak{s}.\mathcal{P}R))$, and we have $\tilde{f}(z_1) = y_1$ and $\tilde{f}(z_2) = y_2$. But also

$$f(z_1)f(z_2) = d_\sharp(y_1)d_\sharp(y_2) = d_\sharp(y_1y_2),$$

so now $d_\sharp(y) = d_\sharp(y_1y_2)$, in which case $y = y_1y_2$ since $d_\sharp$ is injective. Therefore

$$\tilde{f}(z_1z_2) = \tilde{f}(z_1)\tilde{f}(z_2)$$

and $\tilde{f}$ is a homomorphism. Now suppose $\tilde{f}(z_1) = y_1$, $\tilde{f}(z_2) = y_2$ and $y_1 = y_2$. Then $d_\sharp(y_1) = f(z_1)$, $d_\sharp(y_2) = f(z_2)$ and $d_\sharp(y_1) = d_\sharp(y_2)$, hence $f(z_1) = f(z_2)$. But f is an isomorphism, so it follows that $z_1 = z_2$ and thus $\tilde{f}$ is injective.

Since f is an isomorphism, if $v \in \pi_2(G(\mathfrak{s}.\mathcal{P}R))$ then there is an element $q \in St(R)$ for which $f(q) = d_\sharp(v) \in \pi_1(Y(R))$ (we must show that $q \in ker(\phi) := K_2(R)$). By Theorem 1.0.1 and exactness we have

$$\lambda_{1*}(\phi(q)) = i_{Y*}(f(q)) = i_{Y*}(d_\sharp(v)) = 1.$$

Thus $\lambda_{1*}(\phi(q)) = 1$, hence $\phi(q) = 1$ since λ_{1*} is an isomorphism and it follows that $q \in ker(\phi) := K_2(R)$, so that $\tilde{f}$ is surjective. We have now proven our concluding result:

Theorem 1.0.3 *The map $\tilde{f} : K_2(R) \to \pi_2(G(\mathfrak{s}.\mathcal{P}R))$ defined by*

$$\tilde{f}(z) = v \text{ such that } d_\sharp(v) = f(z) \text{ in the diagram}$$

$$\begin{tikzcd}
1 \arrow[r] & K_2(R) \arrow[r, "i_K"] \arrow[d, dashed, "\tilde{f}"] & St(R) \arrow[r, "\phi"] \arrow[d, "f"] & GL(R) \arrow[d, "\lambda_{1*}"] \\
1 \arrow[r] & \pi_2(G(\mathfrak{s}.\mathcal{P}R)) \arrow[r, "d_\sharp"] & \pi_1(Y(R)) \arrow[r, "i_{Y*}"] & \pi_1(\mathfrak{I}(R))
\end{tikzcd}$$

is an isomorphism.

Bibliography

[1] J. Rosenberg, Algebraic K-Theory and its Applications, Graduate Texts in Mathematics, Springer-Verlag, New York, 1994.

[2] J. P. May, Simplicial Objects in Algebraic Topology, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, 1967.

[3] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, Progress in Mathematics, Volume 174, Birkhauser-Verlag, Boston, 1990.

[4] G. Segal, Configuration Spaces and Iterated-loop Spaces, Inventiones Math., Volume 21(1973), pg 213-221, Springer-Verlag.

[5] G. Segal, Classifying Spaces and Spectral Sequences, I.H.E.S, Volume 34(1968), pg 105-112.

[6] J.J. Rotman, An Introduction to Algebraic Topology, Springer-Verlag, New York, 1988.

[7] F. Waldhausen, Algebraic K-Theory of Spaces, Lecture Notes on Math., Volume 1126(1985), Springer-Verlag, Berlin.

[8] J. Milnor, Introduction To Algebraic K-Theory, Annals of Math., Volume 72, Princeton University Press, 1971.

[9] C. Weibel, An Introduction to Homological Algebra, Cambridge Studies on Advanced Math., Volume 38, Cambridge University Press, 1994.

[10] C. Weibel, An Introduction to Algebraic K-Theory, Forthcoming; available at http://www.math.rutgers.edu/∼weibel/Kbook.html.

[11] J. Duflot, Simplicial Groups That Are Models for Algebraic K-Theory, Manuscripta Math., volume 113(2004), Springer-Verlag.

[12] A. Nenashev, Double Short Exact Sequences and K_1 of an Exact Category, K-Theory, volume 14(1998), Kluwer Academic Publishers, Netherlands.

[13] A. Nenashev, K_2 of a Ring via the G-Construction, K-Theory, volume 34(2005), Springer.

[14] A. Nenashev, K_1 by Generators and Relations, Journal of Pure and Applied Algebra, volume 131(1998), Elsevier.

[15] A.R. Garzon and J.G. Miranda, Homotopy Theory for Truncated Weak Equivalences of Simplicial Groups, Math. Proc. Camb. Phil. Soc., volume 121(1997), pg 51-74.

[16] D. Quillen, Higher Algebraic K-Theory, I. Lecture Notes in Mathematics, volume 341(1973), Springer-Verlag, Berlin.

[17] D. Kan, On Homotopy and c.s.s. Groups, Annals of Mathematics, volume 68(1958).

[18] D. Kan, A Combinatorial Definition of Homotopy Groups, Annals of Mathematics, volume 67(1958)

[19] J.R. Munkres, Elements of Algebraic Topology, Addison Wesley, Menlo Park, California, 1984.

[20] S. Lang, Algebra, Springer, New York, 2002.

[21] D.S. Dummit and R.M. Foote, Abstract Algebra (2nd Ed.), John Wiley and Sons, 1999.