Thesis

Matter Effects On Neutrino Oscillations

Submitted by

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Abstract

Matter Effects On Neutrino Oscillations

An introduction to neutrino oscillations in vacuum is presented, followed by a survey of various techniques for obtaining either exact or approximate expressions for $\nu_\mu \to \nu_e$ oscillations in matter. The method developed by Arafune, Koike, and Sato uses a perturbative analysis to find an approximation for the evolution operator. The method used by Freund yields an approximate oscillation probability by diagonalizing the Hamiltonian, finding the eigenvalues and eigenvectors, and then using those to find modified mixing angles with the matter effect taken into account. The method devised by Mann, Kafka, Schneps, and Altinok produces an exact expression for the oscillation by determining explicitly the evolution operator. These methods are compared to each other using the T2K, MINOS, NOνA, and LBNE parameters.

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CHAPTER 1

Motivation and Neutrino Oscillations in Vacuum

The motivation for this thesis was to provide an overview of three different methods for determining the oscillation probability in matter for $\nu_\mu \to \nu_e$. Taking into account these effects is important as the values of the mixing parameters are altered by them. The three papers studied were chosen to provide not only 3 different methods for modeling these matter effects, but to also give the reader some insight into the state of knowledge of matter effects at 3 different points in time. The first method, published in 1999 and formulated by Arafune, Koike and Sato involved a perturbative expansion of the Hamiltonian in order to find the evolution operator. This method was an early one, at which time the value of θ_{13} was not known and presumed to be extremely small. This assumption has since been shown to be invalid. The second method, published in 2001 and devised by Freund, attempts to find modified mixing parameters with the matter effect taken into account in order to use them in the standard vacuum oscillation expansion. At this time, there was a better understanding of the values of the oscillation parameters. The third method, used by Mann, Kafka, Schneps and Altinok and published in 2012, finds an exact expression for the oscillation probability. By the time of the publication of this paper, values for most of the mixing parameters had already been determined. The notation used in each chapter corresponds to the notation used in the particular papers, which is different for each one, so a table relating the variables used in that chapter to standard parameters is provided at the end of each chapter.

Neutrinos can be described by either a mass eigenstate $|\nu_i\rangle$ or a flavor eigenstate $|\nu_\alpha\rangle$. One can convert from one to the other by using a specific unitary mixing matrix:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \tag{1.1}$$

The mixing matrix is known as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. The PMNS matrix is defined as the product of the following unitary matrices:

$$U_{\alpha i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

where $s_{ij} \equiv \sin(\theta_{ij})$ and $c_{ij} \equiv \cos(\theta_{ij})$.

This is often written as $\hat{U} = \hat{R}_1(\theta_{23})\hat{I}_{\delta CP}\hat{R}_2(\theta_{13})\hat{I}_{-\delta CP}\hat{R}_3(\theta_{12})$ for short, where $\hat{I}_{\delta CP} \equiv diag(1, 1, e^{i\delta CP})$ and $\hat{I}_{-\delta CP} = \hat{I}^{\dagger}_{\delta CP}$.

Multiplying the matrices in (1.2) together, the PMNS matrix reads:

$$\hat{U} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{1.3}$$

The time-dependence of a plane wave flavor eigenstate, given the initial mass eigenstate $|\nu_i(t=0)\rangle$ is:

$$|\nu_\alpha(t>0)\rangle = \sum_i U_{\alpha i} e^{-iE_i t/\hbar} |\nu_i(t=0)\rangle \tag{1.4}$$

We can write a mass eigenstate in terms of a flavor eigenstate by using:

$$|\nu_i\rangle = \sum_\beta U_{\beta i} |\nu_\beta\rangle \tag{1.5}$$

Applying it to (1.4), we see that:

$$|\nu_\alpha(t>0)\rangle = \sum_i U_{\alpha i} e^{-iE_i t/\hbar} |\nu_i(t=0)\rangle = \sum_i \sum_\beta U_{\alpha i} e^{-iE_i t/\hbar} U_{\beta i} |\nu_\beta\rangle \tag{1.6}$$

The amplitude of a neutrino in flavor eigenstate α at t=0 being observed in eigenstate β at a later time $t > 0$ is:

$$\langle \nu_\beta | \, \nu_\alpha(t>0)\rangle = \left\langle \nu_\beta \middle| \sum_i \sum_\beta U_{\alpha i} e^{-iE_i t/\hbar} U_{\beta i} \middle| \nu_\beta \right\rangle = \sum_i U_{\alpha i} e^{-iE_i t/\hbar} U_{\beta i} \tag{1.7}$$

The corresponding probability is then:

$$| \langle \nu_\beta | \, \nu_\alpha(t>0)\rangle |^2 = | \sum_i U_{\alpha i} e^{-iE_i t/\hbar} U_{\beta i}|^2 = \sum_{i,j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} e^{-i(E_i - E_j)t} \tag{1.8}$$

Energy in relativity is can be approximated as:

$$E = (p^2 + m^2)^{1/2} \approx p + \tfrac{m^2}{2E} \text{ (for } m \ll E)$$

so we can write:

$$E_i - E_j = \tfrac{m_i^2 - m_j^2}{2E} = \tfrac{\Delta m_{ij}^2}{2E}$$

Each of the mass-squared splittings and mixing angles have been measured and listed in PDG and have the values given in Table (1.1).

Table 1.1. Measured Values of Neutrino Mixing Angles and Mass-Squared Splittings

Parameter	Value
θ_{13}	9.22^o
θ_{23}	45^o
θ_{12}	34.4^o
Δm_{21}^2	$7.59 * 10^{-5} eV^2$
Δm_{32}^2	$2.43 * 10^{-3} eV^2$
Δm_{31}^2	$2.51 * 10^{-3} eV^2$

According to Barger[1], using our mass-squared splittings, the probability becomes:

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i,j} U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} e^{-i\frac{\Delta m_{ij}^2}{2E}t} \tag{1.9}$$

According to Kayser [2], this can be rewritten as:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\textstyle\sum_{i>j} \Re[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin^2(\frac{\Delta m_{ij}^2 L}{4E})$$

$$+2\sum_{i>j} \Im[U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}] \sin(\frac{\Delta m_{ij}^2 L}{2E}) \tag{1.10}$$

According to Freund [4], expansion of (1.10) to order α^2, where $\alpha = \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$ (not to be confused with the subscript α), yields:

$$P(\nu_\alpha \to \nu_\beta) \approx P_0 + P_{\sin\delta} + P_{\cos\delta} + P_3$$

where:

$$P_0 = \sin^2(\theta_{23})\sin^2(2\theta_{13})\sin^2(\frac{\Delta m^2_{31}L}{4E}) \tag{1.11a}$$

$$P_{sin\delta} = \alpha\sin(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})\sin^3(\frac{\Delta m^2_{31}L}{4E}) \tag{1.11b}$$

$$\begin{aligned} P_{cos\delta} = {} & \alpha\cos(\delta)\cos(\theta_{13})\sin(2\theta_{12}) \\ & \sin(2\theta_{13})\sin(2\theta_{23})\cos(\frac{\Delta m^2_{31}L}{4E})\sin^2(\frac{\Delta m^2_{31}L}{4E}) \end{aligned} \tag{1.11c}$$

$$P_3 = \alpha^2\cos^2(\theta_{23})\sin^2(2\theta_{12})\sin^2(\frac{\Delta m^2_{31}L}{4E}) \tag{1.11d}$$

Because we know the mass-squared splittings $|m_i^2 - m_j^2|$ and not the mass values m_i, we do not yet know whether m_3 is significantly higher or lower than m_1 and m_2. We denote "normal" mass hierarchy if $m_3 > m_1, m_2$ and "inverted" mass hierarchy if $m_3 < m_2, m_1$.

As an example, let's calculate the probability of muon to electron neutrino oscillation using the T2K parameters in Table (1.2) and neutrino parameters in Table (1.1). Let us further assume that δ_{CP} is 0. We would then obtain a value of .05047 for the oscillation probability. So if the T2K experiment shoots 100 ν_μ neutrinos at the SK detector that is 295 km from the accelerator, they would expect to measure 5 ν_e neutrinos at SK. Parameters for three other baselines, MINOS, NOνA, and LBNE are also given in Table (1.2) and the corresponding oscillation probabilities are presented in Table (1.3). However, all of these baselines involve travel through matter and not vacuum, so interactions with the electrons in the Earth need to be accounted for. The following three chapters provide 3 different methods for measuring the adjustments needed to account for these extra interactions.

TABLE 1.2. Experimental Parameters for T2K, MINOS, NOνA, and LBNE

Experiment	Baseline (km)	Peak E_ν (GeV)	$n_e(\frac{g}{cm^3})$
T2K	295	.6	2.76
MINOS	735	4	2.76
NOνA	810	2	2.76
LBNE	1300	3	2.76

TABLE 1.3. Oscillation Probabilities for T2K, MINOS, NOνA, and LBNE for $\delta_{CP} = 0$

Experiment	Probability
T2K	0.0504768
MINOS	0.017606
NOνA	0.0487862
LBNE	0.0502134

CHAPTER 2

Arafune, Koike, and Sato (AKS) Method

The publication by J. Arafune, M. Koike, and J. Sato entitled "CP Violation and Matter Effect in Long Baseline Neutrino Oscillation Experiments"[3] provides a method for finding the time evolution operator, using perturbation theory. AKS starts out with a particular Hamiltonian and then decomposes it into an unperturbed part and a perturbed part. An attempt to solve the wave equation for the time-evolution operator using perturbation theory is made and the probability of neutrino oscillation is estimated.

We begin with a slightly modified version of the PMNS matrix, following the AKS notation:

$$U^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\psi & s_\psi \\ 0 & -s_\psi & c_\psi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{pmatrix} \begin{pmatrix} c_\omega & s_\omega & 0 \\ -s_\omega & c_\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_\phi c_\omega & c_\phi s_\omega & s_\phi \\ -c_\psi s_\omega - s_\psi s_\phi c_\omega e^{i\delta} & c_\psi c_\omega - s_\psi s_\phi s_\omega e^{i\delta} & s_\psi c_\phi e^{i\delta} \\ s_\psi s_\omega - c_\psi s_\phi c_\omega e^{i\delta} & -s_\psi c_\omega - c_\psi s_\phi s_\omega e^{i\delta} & c_\psi c_\phi e^{i\delta} \end{pmatrix} \tag{2.1}$$

where $s_\psi = \sin\psi$ and $c_\psi = \cos\psi$, etc. Table (2.1) defines the angles given in Equation (2.1) in terms of more standard notation, as well as other variables used in this chapter.

Table 2.1. Variables used in AKS

AKS Variable	Definition
a	$2^{3/2}G_F n_e E_\nu$
ψ	θ_{23}
ϕ	θ_{13}
ω	θ_{12}

The time-dependent Schrodinger Equation (TDSE) for a flavor eigenstate vector in vacuum is (using natural units, where length and time are treated equally):

$$i\frac{d\nu}{dx} = -U^{(0)}diag(p_1, p_2, p_3)U^{(0)\dagger}\nu \simeq (-p_1\hat{I} + \frac{1}{2E}U^{(0)}diag(0, \delta m_{21}^2, \delta m_{31}^2)U^{(0)\dagger})\nu \tag{2.2}$$

where p_i are the momenta of the 3 mass eigenstates, $\delta m_{ij}^2 \equiv m_i^2 - m_j^2$ are the mass squared splittings of the different neutrinos and E is the energy. The second line of (2.2) is calculated by expanding the relativistic energy formula about small mass m. We can neglect the $-p_1\hat{I}$ term since it just gives an overall global phase.

The TDSE for a flavor eigenstate vector in matter is given by the similar formula:

$$i\frac{d\nu}{dx} = H\nu \tag{2.3}$$

where $H = \frac{1}{2E}Udiag(\mu_1^2, \mu_2^2, \mu_3^2)U^\dagger$.

The mixing matrix U and the masses μ_i are given by:

$$U\begin{pmatrix}\mu_1^2 & 0 & 0\\ 0 & \mu_2^2 & 0\\ 0 & 0 & \mu_3^2\end{pmatrix}U^\dagger = U^{(0)}\begin{pmatrix}0 & 0 & 0\\ 0 & \delta m_{21}^2 & 0\\ 0 & 0 & \delta m_{31}^2\end{pmatrix}U^{(0)\dagger} + \begin{pmatrix}a & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{2.4}$$

where $a \equiv 2^{3/2} G_F n_e E$, G_F is the Fermi coupling constant and n_e is the electron density. The solution of equation (2.3) is:

$$\nu(x) = S(x)\nu(0) \tag{2.5}$$

where $S(x) \equiv Te^{-iHx}$, assuming that the matter density is independent of position and time. The oscillation probability $P(\nu_\alpha \to \nu_\beta; L)$ is then just $|S_{\alpha\beta}(L)|^2$.

If we assume that both a and δm_{21}^2 are very small compared to δm_{31}^2, we can proceed with the following peturbative analysis:

We can separate H into two parts, a main part H_0:

$$H_0 = \frac{1}{2E}U^{(0)}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \delta m_{31}^2\end{pmatrix}U^{(0)\dagger} \tag{2.6}$$

and a perturbation H_1:

$$H_1 = \frac{1}{2E}(U^{(0)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^{(0)\dagger} + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) \tag{2.7}$$

If we make the following substitutions:

$$\Omega(x) = e^{iH_0 x} S(x) \text{ and } H_1(x) = e^{iH_0 x} H_1 e^{-iH_0 x},$$

we can write for our TDSE:

$$i\frac{d\Omega}{dx} = H_1(x)\Omega(x) \tag{2.8}$$

with

$$\Omega(0) = 1$$

If we assume $\frac{ax}{2E} \ll 1$ and $\frac{\delta m_{21}^2 x}{2E} \ll 1$, then we can obtain, as an approximate solution for our TDSE:

$$\Omega(x) \simeq 1 - i\int_0^x H_1(s)\, ds. \tag{2.9}$$

This, combined with our definition of $\Omega(x)$ yields:

$$S(x) \simeq e^{-iH_0x} + e^{-iH_0x}(-i)\int_0^x H_1(s)\,ds \tag{2.10}$$

If we call the first term $S_0(x)$ and the second term $S_1(x)$, it can be shown that:

$$S_0(x)_{\beta\alpha} = \delta_{\alpha\beta} + U^{(0)}_{\beta 3}U^{(0)*}_{\alpha 3}(e^{-i\frac{\delta m^2_{31}x}{2E}} - 1) \tag{2.11}$$

$$\begin{aligned} S_1(x)_{\beta\alpha} = &\ -iU^{(0)}_{\beta i}U^{(0)*}_{\gamma i}(H_1)_{\gamma\delta}U^{(0)}_{\delta j}U^{(0)*}_{\alpha j}(\delta_{i3}\delta_{j3}xe^{-i\frac{\delta m^2_{31}x}{2E}} \\ &+ ((1-\delta_{i3})\delta_{j3} + \delta_{i3}(1-\delta_{j3}))(-i\frac{\delta m^2_{31}x}{2E})^{-1} \\ &\times (e^{-i\frac{\delta m^2_{31}x}{2E}} - 1) + (1-\delta_{i3})(1-\delta_{j3})x) \end{aligned} \tag{2.12}$$

If we invoke the following identities:

$$\begin{aligned} U^{(0)*}_{\gamma i}(H_1)_{\gamma\delta}U^{(0)}_{\delta j} &= \tfrac{1}{2E}(diag(0,\delta m^2_{21},0) + U^{(0)\dagger}diag(a,0,0)U^{(0)})_{ij} \\ &= \frac{\delta m^2_{21}}{2E}\delta_{i2}\delta_{j2} + \frac{a}{2E}U^{(0)*}_{1i}U^{(0)}_{1j} \end{aligned} \tag{2.13}$$

and

$$\sum_{k=1}^{2} U_{\alpha k}^{(0)*} U_{1k}^{(0)} = \delta_{\alpha 1} - U_{\alpha 3}^{(0)*} U_{13}^{(0)} \tag{2.14}$$

we can extract the T-matrix from the S-matrix:

$$S(x)_{\beta\alpha} = \delta_{\beta\alpha} + iT(x)_{\beta\alpha} \tag{2.15}$$

where

$$iT(x) = -2ie^{\frac{-i\delta m_{31}^2 x}{4E}} sin(\tfrac{\delta m_{31}^2 x}{4E}) U_{\beta 3}^{(0)} U_{\alpha 3}^{(0)*} [1 - \tfrac{a}{\delta m_{31}^2}(2|U_{13}^{(0)}|^2 - \delta_{\alpha 1} - \delta_{\beta 1}) - i\tfrac{ax}{2E}|U_{13}^{(0)}|^2]$$
$$-i\tfrac{\delta m_{31}^2 x}{2E}[\tfrac{\delta m_{21}^2}{\delta m_{31}^2} U_{\beta 2}^{(0)} U_{\alpha 2}^{(0)*} + \tfrac{a}{\delta m_{31}^2}[\delta_{\alpha 1}\delta_{\beta 1}|U_{13}^{(0)}|^2 + U_{\beta 3}^{(0)} U_{\alpha 3}^{(0)*}(2|U_{13}^{(0)}|^2 - \delta_{\alpha 3} - \delta_{\beta 3})]]$$

To lowest order, the probability of a muon neutrino oscillating into an electron neutrino can be shown to be:

$$\begin{aligned} P(\nu_\mu \to \nu_e; L) = {} & 4sin^2(\frac{\delta m_{31}^2 L}{4E}) c_\phi^2 s_\phi^2 s_\psi^2 [1 + \frac{a}{\delta m_{31}^2} \times 2(1 - 2s_\phi^2)] \\ & + 2\frac{\delta m_{31}^2 L}{2E} sin(\frac{\delta m_{31}^2 L}{2E}) c_\phi^2 s_\phi s_\psi [-\frac{a}{\delta m_{31}^2} s_\phi s_\psi (1 - 2s_\phi^2) \\ & + \frac{\delta m_{21}^2}{\delta m_{31}^2} s_\omega (-s_\phi s_\psi s_\omega + c_\delta c_\psi c_\omega)] \\ & - 4\frac{\delta m_{21}^2 L}{2E} sin^2(\frac{\delta m_{31}^2 L}{4E}) s_\delta c_\phi^2 s_\phi c_\psi s_\psi c_\omega s_\omega \end{aligned} \tag{2.16}$$

If we assume the parameters given in Table (1.2) and $\delta_{CP} = 0$, we find the following oscillation probabilities for each baseline, using AKS' formula:

Table 2.2. Oscillation Probabilities for T2K, MINOS, NOνA, and LBNE for $\delta_{CP} = 0$

Experiment	Probability
T2K	0.0579066
MINOS	0.0176483
NOνA	0.0534306
LBNE	0.0558083

CHAPTER 3

Martin Freund Method: Eigenvectors of the PMNS Matrix

The calculation used by M. Freund in "Analytic Approximations for Three Neutrino Oscillation Parameters and Probabilities in Matter"[4] determines the oscillation probability in matter by finding modified mixing parameters, with the matter effects taken into account, and substituting them into Barger's approximate oscillation probability for the vacuum case given in the introduction. This is accomplished by directly finding the eigenvalues and eigenvectors of the PMNS matrix with matter effects included, and then comparing with the vacuum PMNS matrix. As a reminder, the PMNS Matrix is:

$$\hat{U}_{(mix)} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{3.1}$$

According to Freund, the oscillation probability in vacuum is given by:

$$P(\nu_{e_l} \to \nu_{e_m}) = \delta_{lm} - 4\sum_{i>j} \Re J_{ij}^{lm} \sin^2(\hat{\Delta}_{ij}) - 2\sum_{i>j} \Im J_{ij}^{lm} \sin(2\hat{\Delta}_{ij}) \tag{3.2}$$

where $J_{ij}^{lm} = U_{li}U_{lj}^{*}U_{mi}^{*}U_{mj}$. These and other variables used by Freund are summarized in Table (3.1).

Table 3.1. Variables used in Freund

Freund Variable	Definition
α	$\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$
Δ	m^2_{31}
$\hat{\Delta}$	$\frac{\Delta L}{4E}$
A	$2^{3/2} G_F n_e E_\nu$
$\hat{A}$	$\frac{A}{\Delta}$
$\hat{C}$	$((\hat{A} - cos(2\theta_{13}))^2 + sin^2(2\theta_{13}))^{1/2}$

This oscillation probability for neutrinos in vacuum can be approximated by:

$$P(\nu_e \rightarrow \nu_\mu) = P_0 + P_{\sin\delta} + P_{\cos\delta} + P_3 \tag{3.3}$$

where

$$P_0 = \sin^2(\theta_{23}) \sin^2(2\theta_{13}) \sin^2 \hat{\Delta},$$

$$P_{\sin\delta} = \alpha \sin(\delta) \cos(\theta_{13}) \sin(2\theta_{13}) \sin(2\theta_{23})) \sin^3 \hat{\Delta},$$

$$P_{cos\delta} = \alpha \cos(\delta) \cos(\theta_{13}) \sin(2\theta_{12}) \sin(2\theta_{13}) \sin(2\theta_{23}) \cos \hat{\Delta} \sin^2 \hat{\Delta},$$

$$P_3 = \alpha^2 \cos^2 \theta_{23} \sin^2 2\theta_{12} \sin^2 \hat{\Delta}$$

$$\Delta = \Delta m^2_{31},\ \alpha\Delta = \Delta m^2_{21},\ \text{and}\ \hat{\Delta} = \tfrac{\Delta L}{4E}$$

The full Hamiltonian with matter effects is:

$$H = \frac{1}{2E}[U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}] \tag{3.4}$$

where $U = U_{23}(\theta_{23})U_{13}(\theta_{23}, \delta)U_{12}(\theta_{12})$, and $A = 2^{3/2}G_F n_e E_\nu$. G_F is the Fermi coupling constant, n_e is the electron density in matter and E_ν is the neutrino beam energy.

By extracting $m_1^2\hat{I}$ from (3.4) and using the relations:

$$U_\delta^\dagger U_{13}(\theta_{13}, \delta)U_\delta = U_{13}(\theta_{13}, 0),$$

$$U_\delta^\dagger U_{12}(\theta_{12})U_\delta = U_{12}(\theta_{12}), \text{ and}$$

$$U_\delta^\dagger \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} U_\delta = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where

$$U_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}$$

it can be shown that:

$$H = \frac{\Delta}{2E} U_{23} U_\delta [U_{13}(\theta_{13}, 0)U_{12} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} U_{12}^\dagger U_{13}(\theta_{13}, 0)^\dagger + \begin{pmatrix} \frac{A}{\Delta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}]\, U_\delta^\dagger U_{23}^\dagger \quad (3.5)$$

where we shall denote the term in brackets as M.

Let's now diagonalize M with $\hat{U} = U_{23}(\hat{\theta}_{23})U_{13}(\hat{\theta}_{13})U_{12}(\hat{\theta}_{12})$, with eigenvalues λ_i. Then:

$$H = \frac{\Delta}{2E} U_{23} U_\delta \hat{U} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \hat{U}^\dagger U_\delta^\dagger U_{23}^\dagger = \frac{\Delta}{2E} U' \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U'^\dagger$$

gives us the mixing matrix U' in matter, where:

$$U' = U_{23}(\theta_{23}) U_\delta U_{13}(\hat{\theta}_{13}) U_{12}(\hat{\theta}_{12}).$$

This has the same form as the vacuum mixing matrix.

To bring U' to standard parameterized form, with $U' = U(\theta'_{23})U_{13}(\hat{\theta}, \delta')U_{12}(\hat{\theta}_{12})$, we can make the matrix:

$$U_{23}(\theta_{23}) U_\delta U(\hat{\theta}_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C & S \\ 0 & -e^{i\delta} S^* & e^{i\delta} C^* \end{pmatrix} \tag{3.6}$$

where:

$$C = \cos(\theta_{23})\cos(\hat{\theta}_{23}) - e^{i\delta}\sin(\theta_{23})\sin(\hat{\theta}_{23})$$

and

$$S = \cos(\theta_{23})\sin(\hat{\theta}_{23}) + e^{i\delta}\sin(\theta_{23})\cos(\hat{\theta}_{23})$$

real by introducing the following phase rotations:

$$\beta = -argC,\, \gamma = argS,\, \gamma' = argC - argS$$

This gives us:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\beta} & 0 \\ 0 & 0 & -e^{i(-\delta-\gamma)} \end{pmatrix} U_{23}(\theta_{23}) U_\delta U_{23}(\hat{\theta}_{23}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & |C| & |S| \\ 0 & -|S| & |C| \end{pmatrix} \tag{3.7}$$

From this, we can write U' as:

$$U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & -e^{i(\delta+\gamma)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & |C| & |S| \\ 0 & -|S| & |C| \end{pmatrix} U_{\delta'} U_{13}(\hat{\theta}_{13}) U^\dagger_{\delta'} U_{12}(\hat{\theta}_{12}) U_{\delta'} \tag{3.8}$$

By absorbing the phase rotations into the other matrices, we are left with U' in standard parameterized form.

From this, we can read off the conversions from the diagonalization matrix angles to the modified matter effect-corrected angles:

$$\theta'_{13} = \hat{\theta}_{13},$$

$$\theta'_{12} = \hat{\theta}_{12},$$

$$\sin^2(\theta'_{23}) = \cos^2(\theta_{23})\sin^2(\hat{\theta}_{23}) + \sin^2(\theta_{23})\cos^2(\hat{\theta_{23}})$$

$$+2\cos(\delta)\sin(\theta_{23})\cos(\theta_{23})\sin(\hat{\theta}_{23})\cos(\hat{\theta}_{23}),$$

$$\sin(\delta') = \sin(\delta)\tfrac{\sin(2\theta_{23})}{\sin(2\theta'_{23})}$$

This yields the matrix M, defined as the term in brackets in equation (3.5):

$$M = \begin{pmatrix} s_{13}^2 + \hat{A} + \alpha c_{13}^2 s_{12}^2 & \alpha s_{12} c_{12} c_{13} & s_{13} c_{13} - \alpha s_{13} c_{13} s_{12}^2 \\ \alpha s_{12} c_{12} c_{13} & \alpha c_{12}^2 & -\alpha s_{12} c_{12} s_{13} \\ s_{13} c_{13} - \alpha s_{13} c_{13} s_{12}^2 & -\alpha s_{12} c_{12} s_{13} & c_{13}^2 + \alpha s_{12}^2 s_{13}^2 \end{pmatrix} \tag{3.9}$$

where $\hat{A} \equiv \frac{A}{\Delta}$

We can now obtain the eigenvalues and eigenvectors of M. To first order, the eigenvalues are:

$$\lambda_1 = \frac{1}{2}(\hat{A} + 1 - \hat{C}) + \alpha\frac{(\hat{C} + 1 - \hat{A}\cos(2\theta_{13}))\sin^2\theta_{12}}{2\hat{C}} \tag{3.10}$$

$$\lambda_2 = \alpha\cos^2(\theta_{12}) \tag{3.11}$$

$$\lambda_3 = \frac{1}{2}(\hat{A} + 1 + \hat{C}) + \alpha\frac{(\hat{C} - 1 + \hat{A}\cos(2\theta_{13}))\sin^2\theta_{12}}{2\hat{C}} \tag{3.12}$$

where $\hat{C} = ((\hat{A} - \cos(2\theta_{12}))^2 + \sin^2(2\theta_{13}))^{1/2}$

To first order, then, the corresponding eigenvectors are:

$$v_1 = \begin{pmatrix} \frac{\sin(2\theta_{13})}{(2\hat{C}(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} - \frac{\alpha\hat{A}\sin^2(\theta_{12})\sin^2(2\theta_{13})}{2\hat{C}(2\hat{C}^2(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} \\ \frac{\alpha(1+\hat{A}-\hat{C})\sin(2\theta_{12})\sin(\theta_{13})}{(1+\hat{A}+\hat{C})(2\hat{C}(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} \\ \frac{-\sin(2\theta_{13})}{(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} - \frac{\alpha\hat{A}\sin^2(\theta_{12})\sin^2(2\theta_{13})}{2\hat{C}(2\hat{C}^2(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} \end{pmatrix} \tag{3.13}$$

$$v_2 = \begin{pmatrix} \frac{-\alpha\cos(\theta_{12})\sin(\theta_{12})}{\hat{A}\cos(\theta_{13})} \\ 1 \\ \frac{\alpha(1+\hat{A})\cos(\theta_{12})\sin(\theta_{12})\sin(\theta_{13})}{\hat{A}\cos^2(\theta_{13})} \end{pmatrix} \tag{3.14}$$

$$v_3 = \begin{pmatrix} \frac{\sin(2\theta_{13})}{(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} + \frac{\alpha\hat{A}\sin^2(\theta_{12})\sin^2(2\theta_{13})}{2\hat{C}(2\hat{C}^2(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} \\ \frac{\alpha(1+\hat{A}-\hat{C})\sin(2\theta_{12})\sin(\theta_{13})}{(1+\hat{A}+\hat{C})(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} \\ \frac{\sin(2\theta_{13})}{(2\hat{C}(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} - \frac{\alpha\hat{A}\sin^2(\theta_{12})\sin^2(2\theta_{13})}{2\hat{C}(2\hat{C}^2(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} \end{pmatrix} \tag{3.15}$$

Using our eigenvectors, we can construct $\hat{U}$. The first order of business is to identify the correct order of the eigenvectors. According to Freund, when $\hat{A} < cos(2\theta_{13})$, which is the case for both T2K and LBNE, the correct order is:

$$\hat{U} = (v_1 v_2 v_3)^T \tag{3.16}$$

Next, we must bring U' to a form consistent with the standard parameterization. As an example, we will now examine the case of $\hat{A} < 0$.

By looking at the $(\mu,3)$ element of $\hat{U}$, it can be seen that the matter perturbation angle $\hat{\theta}_{23}$ will be of order α. Also, by looking at the $(e,2)$ element of $\hat{U}$, it can be seen that the matter perturbation angle $\hat{\theta}_{12}$ is also of order α.

If we make the following replacements:

$$\hat{s}_{12} = \alpha\hat{s}_{12}^{(\alpha)},\ \hat{s}_{23} = \alpha\hat{s}_{23}^{(\alpha)}, \hat{s}_{13} = \hat{s}_{13}^{(0)} + \alpha\hat{s}_{23}^{(\alpha)}$$

and assume that θ_{13} is very close to 0, we can write $\hat{U}$ as:

$$\hat{U} = \begin{pmatrix} \hat{c}_{13} & \alpha\hat{c}_{13}^{(0)}\hat{s}_{12}^{(\alpha)} & \hat{s}_{13} \\ -\alpha(\hat{s}_{12}^{(\alpha)} + \hat{s}_{13}^{(0)}\hat{s}_{23}^{(\alpha)}) & 1 & \alpha\hat{c}_{13}^{(0)}\hat{s}_{23}^{(\alpha)} \\ -\hat{s}_{13} & -\alpha(\hat{s}_{12}^{(\alpha)}\hat{s}_{13}^{(0)} + \hat{s}_{23}^{(\alpha)}) & \hat{c}_{13} \end{pmatrix} \tag{3.17}$$

From U_{e3}, $U_{\mu 3}$, and $U_{\tau 3}$, we can directly read off $\sin(\hat{\theta}_{13})$ and $\sin(\hat{\theta}_{23})$:

$$\sin(\hat{\theta}_{13}) = \frac{\sin(2\theta_{13})}{(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} + \frac{\alpha\hat{A}\sin^2(\theta_{12})\sin^2(2\theta_{13})}{2\hat{C}(2\hat{C}^2(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} \tag{3.18}$$

$$\sin(\hat{\theta}_{23})\frac{\alpha(1+\hat{A}-\hat{C})\sin(2\theta_{12})\sin(\theta_{13})}{2(1-\hat{A}+\hat{C})\cos^2(\theta_{13})} \tag{3.19}$$

To find $\sin(\hat{\theta}_{12})$, we need to separate $\hat{\theta}_{23}$ from $\hat{U}$. The remainder of $\hat{U} = U_{23}^T(\hat{\theta_{23}})\hat{U}'$ needs to be brought to the form:

$$\begin{pmatrix} \hat{c}_{13} & \alpha\hat{c}_{13}^{(0)}\hat{s}_{12}^{(\alpha)} & \hat{s}_{13} \\ -\alpha\hat{s}_{12}^{(\alpha)} & 1 & 0 \\ -\hat{s}'_{13} & -\alpha\hat{s}_{12}^{(\alpha)}\hat{s}_{13}^{(0)} & \hat{c}_{13} \end{pmatrix} \tag{3.20}$$

The angle $\hat{\theta}_{12}$ can then be read off from $\hat{U}'_{\mu 1}$:

$$\sin(\hat{\theta}_{12}) = -\frac{\alpha\hat{C}\sin(2\theta_{12})}{\hat{A}\cos(\theta_{13})(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} \tag{3.21}$$

We can now assemble formulas to convert from the vacuum angles to the modified angles:

$$\sin(\theta'_{13}) = \frac{\sin(2\theta_{13})}{(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} + \frac{\alpha\hat{A}\sin^2(\theta_{12})\sin^2(2\theta_{13})}{2\hat{C}(2\hat{C}^2(\hat{A}+\hat{C}-\cos(2\theta_{13})))^{1/2}} \tag{3.22}$$

$$\sin(\theta'_{12}) = -\frac{\alpha\hat{C}\sin(2\theta_{12})}{|\hat{A}|\cos(\theta_{13})(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13})))^{1/2}} \tag{3.23}$$

$$sin(\theta'_{23}) = sin(\theta_{23}) + \alpha cos(\delta)\frac{\hat{A}sin(2\theta_{12})sin(\theta_{13})cos(\theta_{23})}{1+\hat{C}-\hat{A}cos(2\theta_{13})} \tag{3.24}$$

$$\sin(\delta') = \sin(\delta)(1-\alpha\frac{\cos(\delta)}{\tan(2\theta_{23})}\frac{2\hat{A}\sin(2\theta_{12})\sin(\theta_{13})}{1+\hat{C}-\hat{A}\cos(2\theta_{13})}) \tag{3.25}$$

These other expressions can also be derived from the above parameter mapping:

$$\sin^2(2\theta'_{13}) = \frac{\sin^2(2\theta_{13})}{\hat{C}^2} + \alpha\frac{2\hat{A}(-\hat{A}+\cos(2\theta_{13})\sin^2\theta_{12}\sin^2(2\theta_{13})}{\hat{C}^4} \tag{3.26}$$

$$\sin(2\theta'_{12}) = \alpha\frac{2\hat{C}\sin(2\theta_{12})}{|\hat{A}|\cos(\theta_{13})(2\hat{C}(-\hat{A}+\hat{C}+\cos(2\theta_{13}))^{1/2}} \tag{3.27}$$

$$\sin(2\theta'_{23}) = \sin(2\theta_{23}) + \alpha\cos(\delta)\frac{2\hat{A}\sin(2\theta_{12})\sin(\theta_{13})\cos(2\theta_{23})}{1+\hat{C}-\hat{A}\cos(2\theta_{13})} \tag{3.28}$$

We also obtain for the mass squared differences, with the order again stipulated by Freund:

$$(\Delta m'^2_{21}, \Delta m'^2_{31}, \Delta m'^2_{32}) = (\Delta m^2_3, \Delta m^2_2, \Delta m^2_1) \tag{3.29}$$

with $\Delta m'^2_1 = \Delta(\lambda_3 - \lambda_2)$, $\Delta m'^2_2 = \Delta(\lambda_3 - \lambda_1)$, and $\Delta m'^2_3 = \Delta(\lambda_2 - \lambda_1)$

We can now write (to second order):

$$\begin{aligned}\Re J'^{e\mu}_{12} = & -\cos(\delta')\sin(\theta'_{12})\cos^2(\theta'_{13})\sin(\theta'_{13})\cos(\theta'_{23})\sin(\theta'_{23}) \\ & -\sin^2(\theta'_{12})\cos^2(\theta'_{23})\end{aligned} \tag{3.30}$$

$$\begin{aligned}\Re J'^{e\mu}_{13} = & -\cos(\delta')\sin(\theta'_{12})\cos^2(\theta'_{13})\sin(\theta'_{13})\cos(\theta'_{23})\sin(\theta'_{23}) \\ & -\sin^2(2\theta'_{13})\sin^2(\theta'_{23})\end{aligned} \tag{3.31}$$

$$\Re J'^{e\mu}_{23} = \cos(\delta')\sin(\theta'_{12})\cos^2(\theta'_{13})\sin(\theta'_{13})\cos(\theta'_{23})\sin(\theta'_{23}) \tag{3.32}$$

$$\Im J'^{e\mu}_{12} = -\Im J'^{e\mu}_{13} = \Im J'^{e\mu}_{23}$$

$$= \cos(\delta')\sin(\theta'_{12})\cos^2(\theta'_{13})\sin(\theta'_{13})\cos(\theta'_{23})\sin(\theta'_{23}) \tag{3.33}$$

We need to go to second order because the second term of $\Re J'^{e\mu}_{12}$ isn't suppressed by θ_{13}, so it is not negligible.

We can obtain, then from these, $P(\nu_e \to \nu_\mu)$:

$$P_0 = \sin^2(\theta_{23})\frac{\sin^2(2\theta_{13})}{\hat{C}^2}\sin^2(\hat{\Delta}\hat{C}) \tag{3.34}$$

$$P_{\sin\delta} = \tfrac{1}{2}\alpha\tfrac{\sin(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})}{\hat{A}\hat{C}\cos(\theta_{13}^2)}\sin(\hat{C}\hat{\Delta})$$

$$\times[\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta}] \tag{3.35}$$

$$P_{\cos\delta} = \tfrac{1}{2}\alpha\tfrac{\cos(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})}{\hat{A}\hat{C}\cos(\theta_{13}^2)}\sin(\hat{C}\hat{\Delta})$$

$$\times[\sin((1+\hat{A})\hat{\Delta} - \sin(\hat{C}\hat{\Delta})] \tag{3.36}$$

$$P_1 = -\alpha\tfrac{1-\hat{A}\cos(2\theta_{13})}{\hat{C}^3}\sin^2(\theta_{12})\sin^2(2\theta_{13})\sin^2(\theta_{23})\hat{\Delta}$$
$$\times\sin(2\hat{\Delta}\hat{C}) + \alpha\tfrac{2\hat{A}(-\hat{A}+\cos(2\theta_{13}))}{\hat{C}^4}$$

$$\times\sin^2(\theta_{12})\sin^2(2\theta_{13})\sin^2(\theta_{23})\sin^2(\hat{\Delta}\hat{C}) \tag{3.37}$$

$$\begin{aligned} P_2 = \alpha\frac{-1+\hat{C}+\hat{A}\cos(2\theta_{13})}{2\hat{C}^2\hat{A}\cos^2(\theta_{13})} \\ \cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})\sin^2(\hat{\Delta}\hat{C}) \end{aligned} \tag{3.38}$$

$$P_3 = \alpha^2\frac{2\hat{C}\cos^2(\theta_{23})\sin^2(2\theta_{12})}{\hat{A}^2\cos^2(\theta_{13})(-\hat{A}+\hat{C}+\cos(2\theta_{13})}\sin^2(\frac{1}{2}(1+\hat{A}-\hat{C})\hat{\Delta}) \tag{3.39}$$

We can expand the $\hat{A}$-dependent parts of P_1, P_2, and P_3 to first order in θ_{13} to obtain:

$$\frac{1-\hat{A}\cos(2\theta_{13})}{\hat{C}^3} = +\frac{1}{(\hat{A}-1)^2} \tag{3.40}$$

$$\frac{2\hat{A}(-\hat{A}+\cos(2\theta_{13}))}{\hat{C}^4} = -\frac{2\hat{A}}{(\hat{A}-1)^3} \tag{3.41}$$

$$\frac{-1+\hat{C}+\hat{A}\cos(2\theta_{13})}{2\hat{C}^2\hat{A}\cos^2(\theta_{13})} = 0 \tag{3.42}$$

$$\frac{2\hat{C}}{\cos^2(\theta_{13})(-\hat{A}+\hat{C}+\cos(2\theta_{13}))} = 1 \tag{3.43}$$

Because P_1 is quadratic in $\sin(\theta_{13})$ and P_2 is 0 to first order, we can conclude that they are negligibly small compared to $P_{\sin\delta}$ and $P_{\cos\delta}$ and can be dropped. However, we need to keep P_3 because it isn't suppressed by θ_{13}.

The expressions for the eigenvalues and eigenvectors are not good at the atmospheric resonance. The source of this problem is second order in θ_{13}. This issue only affects the $P_{\cos\delta}$ term and only for large values of θ_{13}. This problem can be mitigated by neglecting the subleading terms. The modified $P_{\sin\delta}$ and $P_{\cos\delta}$ are then:

$$P_{\sin\delta} = \alpha\tfrac{\sin(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})}{\hat{A}\hat{C}\cos(\theta_{13}^2)}$$

$$\times\sin(\hat{C}\hat{\Delta})\sin(\hat{\Delta})\sin(\hat{A}\hat{\Delta}) \tag{3.44}$$

$$P_{\cos\delta} = \alpha \tfrac{\cos(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})}{\hat{A}\hat{C}\cos(\theta_{13}^2)}$$

$$\times \sin(\hat{C}\hat{\Delta})\sin(\hat{\Delta})\sin(\hat{A}\hat{\Delta}) \tag{3.45}$$

Neglecting all subleading terms in θ_{13}, we obtain as our final probability:

$$P_0 = \sin^2(\theta_{23})\frac{\sin^2(2\theta_{13})}{(\hat{A}-1)^2}\sin^2((\hat{A}-1)\hat{\Delta}) \tag{3.46}$$

$$\begin{aligned} P_{\sin\delta} = \alpha \frac{\sin(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})}{\hat{A}(1-\hat{A})}\sin(\hat{\Delta}) \\ \sin(\hat{A}\hat{\Delta})\sin((1-\hat{A})\hat{\Delta}) \end{aligned} \tag{3.47}$$

$$\begin{aligned} P_{\cos\delta} = \alpha \frac{\cos(\delta)\cos(\theta_{13})\sin(2\theta_{12})\sin(2\theta_{13})\sin(2\theta_{23})}{\hat{A}(1-\hat{A})} \\ \cos(\hat{\Delta})\sin(\hat{A}\hat{\Delta})\sin((1-\hat{A})\hat{\Delta}) \end{aligned} \tag{3.48}$$

$$P_3 = \alpha^2\frac{\cos^2(\theta_{23})\sin^2(2\theta_{12})}{\hat{A}^2}\sin^2(\hat{A}\hat{\Delta}) \tag{3.49}$$

Using the experimental parameters in Table (1.2) and the neutrino physics parameters in Table (1.1), with $\delta_{CP} = 0$, we obtain the following probabilities:

TABLE 3.2. Oscillation Probabilities for T2K, MINOS, NOνA, and LBNE for $\delta_{CP} = 0$

Experiment	Probability
T2K	0.0594878
MINOS	0.0180609
NOνA	0.0547998
LBNE	0.0572652

CHAPTER 4

Mann, Kafka, Schneps, and Altinok (MKSA) Method

4.1. Preliminaries

The goal of the paper "Exact Probability with Perturbative Form for $\nu_\mu \to \nu_e$ Oscillations in Matter of Constant Density" by W. Mann, T. Kafka, J. Schneps, and O. Altinok[5] is to obtain the exact oscillation probability of neutrinos in matter by determining the evolution operator. MKSA starts with the Hamiltonian for vacuum oscillations in the mass basis and then transforms into the flavor basis, before adding a matter perturbation to it. They then transform into the propagation basis and finally into the interaction picture. After exponentiating the Hamiltonian, for which a closed form can be found, the resultant evolution operator is transformed back into the flavor basis. The probability amplitude can be read from this evolution operator.

Due to the large number of variables in this section, the table which summarizes them all is given at the end of the chapter.

For neutrino propagation in vacuum, the Hamiltonian in the mass basis $\vec{\nu}_i$ (i=1,2,3) is:

$$\hat{H}_0^{(i)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_{21}^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_{31}^2}{2E} \end{pmatrix} \tag{4.1}$$

One can transform from the mass basis $|\nu_i\rangle$ to the flavor basis $|\nu_\alpha\rangle$ by using the PMNS matrix in the following manner:

$$\vec{\nu}^{(\alpha)} = \hat{U}_{(mix)}\vec{\nu}^{(i)} \tag{4.2}$$

The PMNS matrix is defined as the product of the following matrices:

$$\hat{U}_{(mix)} \equiv \hat{R}_1(\theta_{23})\hat{I}_{\delta_{CP}}\hat{R}_2(\theta_{13})\hat{I}_{-\delta_{CP}}\hat{R}_3(\theta_{12})$$

where:

$$\hat{I}_{\delta_{CP}} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_{CP}} \end{pmatrix}$$

and $\hat{R}_1(\theta_{23}), \hat{R}_2(\theta_{13})$, and $\hat{R}_3(\theta_{12})$ are defined in Chapter 1.

Together, this can be written as:

$$\hat{U}_{(mix)} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix} \tag{4.3}$$

The time-dependent Schrodinger equation (TDSE) for flavor eigenstates is:

$$i\frac{d}{dt}\vec{\nu}^{(\alpha)}(t) = \hat{H}_0^{(\alpha)}\vec{\nu}^{(\alpha)}(t) \tag{4.4}$$

The Hamiltonian in the flavor basis is then given by the following rotation:

$$\hat{H}_0^{(\alpha)} = (\hat{R}_1\hat{I}_{\delta_{CP}}\hat{R}_2\hat{I}_{-\delta_{CP}}\hat{R}_3)\hat{H}_0^{(i)}(\hat{R}_3^T\hat{I}_{\delta_{CP}}\hat{R}_2^T\hat{I}_{-\delta_{CP}}\hat{R}_1^T) \tag{4.5}$$

Because $\hat{I}_{-\delta_{CP}}$ commutes with $\hat{R}_3$ and $\hat{R}_3^T$ commutes with $\hat{I}_{\delta_{CP}}$, and because $\hat{I}_{-\delta_{CP}}\hat{H}_0^{(i)}\hat{I}_{\delta_{CP}} = \hat{H}_0^{(i)}$, we can rewrite the Hamiltonian in the flavor basis as:

$$\hat{H}_0^{(\alpha)} = (\hat{R}_1\hat{I}_{\delta_{CP}})\hat{H}_0^{(23)}(\hat{I}_{-\delta_{CP}}\hat{R}_1^T) \tag{4.6}$$

where $\hat{H}_0^{(23)} = \hat{R}_2\hat{R}_3\hat{H}_0^{(i)}\hat{R}_3^T\hat{R}_2^T$

Written out explicitly:

$$\hat{H}_0^{(23)} = \begin{pmatrix} s_{12}^2c_{13}^2\alpha + s_{13}^2 & \frac{1}{2}c_{13}\alpha' & \frac{1}{2}\sin 2\tilde{\theta}_{13} \\ \frac{1}{2}c_{13}\alpha' & c_{12}^2\alpha & -\frac{1}{2}s_{13}\alpha' \\ \frac{1}{2}\sin 2\tilde{\theta}_{13} & -\frac{1}{2}s_{13}\alpha' & s_{12}^2s_{13}^2\alpha + c_{13}^2 \end{pmatrix} \tag{4.7}$$

where $\alpha' \equiv sin2\theta_{12}\alpha$ and $\alpha \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \simeq 1/32$

The discussion of this method has, until now, concerned only oscillations in vacuum. For oscillations in matter, a matter interaction perturbation term is added to the main Hamiltonian:

$$\hat{H}_{matter}^{(\alpha)} = \begin{pmatrix} V_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.8}$$

where $V_e \equiv \frac{A}{2l_\nu}$, $l_v \equiv \frac{E_\nu}{\Delta m_{31}^2}$ is the vacuum oscillation length, $A \equiv \pm\frac{2^{3/2}G_f n_e E_\nu}{\Delta m_{31}^2}$ is the matter potential, G_f is the Fermi coupling constant, and n_e is the electron density in matter. The reason why we have an interaction term in the 1,1 position in the matrix is that while all three flavors of neutrinos can react with the electrons in the earth via a neutral current interaction, electron neutrinos can also interact via a charged current interaction. This extra interaction pathway means that electron neutrinos will interact with electrons in the earth far more than other types of neutrinos.

4.2. Oscillations in Matter

The Hamiltonian must now be transformed into the propagation basis. The TDSE for such an eigenstate is:

$$i\frac{d}{dt}\vec{\nu}^{(p)} = \hat{H}^{(p)}\vec{\nu}^{(p)} \tag{4.9}$$

where $\hat{H}^{(p)} \equiv \hat{H}_0^{(23)} + \hat{H}_{matter}^{(\alpha)}$

An eigenstate in the propagation basis can be obtained from an eigenstate in the flavor basis by the following transformation, which can be derived by using Equations (4.6) and (4.8) in (4.4):

$$\vec{\nu}^{(p)} = \hat{I}_{-\delta_{CP}}\hat{R}_1^T(\theta_{23})\vec{\nu}^{(\alpha)} \tag{4.10}$$

We can "re-phase" this Hamiltonian by subtracting out the following terms, all of which are proportional to the identity matrix and which just give a global phase:

$$\tfrac{c_{13}^2}{2l_\nu}\hat{I},\ \tfrac{1}{4l_\nu}(A - \cos 2\theta_{13})\hat{I},\ \tfrac{1}{4l_\nu}s_{12}^2\alpha\hat{I}$$

This then yields:

$$\hat{H}^{(p)} = \frac{1}{4l_\nu}\begin{pmatrix} -(\cos 2\tilde{\theta}_{13} - A) & c_{13}\alpha' & \sin 2\tilde{\theta}_{13} \\ c_{13}\alpha' & -[(1+A)+\alpha''] & -s_{13}\alpha' \\ \sin 2\tilde{\theta}_{13} & -s_{13}\alpha' & +(\cos 2\tilde{\theta}_{13} - A) \end{pmatrix} \tag{4.11}$$

where $\alpha'' \equiv (1 - 3c_{13}^2)\alpha$

We can simplify this by defining the following five new variables:

$$G \equiv \tfrac{1}{4l_\nu}[(1+A)+\alpha''],\ Q \equiv \tfrac{1}{4l_\nu}[\cos 2\tilde{\theta}_{13} - A],\ f \equiv \tfrac{1}{4l_\nu}\sin 2\tilde{\theta}_{13},$$

$$a \equiv \tfrac{1}{4l_\nu}[c_{13}\alpha'],\ b \equiv \tfrac{1}{4l_\nu}[-s_{13}\alpha']$$

yielding:

$$\hat{H}^{(p)} = \begin{pmatrix} -Q & a & f \\ a & -G & b \\ f & b & +Q \end{pmatrix} \tag{4.12}$$

Now that the Hamiltonian is in the propagation basis, it must be formulated in the interaction picture. To do so, we separate $\hat{H}^{(p)}$ into the unperturbed piece $\hat{H}_0^{(p)}$ and the perturbed piece $\hat{V}$:

$$\hat{H}^{(p)} = \hat{H}_0^{(p)} + \hat{V} = \begin{pmatrix} -Q & 0 & f \\ 0 & -G & 0 \\ f & 0 & +Q \end{pmatrix} + \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{pmatrix} \tag{4.13}$$

An eigenstate in the propagation basis can be transformed into one in the interaction picture by:

$$\vec{\nu}^{(I)}(t) = e^{i\hat{H}_0^{(p)}t}\vec{\nu}^{(p)}(t), \tag{4.14}$$

yielding a TDSE of :

$$i\frac{d}{dt}\vec{\nu}^{(I)}(t) = \hat{V}_I\vec{\nu}^{(I)}(t) \tag{4.15}$$

where $\hat{V}_I \equiv e^{i\hat{H}_0^{(p)}t}\hat{V}e^{-i\hat{H}_0^{(p)}t}$

As stated earlier, we are interested in the time evolution operator in the interaction picture:

$$\vec{\nu}^{(I)}(t) = \hat{U}_I(t,0)\vec{\nu}^{(I)}(0) \tag{4.16}$$

Using this, we can rewrite our wave equation as:

$$i\frac{d}{dt}\hat{U}_I(t,0) = \hat{V}_I(t)\hat{U}_I(t,0) \tag{4.17}$$

To obtain our evolution operator, we must exponentiate our unperturbed Hamiltonian in the propagation basis. We use the following expansion:

$$\hat{W} \equiv e^{i\hat{H}_0^{(p)}t} = \sum_{i=0}^{\infty} \frac{(i\hat{H}_0^{(p)}t)^n}{n!} \tag{4.18}$$

This yields:

$$e^{i\hat{H}_0^{(p)}t} = \begin{pmatrix} W_{11} & 0 & W_{13} \\ 0 & e^{-iGt} & 0 \\ W_{31} & 0 & W_{33} \end{pmatrix} \tag{4.19}$$

Because neither the middle row, nor the middle column of $\hat{H}_0^{(p)}$ mix with the other rows and columns, we can work in a reduced 2×2 space:

$$\hat{H}_R^{(p)} = \begin{pmatrix} -Q & f \\ f & +Q \end{pmatrix} = f\hat{\sigma}_x - Q\hat{\sigma}_z \tag{4.20}$$

where we have invoked the Pauli matrices. We can write $\hat{H}_R^{(p)}$ as $\vec{N} \bullet \hat{\sigma}$ where $\vec{N} = (f, 0, -Q)$. Further defining $\hat{n}$ as $\frac{\vec{N}}{N}$ and noting that in natural units, t=l, we obtain:

$$e^{i\hat{H}_R^{(p)}(t=l)} = e^{i\hat{n}\bullet\vec{\sigma}(Nl)} = e^{i\hat{n}\bullet\vec{\sigma}\phi} \tag{4.21}$$

where $\phi \equiv nl$ is the rotation angle about $\hat{n}$, which serves as our axis of rotation in this reduced space.

With $\hat{n} = (n_x, 0, n_z)$, we can now write:

$$e^{i\hat{n}\bullet\vec{\sigma}\phi} = \begin{pmatrix} \cos\phi + in_z sin\phi & in_x \sin\phi \\ in_x \sin\phi & \cos\phi - in_z \sin\phi \end{pmatrix} \tag{4.22}$$

If we define $\gamma \equiv \cos\phi + in_z \sin\phi$ and $\beta \equiv n_x \sin\phi$, then we can write:

$$e^{i\hat{H}_0^{(p)}t} = \begin{pmatrix} \gamma & 0 & i\beta \\ 0 & e^{-iGt} & 0 \\ i\beta & 0 & \gamma^* \end{pmatrix} \tag{4.23}$$

Using this, we can now express $\hat{V}_I(t)$ as:

$$\hat{V}_I(l) = \begin{pmatrix} 0 & (\gamma a + i\beta b)e^{iGl} & 0 \\ (\gamma^* a - i\beta b)e^{-iGl} & 0 & (\gamma b - i\beta a)e^{-iGl} \\ 0 & (\gamma^* b + i\beta a)e^{iGl} & 0 \end{pmatrix} \tag{4.24}$$

If we define $u \equiv (\gamma a + i\beta b)e^{iGl}$ and $v \equiv (\gamma b - i\beta a)e^{-iGl}$, we can write:

$$\hat{V}_I(l) = \begin{pmatrix} 0 & u & 0 \\ u^* & 0 & v \\ 0 & v^* & 0 \end{pmatrix} \tag{4.25}$$

To obtain the evolution operator, we must exponentiate (4.25). It can be shown that $(\hat{V}_I)^{n=odd} = \eta^{n-1}\hat{V}_I$ and $(\hat{V}_I)^{n=even} = \eta^{n-2}\hat{V}_I^2$ where $\eta \equiv \frac{\alpha'}{4l_v}$ and $\eta^2 = |u|^2 + |v|^2$. Therefore:

$$e^{i\hat{V}_I l} = \sum_{n=0}^{\infty} \frac{(-i\hat{V}_I l)^n}{n!} = \hat{1} - (\frac{\hat{V}_I}{\eta})^2(1 - \cos(\eta l)) - i\frac{\hat{V}_I}{\eta}\sin(\eta l) \tag{4.26}$$

If we make the following substitutions:

$$\theta \equiv \eta l,\ \bar{u} \equiv \tfrac{u}{\eta},\ \bar{v} \equiv \tfrac{v}{\eta}, (1 - \cos\theta) = 2\sin^2\tfrac{\theta}{2}$$

We can write the evolution operator in the interaction picture as:

$$\hat{U}_I(l,0) = \begin{pmatrix} 1 - 2|\bar{u}|^2\sin^2\frac{\theta}{2} & -i\bar{u}\sin\theta & -2\bar{u}\bar{v}\sin^2\frac{\theta}{2} \\ -i\bar{u}^*\sin\theta & \cos\theta & -i\bar{v}\sin\theta \\ -2(\bar{u}\bar{v})^*\sin^2\frac{\theta}{2} & -i\bar{v}^*\sin\theta & 1 - 2|\bar{v}|^2\sin^2\frac{\theta}{2} \end{pmatrix} \tag{4.27}$$

Before continuing, it is helpful to make the following substitutions to simplify the algebra:

$$D_u \equiv 1 - 2|\bar{u}|^2 \sin^2 \tfrac{\theta}{2},\ D_v \equiv 1 - 2|\bar{v}|^2 \sin^2 \tfrac{\theta}{2},$$
$$d \equiv \cos\theta,\ e \equiv \bar{u}\sin\theta,\ p \equiv -2\bar{u}\bar{v}\sin^2 \tfrac{\theta}{2},\ k \equiv \bar{v}\sin\theta$$

Now that we have the evolution operator in the interaction picture, we can now transform it back to the flavor basis. The transformation from the interaction picture to the propagation basis is:

$$\hat{U}^{(p)}(l,0) = e^{-i\hat{H}_0^{(p)}}\hat{U}_I(l,0) \tag{4.28}$$

The evolution operator in the propagation basis, using the substitutions preceding (4.28) is:

$$\hat{U}^{(p)}(l,0) = \begin{pmatrix} (\gamma^* D_u - i\beta p^*) & (\gamma^*(-ie) - \beta k^*) & (\gamma^* p - i\beta D_v) \\ (-ie^*)e^{iGl} & de^{iGl} & (-ik)e^{iGl} \\ (\gamma p^* - i\beta D_u) & (\gamma(-ik^*) - \beta e) & (\gamma D_v - i\beta p) \end{pmatrix} \tag{4.29}$$

To switch into the flavor basis, the following transformation is used:

$$\hat{U}^{(\alpha)}(l,0) = \hat{R}_1(\theta_{23})\hat{I}_{\delta cp}\hat{U}^{(p)}(l,0)\hat{I}_{-\delta cp}\hat{R}_1^T(\theta_{23}) \tag{4.30}$$

The full matrix is presented in Appendix A. For $\nu_\mu \rightarrow \nu_e$, we need element $U_{12}^{(\alpha)}$=$A(\nu_\mu \rightarrow \nu_e)$, which equals, after some substitutions back into earlier notations:

$$A(\nu_\mu \rightarrow \nu_e) = (-i)s_{23}\beta e^{-i\delta cp} + (-i)c_{23}[\gamma^*\bar{u} - i\beta\bar{v}^*]\sin\theta$$

$$+2s_{23}[i\beta|\bar{v}|^2 - \gamma^*\bar{u}\bar{v}]\sin^2(\frac{\theta}{2})e^{-i\delta_{CP}} \tag{4.31}$$

Recalling that $P(\nu_\mu \rightarrow \nu_e) = |A(\nu_\mu \rightarrow \nu_e)|^2$, it can be shown that the probability for muon neutrinos to shapeshift into electron neutrinos is:

$$
\begin{aligned}
P(\nu_\mu \to \nu_e) = {} & (\sin 2\tilde{\theta}_{13})^2 s_{23}^2 \frac{\sin^2(D\Delta)}{D^2} + \sin 2\tilde{\theta}_{13} c_{13} \sin 2\theta_{23} \sin(\alpha'\Delta) \\
& \times \frac{\sin(D\Delta)}{D}[\cos\Delta' \cos\delta_{cp} - \sin\Delta' \sin\delta_{cp}] \\
& + c_{13}^2 c_{23}^2 \sin^2(\alpha'\Delta) \\
& - 2\sin 2\theta_{13} \sin 2\tilde{\theta}_{13} s_{23}^2 F_A \sin^2(\frac{\alpha'\Delta}{2})\frac{\sin^2(D\Delta)}{D^2}]] \\
& + \sin 2\theta_{13} c_{13} \sin 2\theta_{23} \sin(\alpha'\Delta) \sin^2(\frac{\alpha'\Delta}{2}) \\
& [\cos(D\Delta)\sin(\Delta' + \delta_{CP}) - F_A \frac{\sin(D\Delta)}{D} \cos(\Delta' + \delta_{cp})] \\
& + \sin^2 2\theta_{13} s_{23}^2 \sin^4(\frac{\alpha'\Delta}{2})[\cos^2(D\Delta) + F_A^2 \frac{\sin^2(D\Delta)}{D^2}]
\end{aligned}
\tag{4.32}
$$

where

$$
\Delta \equiv \tfrac{\Delta m_{31}^2 l}{4E_\nu} = \tfrac{l}{4l_\nu},\ \sin 2\tilde{\theta}_{13} = (1 - s_{12}^2\alpha)\sin 2\theta_{13},\ \Delta' \equiv Gl = \Delta[(1+A) + \alpha'']
$$

$$
\alpha'' \equiv (1 - 3c_{12}^2)\alpha, \text{ and } F_A \equiv [c_{13}^2(1 - s_{12}^2\alpha) - (cos2\tilde{\theta}_{13} - A)]
$$

Assuming a δ_{CP} of 0, the oscillation probabilities for the four baselines are given in Table (4.2).

Table 4.1. Variables used in MKSA

MKSA Variable	Definition
α	$\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$
Δ	$\frac{\Delta m^2_{31} l}{4E_\nu}$
A	$\frac{2^{3/2} G_F n_e E_\nu}{\Delta m^2_{31}}$
V_e	$\frac{A}{2l_\nu}$
α'	$\sin(2\theta_{12})\alpha$
α''	$(1-3c^2_{12})\alpha$
$\sin(2\tilde{\theta}_{13})$	$(1-s^2_{12}\alpha)\sin(2\theta_{13})$
$\cos(2\tilde{\theta}_{13})$	$(1-s^2_{12}\alpha)\cos(2\theta_{13})$
N	$\frac{1}{4l_\nu}[(\sin(2\tilde{\theta}_{13}))^2 + (\cos(2\tilde{\theta}_{13}) - A)^2]^{1/2}$
η	$\frac{\alpha'}{4l_\nu}$
G	$\frac{1}{4l_\nu}[(1+A)+\alpha'']$
F_A	$[c^2_{13}(1-s^2_{12}\alpha) - (\cos(2\tilde{\theta}_{13}) - A)]$
D	$4l_\nu N$
Δ'	Gl
Q	$\frac{1}{4l_\nu}(\cos(2\tilde{\theta}_{13}) - A)$
f	$\frac{1}{4l_\nu}(\sin(2\tilde{\theta}_{13}))$
a	$\frac{1}{4l_\nu}[c_{13}\alpha']$
b	$\frac{1}{4l_\nu}[-s_{13}\alpha']$
γ	$\cos\phi + in_z\sin\phi$
β	$n_x\sin\phi$
u	$(\gamma a + i\beta b)e^{iGl}$
v	$(\gamma b + i\beta a)e^{-iGl}$
θ	ηl
$\bar{u}$	$\frac{u}{\eta}$
$\bar{v}$	$\frac{v}{\eta}$
D_u	$1-2\lvert\bar{u}\rvert^2\sin^2(\frac{\theta}{2})$
D_v	$1-2\lvert\bar{v}\rvert^2\sin^2(\frac{\theta}{2})$
d	$\cos\theta$
e	$\bar{u}\sin\theta$
p	$-2\bar{u}\bar{v}\sin^2(\frac{\theta}{2})$
k	$\bar{v}\sin\theta$

Table 4.2. Oscillation Probabilities for T2K, MINOS, NOνA, and LBNE for $\delta_{CP} = 0$

Experiment	Probability
T2K	0.0576435
MINOS	0.0177052
NOνA	0.0533319
LBNE	0.0556574

CHAPTER 5

Comparison of the Methods

In this section, we present and compare the probabilities versus δ_{CP} and versus E_ν for $\delta_{CP} = 0$ of $\nu_\mu \to \nu_e$ oscillations for each formula, using the T2K, MINOS, NOνA, and LBNE parameters, given in Table (1.2).

Plots of the methods for Probability vs δ_{CP} for each of the four baselines are given as Figures (5.1)-(5.4).

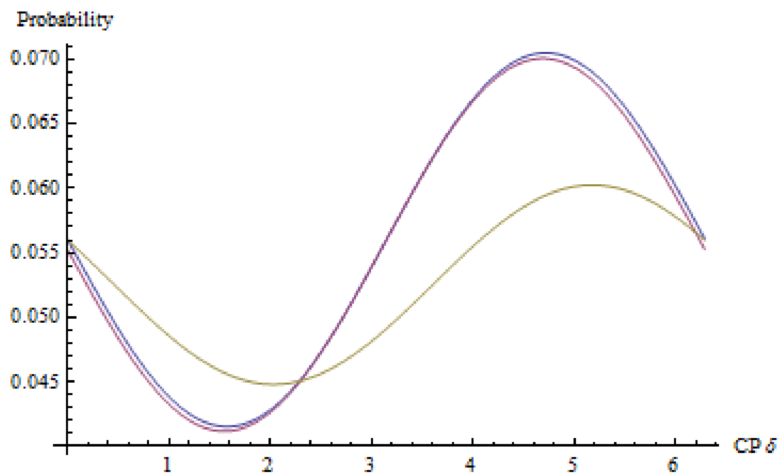

Figure 5.1. Plots of the probability vs δ_{CP} for the three different methods using T2K parameters. The curve colors include Freund (blue, top), MKSA (purple, middle), and AKS (gold, bottom)

Each formula can be simplified down to three terms. This is calculated for the T2K case and shown in Table 5.1:

From Table (5.1), it can be seen that the AKS formula differs significantly from the others in that it has the lowest constant value and has a smaller sin(δ) value than the others. It is a trivial exercise to show that if one combines the cos and sin terms in each formula into

Table 5.1. Formulas using the T2K parameters for the Values of the Mass-Squared Splittings and Mixing Angles

Source	Formula
AKS	$.0544576 + .00344891 * cos(\delta) - .0068977 * sin(\delta)$
Freund	$.0593976 - .0000902519 * cos(\delta) - .0148669 * sin(\delta)$
MKSA	$.0587505 - .0110705 * cos(\delta) - .0147829 * sin(\delta)$

a single phase-shifted cos term, the amplitudes of the resultant terms are extremely close to .0148 for Freund and MKSA, but not AKS, which has an amplitude of .0077. It is easy

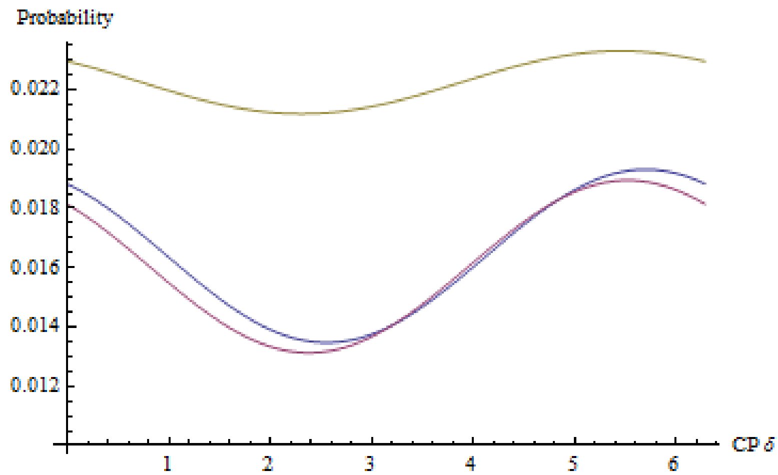

Figure 5.2. Plots of the probability vs δ_{CP} for the three different methods using MINOS parameters. The curve colors include Freund (blue, top), MKSA (purple, middle), and AKS (gold, bottom)

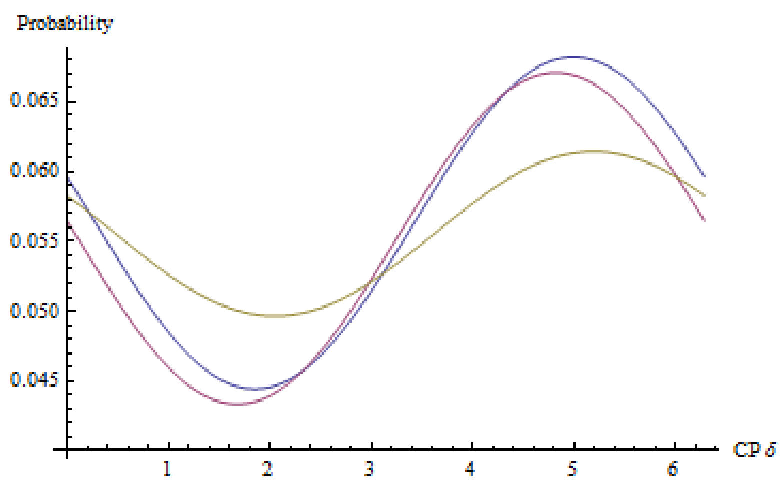

Figure 5.3. Plots of the probability vs δ_{CP} for the three different methods using NOvA parameters. The curve colors include Freund (blue, top), MKSA (purple, middle), and AKS (gold, bottom)

to see in Figure 5.1 that the phase shifts from a pure cos term for the resultant expressions is around 90 degrees for Freund and MKSA, but not AKS, where it is about 63 degrees, since the MKSA and Freund formulas appear very close to $-\cos(\delta)$. When plotting each of these three formulas, if one "shuts off" the $\cos(\delta)$ term in each one, the Freund and MKSA formulas are very close to each other while the AKS formula differs significantly, suggesting that the $\sin(\delta)$ term plays a very important role in differentiating them. When the $\sin(\delta)$ term is eliminated, though, all three formulas differ from each other significantly, suggesting that the difference is really a combination of both sinusoidal terms.

The MKSA formula, although different from the other fomulas in structure, due to it being exact, does come remarkably close to the others when plotted. As it agrees well with Freund, it suggests that both MKSA and Freund are quite accurate for the T2K case.

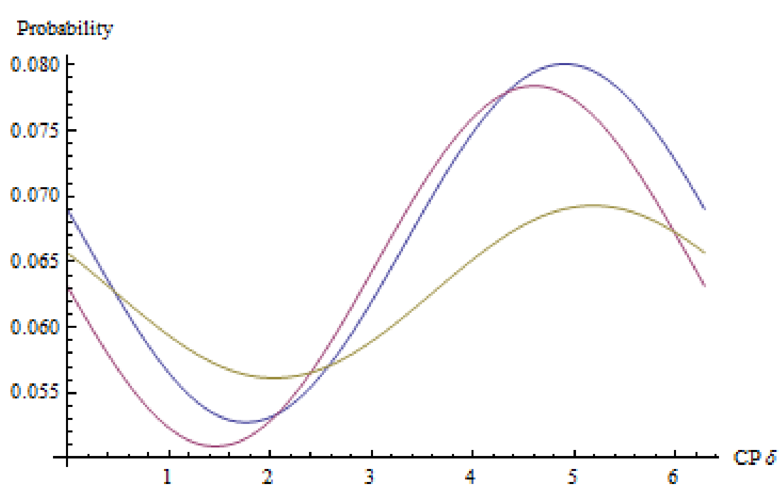

FIGURE 5.4. Plots of the probability vs δ_{CP} for the three different methods using LBNE parameters. The curve colors include Freund (blue, top), MKSA (purple, middle), and AKS (gold, bottom)

Figures (5.5)-(5.8) are plots of the probability vs energy for each of the 4 baselines, using $\delta_{CP} = 0$.

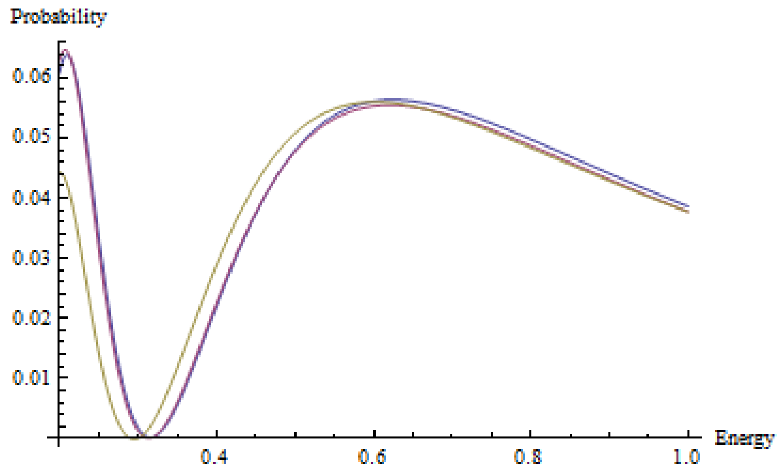

FIGURE 5.5. Probability vs Energy for $\delta_{CP} = 0$ for each of the five methods, using T2K parameters (color coding is the same as in Figure 5.1)

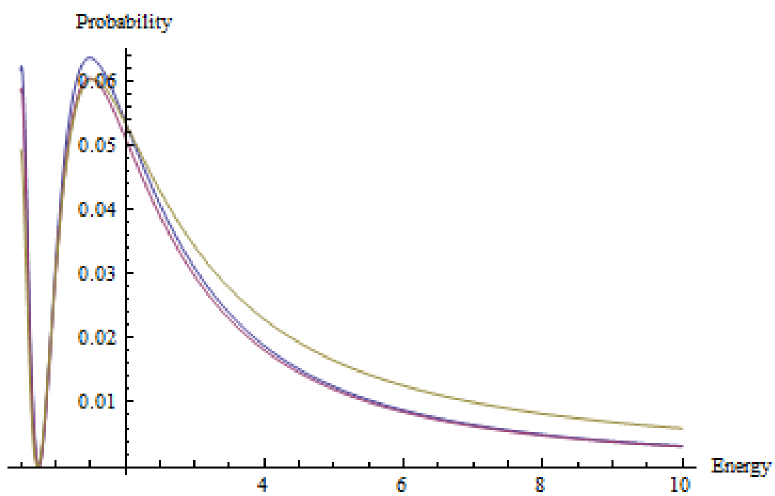

FIGURE 5.6. Probability vs Energy for $\delta_{CP} = 0$ for each of the five methods, using MINOS parameters (color coding is the same as in Figure 5.1)

Based off of Figure (5.5), at the energy and baseline length of T2K, the three formulas are close to each other, so it would seem that it does not matter greatly which formula is used, though Figure (5.1) suggests that AKS should be avoided for other reasons. Figures

(5.6)-(5.8), however, shows that at higher energies and longer baselines, the formulas diverge a fair amount. The MKSA formula is very close to Freund in this circumstance. It would appear that to be safe, the MKSA formula should be used as it is exact, despite being relatively complicated, while the other formulas differ from it under various circumstances.

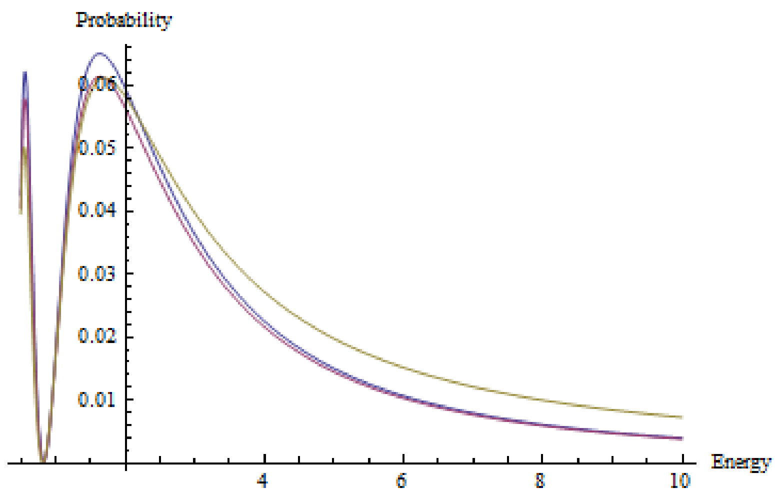

FIGURE 5.7. Probability vs Energy for $\delta_{CP} = 0$ for each of the five methods, using NOvA parameters (color coding is the same as in Figure 5.1)

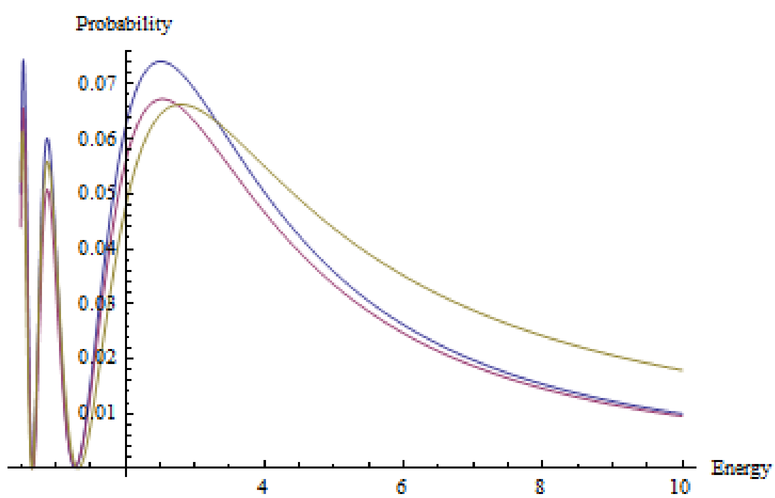

FIGURE 5.8. Probability vs Energy for $\delta_{CP} = 0$ for each of the five methods, using LBNE parameters (color coding is the same as in Figure 5.1)

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APPENDIX A

The Evolution Operator for MKSA

$$\hat{U}^{(\alpha)}_{1,1}(l,0) = -ip^*\beta + D_u\gamma^* \tag{A.1}$$

$$\hat{U}^{(\alpha)}_{1,2}(l,0) = -(k^*\beta + ie\gamma^*)\cos(\theta_{23}) + e^{-i\delta}(-iD_v\beta + p\gamma^*)\sin(\theta_{23}) \tag{A.2}$$

$$\hat{U}^{(\alpha)}_{1,3}(l,0) = e^{-i\delta}(-iD_v\beta + p\gamma^*)\cos(\theta_{23}) + (k^*\beta + ie\gamma^*)\sin(\theta_{23}) \tag{A.3}$$

$$\hat{U}^{(\alpha)}_{2,1}(l,0) = -ie^{iGl}e^*\cos(\theta_{23}) + e^{i\delta}(-iD_u\beta + p^*\gamma)\sin(\theta_{23}) \tag{A.4}$$

$$\hat{U}^{(\alpha)}_{2,2}(l,0) = de^{iGl}\cos^2(\theta_{23}) - ie^{-i\delta}(e^{iGl}k + e^{2i\delta}(-ie\beta + k^*\gamma))\cos(\theta_{23})\sin(\theta_{23}) + (-ip\beta + D_v\gamma)\sin^2(\theta_{23}) \tag{A.5}$$

$$\hat{U}^{(\alpha)}_{2,3}(l,0) = -ie^{i(Gl-\delta)}k\cos^2(\theta_{23}) - (de^{iGl} + ip\beta - D_v\gamma)\cos(\theta_{23})\sin(\theta_{23}) + e^{i\delta}(e\beta + ik^*\gamma)\sin^2(\theta_{23}) \tag{A.6}$$

$$\hat{U}^{(\alpha)}_{3,1}(l,0) = e^{i\delta}(-iD_u\beta + p^*\gamma)\cos(\theta_{23}) + ie^{iGl}e^*\sin(\theta_{23}) \tag{A.7}$$

$$\hat{U}^{(\alpha)}_{3,2}(l,0) = e^{-i\delta}(-e^{2i\delta}(e\beta+ik^*\gamma)\cos^2(\theta_{23})-e^{i\delta}(de^{iGl}+ip\beta-D_v\gamma)\cos(\theta_{23})\sin(\theta_{23})+ie^{iGl}k\sin^2(\theta_{23}) \tag{A.8}$$

$$\hat{U}^{(\alpha)}_{3,3}(l,0) = (-ip\beta+D_v\gamma)\cos^2(\theta_{23})+e^{-i\delta}(ie^{iGl}k+e^{2i\delta}(e\beta+ik^*\gamma))\cos(\theta_{23})\sin(\theta_{23})+de^{iGl}\sin^2(\theta_{23}) \tag{A.9}$$

APPENDIX B

Location of Mathematica Notebooks

The notebook for the Probability vs CP Angle plot is named ProbvsCP.nb and the notebook for the Probability vs Energy plot is named ProbvsEnergy.nb Both notebooks are located at http://hep.colostate.edu/t2k/jmla/